A GENERALIZATION OF EPSTEIN ZETA FUNCTIONS

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(*With a supplementary note by* HERMANN WEYL)

§ 1. An Epstein zeta function (in its simplest form) is a function represented by the Dirichlet's series

$$\sum{}' \frac{\exp i\,(n_1\alpha_1+\cdots+n_k\alpha_k)}{(n_1^2+n_2^2+\cdots+n_k^2)^s}, \qquad s = \sigma + i\tau \tag{1}$$

where $\alpha_1 \ldots \alpha_k$ are real and $n_1, n_2, \ldots n_k$ run through integral values. The properties of this function are well known and the simplest of them were proved by Epstein [2, 3]. The aim of this note is to define a general class of Dirichlet's series, of which the above can be viewed as an instance, and to discuss the problem of analytic continuation of such series.

The point of view is the following: The exponential functions $\exp i(n_1x_1 +\cdots+ n_kx_k)$ may be viewed as the eigenfunctions of

$$\Delta u + \lambda u \equiv \sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} + \lambda u = 0 \tag{2}$$

in the domain $D \equiv (0 \leq x_1, \ldots, x_k \leq 2\pi)$, the boundary condition being

$$\begin{aligned} u(0, x_2, \ldots, x_k) &= u(2\pi, x_2, \ldots, x_k) \\ u(x_1, 0, x_3, \ldots, x_k) &= u(x_1, 2\pi, x_3, \ldots, x_k) \\ \cdots \quad & \quad \cdots \\ u(x_1, \ldots, x_{k-1}, 0) &= u(x_1, \ldots, x_{k-1}, 2\pi). \end{aligned} \tag{3}$$

The associated eigenvalues may be verified to be $n_1^2+\ldots+n_k^2$. If we arrange the eigenfunctions and eigenvalues into sequences $\omega_n(x)$, λ_n we observe that (1) may be written as

$$\sum{}' \frac{\omega_n(x)\overline{\omega_n(y)}}{\lambda_n^s}, \qquad x = (x_1, \ldots, x_k)$$

(we have only to set $\alpha_i = x_i - y_i$ in (1)), where Σ' indicates that the zero eigenvalue is omitted in the summation.

Taking this point of view, we shall consider the eigenfunctions of (2) for a general class of domains and for the I and II boundary value problems and study the corresponding Dirichlet's series which may be easily defined.

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§ **2.** In the k-dimensional Euclidean space, let D denote a bounded domain with a regular boundary B, so that the following boundary value problem

$$(4) \qquad \Delta u + \lambda u \equiv \sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} + \lambda u = 0,$$

$$\text{I} \qquad u(x) \equiv u(x_1, \ldots, x_k) = 0 \qquad \text{on } B$$

or

$$\text{II} \qquad \frac{\partial u}{\partial n}(x) = 0 \qquad \text{on } B$$

where $\frac{\partial}{\partial n}$ denotes the derivative along the normal, has an infinite number of non-negative eigenvalues with the associated eigenfunctions. We denote the eigenvalues and eigenfunctions by λ_n and $\omega_n(x)$ respectively[1] and assume that the eigenvalues are arranged according to non-decreasing order of magnitude and that the eigenfunctions form a complete normal orthogonal set. We observe, that in the case of the second boundary value problem viz. II, $\lambda_0 = 0$ is a simple eigenvalue with the eigenfunction $\omega_0(x) = \frac{1}{\sqrt{V}}$, where V is the volume of D, while in I, all the eigenvalues are positive. So it will be convenient to assume that the suffix n runs through $1, 2, 3, \ldots$ in case I and through $0, 1, 2, \ldots$ in case II and that $\lambda_1 > 0$.

With the above notations we have the following

THEOREM. *The series* $\sum_{n=1}^{\infty} \omega_n(x)\omega_n(y)/\lambda_n^s$, $s = \sigma + i\tau$, *which has a finite abscissa of absolute convergence, represents an entire function of* s, *if* x *and* y *are distinct points in* D, *with the so-called "trivial zeros" at* $s = 0, -1, -2, \ldots$ *in case* I *and at* $s = -1, -2, -3, \ldots$ *in case* II.

