INTEGRABILITY AND L-CONVERGENCE OF
MULTIPLE TRIGONOMETRIC SERIES

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It is proved that under the following condition, the sum f of the double trigonometric series with coefficients c_{jk} is integrable and the rectangular partial sums $s_{mn}(f, x, y)$ converge to f in L^1-norm:

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (\ln |j|)^{\top} (\ln |k|)^{\top} |\Delta_{11} c_{jk}| < \infty.$$

This generalises the corresponding results of Móricz [3]. We also prove that the aforementioned condition is sharp. A more general version of this result is established for double series of orthonormal functions, which generalises Móricz-Schipp-Wade [4]. An extension to higher-dimensions is given.

1. INTRODUCTION

Let $T^2 = \{(x, y) \in R^2 : -\pi \leqslant x, y < \pi\}$. Consider the double trigonometric series

$$(1.1) \qquad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)}.$$

We assume that the coefficients c_{jk} satisfy the following two conditions:

$$(1.2) \qquad c_{jk} \to 0 \quad \text{as} \quad \max(|j|, |k|) \to \infty,$$

$$(1.3) \qquad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (\ln |j|)^{\top} (\ln |k|)^{\top} |\Delta_{11} c_{jk}| < \infty,$$

where $\xi^{\top} = \max(1, \xi)$ and $\Delta_{11} c_{jk} = c_{jk} - c_{j,k+1} - c_{j+1,k} + c_{j+1,k+1}$. Let $s_{mn}(x, y)$ denote the rectangular partial sums of series (1.1), that is,

$$s_{mn}(x, y) = \sum_{|j| \leqslant m} \sum_{|k| \leqslant n} c_{jk} e^{i(jx+ky)}.$$

If the coefficients c_{jk} are the Fourier coefficients of $f \in L^1(T^2)$, then we write $s_{mn}(f, x, y)$ instead of $s_{mn}(x, y)$. Recently, Móricz [3] considered the following two kinds of double series:

$$(1.4) \qquad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \lambda_j \lambda_k a_{jk} \cos jx \cos ky,$$

$$(1.5) \qquad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \sin jx \sin ky,$$

where $\lambda_0 = 1/2$ and $\lambda_j = 1$ for $j \geqslant 1$. He proved:

THEOREM. (Móricz [3, Theorems 2, 4 and 5].) *Assume that (1.1) is either (1.4) or (1.5). If both of conditions (1.2) and (1.3) are satisfied, then the sum f of series (1.1) is integrable, (1.1) is the Fourier series of f, and $\|s_{mn}(f) - f\|_1 \to 0$ as $\min(m, n) \to \infty$.*

This result does not deal with the general case (1.1), in particular, the double cosine-sine series

$$(1.6) \qquad \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \lambda_j a_{jk} \cos jx \sin ky,$$

where λ_j are given as above. The purpose of this paper is to extend the aforementioned result of Móricz to any series of type (1.1). Of course, this result will apply to any of (1.4)–(1.6). Our result reads as follows:

THEOREM 1. *If both of (1.2) and (1.3) are satisfied, then the sum f of series (1.1) is integrable, (1.1) is the Fourier series of f, and $\|s_{mn}(f) - f\|_1 \to 0$ as $\min(m, n) \to \infty$.*

It should be noticed that condition (1.3) cannot be replaced by

$$(1.7) \qquad \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \{(\ln|j|)^\top\}^{1-\varepsilon} \{(\ln|k|)^\top\}^{1-\delta} |\Delta_{11} c_{jk}| < \infty,$$

where $\varepsilon, \delta \geqslant 0$ and one of them is greater than 0. The following is a counterexample to the case $\varepsilon > 0$ and $\delta > 0$:

$$(1.8) \qquad \sum_{j=2}^{\infty} \sum_{k=2}^{\infty} \frac{\sin jx \sin ky}{(\ln j)(\ln k)}.$$

