Proceedings of the Edinburgh Mathematical Society (2001) **44**, 455–478 ©

THE SECOND DERIVATIVE OF A MEROMORPHIC FUNCTION

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(Received 19 September 1999)

Abstract Let f be meromorphic of finite order in the plane, such that $f^{(k)}$ has finitely many zeros, for some $k \geqslant 2$. The author has conjectured that f then has finitely many poles. In this paper, we strengthen a previous estimate for the frequency of distinct poles of f. Further, we show that the conjecture is true if either

(i) f has order less than $1 + \varepsilon$, for some positive absolute constant ε, or

(ii) $f^{(m)}$, for some $0 \leqslant m < k$, has few zeros away from the real axis.

Keywords: meromorphic function; Nevanlinna theory; zeros of derivatives

AMS 2000 *Mathematics subject classification:* Primary 30D35

1. Introduction

Suppose that f is a function transcendental and meromorphic in the plane. By a theorem of Pólya [**9**, **26**], if f has at least two poles, then for each sufficiently large k the kth derivative $f^{(k)}$ has at least one zero. The following theorem confirmed a conjecture of Hayman [**8**] from 1959.

Theorem 1.1 (see [5, 7, 18]). *Suppose that $m \geqslant 0$ and $k \geqslant 2$ and that f is meromorphic in the plane such that $f^{(m)}$ and $f^{(m+k)}$ each have finitely many zeros. Then $f^{(m+1)}/f^{(m)}$ is a rational function. In particular, f has finite order and finitely many poles.*

We refer the reader to [**2**, **6**, **13**, **19**, **23**] for related results. Now, Gol'dberg has conjectured that the frequency of distinct poles of f is controlled by the frequency of zeros of a single derivative $f^{(k)}$, provided $k \geqslant 2$, and the author made the following, related conjecture in [**21**].

Conjecture 1.2. *Suppose that $k \geqslant 2$ and f is meromorphic of finite order in the plane and that $f^{(k)}$ has finitely many zeros. Then f has finitely many poles.*

Obviously, if Conjecture 1.2 is true for $k = 2$, then it is true for $k \geqslant 2$. On the other hand, Conjecture 1.2 is false for functions of infinite order, as shown in [**21**] by examples of the form $f''/f' = \mathrm{e}^h g^{-1}$ with g, h entire, for which both f' and f'' are zero-free. The

https://doi.org/10.1017/S0013091599001029 Published online by Cambridge University Press

following theorem, in which the notation is that of [**9**], summarizes some results in the direction of Conjecture 1.2.

Theorem 1.3 (see [22, 23]). *Suppose that f is meromorphic of finite order ρ in the plane and that f'' has finitely many zeros. Then*

$$\bar{N}(r,f) = O(\log r)^3, \quad r \to \infty. \tag{1.1}$$

If, in addition, f satisfies any one of the following, then f has finitely many poles:

(i) $N(r, 1/f') = o(r^{1/2})$ *as* $r \to \infty$;

(ii) $T(r,f) = O(r(\log r)^{\delta})$ *as* $r \to \infty$, *with* δ *a constant satisfying* $0 < 3200\mathrm{e}^{16}\delta < 1$;

(iii) *there exists* $\varepsilon > 0$ *such that all but finitely many poles* w *of* f *have multiplicity* $\mu(w) \leqslant |w|^{\rho-\varepsilon}$.

The main results of this paper are substantial improvements of (1.1) and of part (ii) of Theorem 1.3. First we have the following theorem.

Theorem 1.4. *Suppose that f is meromorphic of finite order ρ in the plane, and that f'' has finitely many zeros. Then*

$$\bar{N}(r,f) \leqslant \kappa(\log r)^2, \quad r \to \infty, \tag{1.2}$$

in which κ is a positive constant depending only on the asymptotic values of f'.

The key to the proof of Theorem 1.4 is a new way, described in § 4, of estimating f on regions where f' is close to its finite asymptotic values. Theorem 1.4 leads to the next result, establishing Conjecture 1.2 for functions of order not much greater than 1.

Theorem 1.5. *There exists a constant ε with $0 < \varepsilon < \frac{1}{2}$ such that if f is meromorphic of order less than $1 + \varepsilon$ in the plane and f'' has finitely many zeros, then f has finitely many poles.*

Our last result proves Conjecture 1.2 for functions for which some derivative $f^{(m)}$, with $0 \leqslant m < k$, has relatively few zeros away from the real axis.