On the other hand the series $\sum_{n=1}^{\infty} \omega_n^2(x)/\lambda_n^s$ *which also has a finite abscissa of convergence, represents a meromorphic function of* s *with a simple pole at* $s = {}^k/_2$ *and residue* $1/\Gamma(k/2)(2\sqrt{\pi})^k$ *and with the so-called "trivial zeros" at* $s = 0, -1, -2, \ldots$ *in case* I *and at* $s = -1, -2, -3, \ldots$ *in case* II.

§ **3.** The proof of the theorem depends on a lemma on the Green's function for the heat equation

$$(5) \qquad \Delta u - \frac{\partial u}{\partial t} = 0$$

which we state below. Let $G(x, y; t)$ denote the Green's function for the heat

[1]Each eigenvalue being repeated according to its multiplicity and the eigenfunctions being real.

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equation (5) associated with either of the boundary conditions[2] I, II and further satisfying the initial condition

$$\lim_{t=0} G(x, y; t) = 0, \qquad x, y \text{ in } D, x \neq y.$$

It is known that $G(x, y; t)$ can be expressed in the form

$$G(x, y; t) = \frac{1}{(2\sqrt{\pi t})^k} \exp\left(-\frac{\Sigma(x_i - y_i)^2}{4t}\right) - g(x, y; t) \tag{6}$$

where $\dfrac{1}{(2\sqrt{\pi t})^k} \exp\left(-\dfrac{\Sigma(x_i - y_i)^2}{4t}\right)$ is the fundamental solution of (5). We have then the following

LEMMA. If l_y *denotes the minimum distance between* y *and points on* B, *then, in any finite interval of* t,

$$|g(x, y; t)| < \frac{c}{t^{k/2}} e^{-l_y^2/4t} \tag{7}$$

for all x *in* D, *where* c *is a constant independent of* x *and* t.

Assuming the lemma for the present we shall prove the theorem. We first observe that

$$G(x, y; t) = \sum \omega_n(x)\omega_n(y)e^{-\lambda_n t}$$

the summation on the right running from 1 to ∞ or 0 to ∞ according as the first or the second boundary condition is under consideration. So we can write the above equation in a more convenient form, viz.:

$$\sum_{n=1}^{\infty} \omega_n(x)\omega_n(y)e^{-\lambda_n t} = G(x, y; t) - \frac{\delta}{V} \tag{8}$$

where $\delta = 0$ or 1 according as I or II boundary problem is taken into consideration. Multiplying both sides of (8) by t^{s-1} and integrating with respect to t from 0 to ∞, we note that if $R(s)$ is sufficiently large the left side gives

$$\Gamma(s) \sum \omega_n(x)\omega_n(y)/\lambda_n^s \tag{9}$$

the series being absolutely convergent.[3] We now have to evaluate

$$\begin{aligned}
&\int_0^{\infty}\left(G(x, y; t) - \frac{\delta}{V}\right) t^{s-1}dt \\
&\quad = \int_0^1\left(G(x, y; t) - \frac{\delta}{V}\right) t^{s-1}dt + \int_1^{\infty} \sum_{n=1}^{\infty} \omega_n(x)\omega_n(y)e^{-\lambda_n t}t^{s-1}dt \\
&\quad = \int_0^1 G(x, y; t)\, t^{s-1}dt - \frac{\delta}{s.V} + \int_1^{\infty} \sum_{n=1}^{\infty} \omega_n(x)\omega_n(y)e^{-\lambda_n t}t^{s-1}dt \qquad (10)\\
&\quad = -\frac{\delta}{sV} + \int_0^1 G(x, y; t)\, t^{s-1}dt + \sum \omega_n(x)\omega_n(y) \int_1^{\infty} e^{-\lambda_n t}t^{s-1}dt
\end{aligned}$$

if $R(s) > 0$.

[2]Though the Green's functions are different in the two boundary problems we use the same notation $G(x, y; t)$ and there need be no confusion.

[3]The iterated Green's function of Δu will belong to L_2 for a sufficiently high order of iteration, from which it will follow that $\Sigma\omega_n^2(x)/\lambda_n^\sigma$ is convergent if σ is large and $\Sigma\omega_n(x)\omega_n(y)/\lambda_n^\sigma$ will be absolutely convergent for large σ.