Write (1.8) in the form of (1.1) and let c_{jk} be the corresponding coefficients. Then

$$|\Delta_{11} c_{jk}| \leqslant M \left(|j|^\top |k|^\top \right)^{-1} \{(\ln|j|)^\top (\ln|k|)^\top\}^{-2}$$

for all j and for all k, where M is an absolute constant. Thus, (1.7) is satisfied for all $\varepsilon > 0$ and all $\delta > 0$. However $\sum_{j=2}^{\infty} (\sin jx)/(\ln j)$ is not a Fourier series, (see [6, Vol I, p.253]), and so the sum of series (1.8) is not integrable. This says that the conclusion of Theorem 1 fails for this case. On the other hand, we point out that for ε, $\delta \leqslant 1$ condition (1.7) implies

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c_{jk}| < \infty.$$

Thus, it follows from [2] that the sum f of (1.1) is an $L^p(T^2)$-function for all $0 < p < 1$ and $s_{mn}(x, y)$ converges in the $L^p(T^2)$-metric to f as $\min(m, n) \to \infty$.

As will be pointed out later, the theory developed here can be extended to any double series of the following type:

$$(1.9) \qquad\qquad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{jk}\phi_j(x)\phi_k(y),$$

where $\phi_j(t)$ are measurable functions defined on $[0, \alpha)$. The prerequisite is the existence of a sequence $\{\Psi_j(t): j \geqslant -1\}$ of measurable functions such that $\Psi_{-1}(t) = 0$,

$$(1.10) \qquad\qquad |\Psi_j(t)| \leqslant A(|j|)^{\top} \qquad (j \geqslant 0),$$

$$(1.11) \qquad\qquad |\Psi_j(t)| \leqslant B/|t| \qquad (j \geqslant 0),$$

$$(1.12) \qquad\qquad \Psi_j(t) - \Psi_{j-1}(t) = \phi_j(t) \qquad (j \geqslant 0),$$

where A and B are two absolute constants. The precise statement for this is:

THEOREM 2. *If (1.2), (1.3) and (1.10)–(1.12) are satisfied, then the sum f of series (1.9) is integrable, and $\|s_{mn}(x, y) - f(x, y)\|_1 \to 0$ as $\min(m, n) \to \infty$, where*

$$s_{mn}(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} c_{jk}\phi_j(x)\phi_k(y) \qquad (m, n \geqslant 0).$$

In addition, if $\{\phi_j(t): j \geqslant 0\}$ forms an orthonormal system on $[0, \alpha)$, then

$$(1.13) \qquad c_{jk} = \int_0^{\alpha} \int_0^{\alpha} f(x, y)\phi_j(x)\phi_k(y)\,dx\,dy \qquad (j, k \geqslant 0).$$

Let $\phi_j(t) = w_j(t)$, where $\{w_j(t): j \geqslant 0\}$ is the Walsh orthonormal system defined on the interval $[0, 1)$, in the Paley enumeration. Choose $\Psi_j(t) = D_j(t)$, the corresponding Walsh-Dirichlet kernel. Then $\Psi_j(t)$ possesses the properties (1.10)–(1.12), (see [1] or [5]). Hence, the conclusion of Theorem 2 holds for double Walsh series. This says that Theorem 2 generalises Móricz-Schipp-Wade [4, Theorem 1].

2. Auxiliary lemmas

Let $\Psi_j(t)$, $j = 0+, 0-, \pm 1, \pm 2, \ldots$ be the functions defined by: $\Psi_{0+}(t) = \Psi_{0-}(t) = 1/2$,

$$\Psi_j(t) = \frac{1}{2} + \left(e^{it} + e^{2it} + \ldots + e^{jit}\right) \qquad (j \geqslant 1),$$

and $\Psi_{-j}(t) = \Psi_j(-t)$ for $j \geqslant 1$. Then we have

$$|\Psi_j(t)| \leqslant \min\left(2|j|^\top, \pi/|t|\right) \qquad \text{for all } j \text{ and for all } t.$$

Moreover, for $j \geqslant 0+$, we have

$$(2.1) \qquad\qquad \Psi_j(t) = \frac{e^{i(j+1)t}}{2ie^{it/2}\sin t/2} - \frac{\cos t/2}{2i\sin t/2} \qquad (0 < |t| \leqslant \pi).$$

The next lemma is useful in the proof of the main result.