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The last integral on the right of (10) is easily seen to be an entire function of s, while the second is

$$(11) \qquad \frac{1}{(2\sqrt{\pi})^k}\int_0^1 t^{s-k/2-1}\exp\left(-\frac{\sum_{i=1}^{k}(x_i-y_i)^2}{4t}\right)dt - \int_0^1 g(x,y;t)t^{s-1}dt.$$

The first integral in (11) is an entire function of s, if $x \neq y$ and the second is also an entire function of s on account of the lemma since $l_y \neq 0$ as long as y is an interior point. So combining (9) and (10) we note that

$$(12) \qquad \sum_{n=1}^{\infty}\frac{\omega_n(x)\omega_n(y)}{\lambda_n{}^s} + \frac{\delta}{V.\,\Gamma(s+1)} = \frac{E(s)}{\Gamma(s)}$$

where $E(s)$ denotes an entire function of s. (12) proves everything stated in the theorem about the series on the left. Incidentally it shows that while the analytic continuation of the Dirichlet's series vanishes at $s = 0$ in case I, it does not vanish in case II; in fact it takes the value $-\frac{1}{V}$.

If the two points x and y coincide, we have only to put $x = y$ in (9), (10), and (11) and observe that if $R(s)$ is large enough

$$\Gamma(s)\sum_{n=1}^{\infty}\omega_n^2(x)/\lambda_n{}^s = -\frac{\delta}{sV} + \frac{1}{(2\sqrt{\pi})^k}\cdot\frac{1}{s-k/2}$$

$$(13) \qquad -\int_0^1 g(x,x;t)\,t^{s-1} - \sum_{n=1}^{\infty}\omega_n^2(x)\int_1^{\infty}e^{-\lambda_n t}t^{s-1}dt.$$

While the last integral is easily seen to be an entire function of s, the last but one can be seen to be an entire function of s by the lemma as long as x is an interior point. Hence we have

$$(14) \qquad \sum_{n=1}^{\infty}\frac{\omega_n^2(x)}{\lambda_n{}^s} = \frac{1}{(2\sqrt{\pi})^k}\cdot\frac{1}{\Gamma(s)(s-k/2)} - \frac{\delta}{V\Gamma(s+1)} + \frac{E(s)}{\Gamma(s)}$$

which proves everything about $\Sigma\, \omega_n^2(x)/\lambda_n{}^s$ stated in the theorem.[4]

REMARKS: 1. It may be of interest to remark that by applying Ikehara's Tauberian theorem to (14) we obtain the following asymptotic distribution of eigenfunctions:

$$(15) \qquad \sum_{\lambda_n \leq T}\omega_n^2(x) \sim \frac{1}{(2\sqrt{\pi})^k\Gamma(k/2+1)}\,T^{k/2},$$

a result which can also be proved by applying Hardy-Littlewood's Tauberian theorem on Dirichlet's series to (8) with $x = y$ (see (2) in [5]).

[4]This series for the two dimensional domain was first studied by T. Carleman [1].

2. Another point of interest[5] is to study the nature of the series $\Sigma\, 1/\lambda_n{}^s$. If $R(s)$ is large and positive

$$\Gamma(s) \sum_{n=1}^{\infty} 1/\lambda_n{}^s = \int_D dx \int_0^1 \Sigma\omega_n{}^2(x)^{-\lambda_n t} t^{s-1} dt$$

(16)
$$= \int_D dx \int_0^1 + \int_D dx \int_1^\infty .$$

As usual the second integral on the right of (16) is an entire function while the first is

$$-\frac{\delta}{s} + \frac{V}{(2\sqrt{\pi})^k} \cdot \frac{1}{s - {}^k/_2} - \int_D dx \int_0^1 g(x, x; t)\, t^{s-1} dt$$

and by the lemma

$$\left| \int_D dx \int_0^1 g(x, x; t)\, t^{s-1} dt \right| \leq c \int_0^\infty dl_x \int_0^1 t^{\sigma - k/2 - 1} e^{-l_x^2/4t} dt$$

$$\leq c \int_0^\infty e^{-u} u^{-\frac{1}{2}} du \int_0^1 v^{\sigma - k/2 - \frac{1}{2}} dv$$

on using the transformation $l_x^2/4t = u$; $t = v$. The above integral on the extreme right is finite if $\sigma > {}^k/_2 - \frac{1}{2}$, so that the first integral on the right of (16) is regular for $\sigma > {}^k/_2 - \frac{1}{2}$. Thus

(17)
$$\Sigma \frac{1}{\lambda_n{}^s} = -\frac{\delta}{\Gamma(s+1)} + \frac{V}{(2\sqrt{\pi})^k} \cdot \frac{1}{\Gamma(s)(s - {}^k/_2)} + \frac{F(s)}{\Gamma(s)}$$

where $F(s)$ is regular for $R(s) > {}^k/_2 - \frac{1}{2}$. Again by an application of Ikehara's Tauberian theorem to (17) we have the following well-known asymptotic distribution of eigenvalues: If $N(T)$ is the number of eigenvalues $\leq T$,

$$N(T) \sim \frac{V}{(2\sqrt{\pi})^k \Gamma({}^k/_2 + 1)} T^{k/2}.$$

§ **4.** Now it remains to prove the lemma. For this purpose we introduce slightly different notations. We denote by capital letters $P, Q \ldots$ points interior to D and by small letters $p, q, \ldots$ points on its boundary B. If $P = (x_1, \ldots, x_k)$, $Q = (y_1, \ldots, y_k)$ we write

$$r_{PQ}{}^2 = \sum_{i=1}^k (x_i - y_i)^2,$$

$$h_{\alpha\beta}(P, s; Q, t) = \frac{1}{(2\sqrt{\pi})^k} \frac{r_{PQ}{}^\alpha}{(t-s)^\beta} \exp\left(-\frac{r_{PQ}{}^2}{4(t-s)}\right),$$

$\cos r_{pq} n_p = \cos rn =$ cosine of the angle between the radius vector r_{pq} and the normal n_p at r. According to these notations

$$G(x, y; t) = G(P, Q; t) = h_{0,\, k/2}(P, 0; Q, t) - g(P, Q; t).$$

[5]The author is indebted to Professor H. Weyl for this remark.

As a function of (Q, t), $g(P, Q; t)$ is a solution of the heat equation tending to zero as $t \to 0$, and takes the same boundary values as $h_{0, k/2}$ on B for all t in case I, while in the case II its normal derivative has the same value as the normal derivative of $h_{0, k/2}$ on B.

We shall first prove the lemma in the case of the II boundary value problem. In this case we can set

$$g(P, Q; t) = \int_B \int_0^t h_{0, k/2}(p, s, Q, t)\psi(p, s)dpds \tag{18}$$

where $\psi(p, s)$ is an unknown function to be determined. It is clear our lemma will follow if we can show that a function $\psi(p, s)$ which is bounded in every finite interval $0 \leq s \leq T$, uniformly on B can be determined to satisfy (18). To show this we note that $\psi(p, s)$ is a solution of the following integral equation:

$$\begin{aligned} \psi(q, t) + \int_B \int_0^t h_{1, k/2+1}(p, s, q, t) \cos rn\psi(p, s)dpds \\ = \frac{\partial}{\partial n} g(p, q, t) = h_{1, k/2+1}(p, 0, q, t) = \phi(q, t), \end{aligned} \tag{19}$$

say. Then $\psi(q, t)$ can be determined as a Neumann's series:

$$\begin{aligned} \psi(q, t) &= \phi_0(q, t) - \phi_1(q, t) + \phi_2(q, t) - \dots, \\ \phi_0(q, t) &= \phi(q, t), \\ \phi_i(q, t) &= \int_B \int_0^t h_{1, k/2+1}(p, s; q, t) \cos rn\phi_{i-1}(p, s)dpds. \end{aligned} \tag{20}$$

If $|\phi(q, t)| \leq M$, as is certainly the case in $0 \leq t \leq T$, one proves by induction

$$|\phi_i(q, t)| \leq \frac{MN^i t^{i/2}}{\Gamma(i/2 + 1)}, \tag{21}$$

where N is a constant independent of t and i, which can be determined.[6] This shows that if $\phi(q, t)$ is bounded in any finite interval $0 \leq t \leq T$, so is $\psi(q, t)$. Hence the lemma.

Corresponding argument can be made use of in the case of the first boundary value problem.

References

[1] T. Carleman, "Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes," *Scand. Math. Congress* (1934), 34-44.

[2] P. Epstein, "Zur Theorie allgemeiner Zetafunktionen I," *Math. Ann.*, vol. 56 (1903), 615-644.