LEMMA 1. *Let $\Psi_j(t)$ be defined as above. Then*

$$\int_{-\pi}^{\pi} |\Psi_j(t)|\, dt \leqslant 6\pi (\ln|j|)^\top \quad \text{for all } j.$$

PROOF OF LEMMA 1: Let $\rho = |j|^\top$. Then by (2.1), we get

$$\int_{\pi/\rho \leqslant |t| \leqslant \pi} |\Psi_j(t)|\, dt \leqslant \pi \left[\int_{\pi/\rho \leqslant |t| \leqslant \pi} (1/|t|)\, dt\right]$$

$$\leqslant 2\pi(\ln\rho) \leqslant 2\pi(\ln|j|)^\top.$$

On the other hand, we have

$$\int_{|t| < \pi/\rho} |\Psi_j(t)|\, dt \leqslant 4\pi(|j|)^\top/\rho = 4\pi.$$

Combining all that we have done so far, we obtain the desired inequality.   □

From the definition of $\Psi_j(t)$, we find that $\Psi_{0+}(t) + \Psi_{0-}(t) = 1$, and

$$\Psi_j(t) - \Psi_{j-1}(t) = e^{ijt} \qquad (j \geqslant 2); \quad \Psi_1(t) - \Psi_{0+}(t) = e^{it},$$
$$\Psi_{-j}(t) - \Psi_{-j+1}(t) = e^{-ijt} \qquad (j \geqslant 2); \quad \Psi_{-1}(t) - \Psi_{0-}(t) = e^{-it}.$$

These relations lead us to the following summation by parts formula, which can be proved by a direct calculation. We leave it to the reader.

LEMMA 2.

$$\sum_{|j| \leqslant m} a_j e^{ijt} = \sum_{|j|=0\pm}^{m} \Delta a_j \Psi_j(t) + \sum_{|j|=m} a_{\tau(j)} \Psi_j(t).$$

In this lemma, the sum $\sum\limits_{|j|=0\pm}^{m}$ means $\sum\limits_{0+\leqslant j \leqslant m} + \sum\limits_{-m \leqslant j \leqslant 0-}$. The function $\tau(j)$ is defined by the formulas: $\tau(0+) = 1$, $\tau(0-) = -1$, $\tau(j) = j + 1$ for $j \geqslant 1$, and $\tau(j) = j - 1$ for $j \leqslant -1$.

The first-order differences $\Delta_1 c_{jk}$ and $\Delta_2 c_{jk}$ are defined by the formulas:

$$\Delta_1 c_{jk} = c_{jk} - c_{\tau(j),k}, \ \Delta_2 c_{jk} = c_{jk} - c_{j,\tau(k)}$$

for all $j, k = 0\pm, 1, 2, \ldots$, where $c_{0+,k} = c_{0-,k} = c_{0k}$ and $c_{j,0+} = c_{j,0-} = c_{j0}$. The second-order mixed difference $\Delta_{11}^* c_{jk}$ is defined by the formula: $\Delta_{11}^* c_{jk} = \Delta_1 \Delta_2 c_{jk} = \Delta_2 \Delta_1 c_{jk}$. The symbols a_{0+}, a_{0-}, and Δa_j are defined in the same way.

Performing a double summation by parts (that is, applying Lemma 2 twice), we get another representation of $s_{mn}(x, y)$ as follows:

LEMMA 3.

$$s_{mn}(x, y) = \sum_{|j|=0\pm}^{m} \sum_{|k|=0\pm}^{n} \Delta_{11}^* c_{jk} \Psi_j(x) \Psi_k(y)$$

$$+ \sum_{|j|=0\pm}^{m} \sum_{|k|=n} \Delta_1 c_{j,\tau(k)} \Psi_j(x) \Psi_k(y)$$

$$+ \sum_{|j|=m} \sum_{|k|=0\pm}^{n} \Delta_2 c_{\tau(j),k} \Psi_j(x) \Psi_k(y)$$

$$+ \sum_{|j|=m} \sum_{|k|=n} c_{\tau(j),\tau(k)} \Psi_j(x) \Psi_k(y).$$

3. PROOF OF THEOREMS 1 AND 2

PROOF OF THEOREM 1: By comparing the definitions of Δ_{11} and Δ_{11}^*, we find that (1.3) is equivalent to

(3.1) $$\sum_{|j|=0\pm}^{\infty} \sum_{|k|=0\pm}^{\infty} (\ln |j|)^T (\ln |k|)^T |\Delta_{11}^* c_{jk}| < \infty.$$