[6]For a detailed discussion of these inequalities and the determination of the Green's function of the heat equation by Neumann's series cf. E. E. Levi [4, pp. 261-262.] We derive the inequality (21) assuming $\cos(rn) = 0(r)$ for small r. Similar inequalities can also be derived with $\cos(rn) = 0(r^{\alpha})$ $0 < \alpha \leq 1$ which is a regularity condition on B.

[3] P. Epstein, "Zur Theorie allgemeiner Zetafunktionen II," *Math. Ann.*, vol. 63 (1907), 205-246.
[4] E. E. Levi, "Sull' equatione del calore," *Annali di Math.*, (3a), vol. 14 (1902), 187-264.
[5] S. Minakshisundaram, "Notes on Fourier Expansions II," *J. London Math. Soc.*, vol. 20 (1945), 148-153.

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Supplementary Note by Hermann Weyl

At Minakshisundaram's request I add another proof of his fundamental lemma. This alternative proof is restricted to the boundary value problem I, but for this case yields a more complete result. Take for y a fixed interior point y_0 and set $l_{y_0} = l$. The maximum of the main part of Green's function along the boundary is

$$(4\pi t)^{-k/2} \exp\left(-\,l^2/4t\right) = j(t),$$

and this function $j(t)$ is on the increase as long as t moves from 0 to $T_0 = l^2/2k$.

Instead of splitting the integrals $\int_0^\infty$ into $\int_0^1 + \int_1^\infty$, as Minakshisundaram does, one splits into $\int_0^{T_0} + \int_{T_0}^\infty$. I maintain that in the first part, i.e. for $0 < t < T_0$, the compensating function $u(x; t) = g(x, y_0\,; t)$ satifies the inequality

$$0 \leq u(x; t) \leq j(t).$$

Proof. (1) For any t in this interval let $m(t)$ and $M(t)$ denote minimum and maximum of $u(x; t)$ as a function of x. Suppose that, contrary to the statement, $m(t_1)$ is negative for a certain t_1 between 0 and T_0, and that $u(x_1; t_1) = m(t_1)$. Since the boundary values of $u(x; t_1)$ are positive, x_1 must be an interior point, and one must have $\Delta u \geq 0$ for $x = x_1$, $t = t_1$. Hence the equation $\frac{\partial u}{\partial t} = \Delta u$ shows that $\frac{\partial u}{\partial t} \geq 0$ for the same values. Choose a positive constant a and form

$$v = e^{-at} \,.\, u, \quad \frac{\partial u}{\partial t} - au = e^{at} \,.\, \frac{\partial v}{\partial t}\,;$$

one then sees that $\frac{\partial v}{\partial t} > 0$ for $x = x_1, t = t_1$. Hence $v(x_1; t)$ actually decreases during a short time, $t_1 \geq t \geq t'_1$, while t decreases starting with the value t_1. Set $e^{-at}.\, m(t) = m^*(t)$. As $m^*(t) \leq v(x_1; t)$ we have *a fortiori* $m^*(t) \leq m^*(t_1)$ in that interval. Thus $m^*(t)$ goes down and continues to be negative while t decreases. That makes it possible to repeat the argument and to conclude that in the whole interval $t_1 \geq t > 0$ the function $m^*(t)$ decreases along with t

and thus stays less than or equal to $m^*(t_1)$. But this is a contradiction, since $v(x; t) \to 0$ for $t \to 0$.

(2) Suppose that, contrary to the proposition, $M(t_1) > j(t_1)$ for a definite value t_1 in the interval $0 < t_1 < T_0$. Then the maximum $M(t_1)$ is taken on at an interior point $x = x'_1$. We have $\Delta u \leq 0$ and hence $\dfrac{\partial u}{\partial t} \leq 0$ for $x = x'_1$, $t = t_1$, and thus $\dfrac{\partial u^*}{\partial t} < 0$ for the same values and $u^*(x; t) = u(x; t) - j(t)$. As $M(t) \geq u(x'_1; t)$, the "span" $M^*(t) = M(t) - j(t)$ thus increases during a short time while t decreases from t_1 downward. It therefore stays positive, and repetition of the argument shows that the span is increasing with decreasing t during the whole time $t_1 \geq t > 0$, a result that contradicts the limit equations $u(x; t) \to 0$, $j(t) \to 0$ for $t \to 0$.

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