For any j and any k, we have

$$|\Delta_1 c_{jk}| \leqslant \sum_{|v|=|k|}^{\infty} |\Delta_{11}^* c_{jv}|$$

and

$$|\Delta_2 c_{jk}| \leqslant \sum_{|u|=|j|}^{\infty} |\Delta_{11}^* c_{uk}|.$$

From here, we see that (3.1) implies both of the following two conditions:

$$(3.2) \qquad \left[\sum_{|j|=0\pm}^{\infty} (\ln|j|)^{\top}(\ln|k|)^{\top}|\Delta_1 c_{jk}| \right] \to 0 \quad \text{as} \quad |k| \to \infty,$$

$$(3.3) \qquad \left[\sum_{|k|=0\pm}^{\infty} (\ln|j|)^{\top}(\ln|k|)^{\top}|\Delta_2 c_{jk}| \right] \to 0 \quad \text{as} \quad |j| \to \infty.$$

Moreover,

$$|c_{jk}| \leqslant \sum_{|u|=|j|}^{\infty} \sum_{|v|=|k|}^{\infty} |\Delta_{11}^* c_{uv}|,$$

and so (3.1) implies that

$$(3.4) \qquad c_{jk}(\ln|j|)(\ln|k|) = o(1) \quad \text{as} \quad \min(|j|,|k|) \to \infty.$$

We know that $L^1(T^2)$ is complete. By Lemma 1, Lemma 3, and (3.1)–(3.4), we find that the sum f of series (1.1) is integrable, (1.1) is the Fourier series of f, and $\|s_{mn}(f) - f\|_1 \to 0$ as $\min(m,n) \to \infty$. That is what we want. $\square$

PROOF OF THEOREM 2: Following the proof of Lemma 1, we find that conditions (1.10) and (1.11) together imply

$$\int_0^{\alpha} |\Psi_j(t)|\, dt \leqslant (A\alpha + B)(\ln|j|)^{\top} \qquad (\text{all } j).$$

It is assumed that condition (1.12) is satisfied. Hence, results analogous to Lemmas 2 and 3 will hold. Applying the same proof as Theorem 1, with minor modification, we conclude that the sum f of series (1.9) is integrable and

$$\|s_{mn}(x,y) - f(x,y)\|_1 \to 0 \quad \text{as} \quad \min(m,n) \to \infty.$$

We assume that $\{\phi_j(t): j \geqslant 0\}$ forms an orthonormal system on $[0,\alpha)$. Then

$$c_{jk} = \int_0^{\alpha} \int_0^{\alpha} s_{mn}(x,y)\phi_j(x)\phi_k(y)\, dxdy \qquad (m \geqslant j \geqslant 0,\, n \geqslant k \geqslant 0),$$

and so by (1.10) and (1.12), we get

$$\left| c_{jk} - \int_0^{\alpha} \int_0^{\alpha} f(x,y)\phi_j(x)\phi_k(y)\, dxdy \right|$$

$$\leqslant \int_0^{\alpha} \int_0^{\alpha} |s_{mn}(x,y) - f(x,y)| \cdot |\phi_j(x)\phi_k(y)|\, dxdy$$

$$\leqslant \left(4A^2 |j|^{\top} |k|^{\top} \right) \|s_{mn}(x,y) - f(x,y)\|_1$$

$$\to 0 \quad \text{as} \quad \min(m,n) \to \infty.$$

From here, we get (1.13), and the proof is complete. $\square$

REMARK. Following the above proofs, we find that Theorems 1 and 2 can easily be extended to the higher-dimensional case. In this case, condition (1.3) is replaced by

$$\sum_{j_1\lambda=-\infty}^{\infty} \cdots \sum_{j_n\lambda=-\infty}^{\infty} (\ln |j_1|)^{\top} \cdots (\ln |j_n|)^{\top} |\Delta_1, \cdots ,_1 c_{j_1 \cdots j_n}| < \infty$$

or

$$\sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} (\ln j_1)^{\top} \cdots (\ln j_n)^{\top} |\Delta_1, \cdots ,_1 c_{j_1 \cdots j_n}| < \infty.$$

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