Theorem 1.6. *Suppose that $0 \leqslant m < k$ and $k \geqslant 2$ and that $\phi(r)$ is a positive function tending to 0 as $r \to \infty$. Suppose further that f is meromorphic of finite order ρ in the plane, and that $f^{(k)}$ has finitely many zeros. Finally, suppose that*

$$\limsup_{r\to\infty} \frac{\log^+ N^*(r, 1/f^{(m)})}{\log r} < \rho_0 < \frac{1}{2 - 1/\rho}, \tag{1.3}$$

in which $N^*(r, 1/f^{(m)})$ counts the zeros of $f^{(m)}$ which lie outside the set $\{z : |\arg z^2| \leqslant \phi(|z|)\}$. Then f has finitely many poles.

Note that by part (ii) of Theorem 1.3 we may assume that $\rho \geqslant 1$ in Theorem 1.6. Functions satisfying the hypotheses of Theorem 1.6 abound: for example $f(z) = 1 - \mathrm{e}^{\mathrm{i}z}$. There is a substantial literature dealing with entire and meromorphic functions f, some of whose derivatives have only real zeros [**15**, **16**, **29**]. Theorem 1.6 does not really belong to this strand: rather, in addition to improving part (i) of Theorem 1.3, it shows that some extra geometric information on the distribution of zeros of $f^{(m)}$ suffices to prove Conjecture 1.2.

2. Lemmas needed for the theorems

Throughout this paper we denote by $B(z_0, r)$ the Euclidean disc $\{z : |z - z_0| < r\}$, by $S(z_0, r)$ the circle $\{z : |z - z_0| = r\}$, and by $A(z_0, R, S)$ the open annulus $\{z : R < |z - z_0| < S\}$.

Lemma 2.1 (see [17, 22]). *Suppose that $h(z) = \sum_{j=1}^{\infty} a_j z^j$ maps the disc $B(0, s)$ conformally onto a simply connected domain D of finite area A. Then, for real θ and $0 < r < s$, the length $L(r, \theta)$ of the image under h of the line segment $z = t\mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant t \leqslant r$, satisfies*

$$L(r, \theta)^2 \leqslant \frac{A}{\pi} \log\left(\frac{1}{1 - r^2 s^{-2}}\right).$$

Lemma 2.2 (see [24]). *Suppose that $d \geqslant 1$ and that F is transcendental and meromorphic in the plane with $T(r, f) = O(r^d)$ as $r \to \infty$. Then there exist arbitrarily small positive R such that $F(z)$ has no multiple points with $|F(z)| = R$ and the length $L(r, R, F)$ of the level curves $|F(z)| = R$ lying in $|z| \leqslant r$ satisfies $L(r, R, F) = O(r^{(3+d)/2})$ as $r \to \infty$.*

Next we require Tsuji's well-known estimate for harmonic measure [**30**, p. 116].

Lemma 2.3 (see [30]). *Let D be a simply connected domain not containing the origin, and let z_0 lie in D. Let $r \neq |z_0|$. Let $\theta(t)$ denote the angular measure of $D \cap S(0, t)$, and let D_r be the component of $D \backslash S(0, r)$ which contains z_0. Then the harmonic measure of $S(0, r)$ with respect to the domain D_r, evaluated at z_0, satisfies*

$$\omega(z_0, S(0, r), D_r) \leqslant C \exp\left(-\pi \int_I \frac{\mathrm{d}t}{t\theta(t)}\right), \tag{2.1}$$

in which C is an absolute constant, and $I = [2|z_0|, r/2]$ if $r > 4|z_0|$, with $I = [2r, |z_0|/2]$ if $4r < |z_0|$.

Note that (2.1) for $4r < |z_0|$ is obtained from the same estimate for the case $r > 4|z_0|$ by the substitution $\zeta = 1/z$.

Lemma 2.4. *Let $0 < \rho < 10^{-3}$ and let $\Omega = \{z : \rho < |z| < 1,\ \mathrm{Im}(z) > 0\}$. Let $F_0 = \{\mathrm{e}^{\mathrm{i}t} : \pi/3 \leqslant t \leqslant 2\pi/3\}$. Let z_1 lie in Ω with $200\rho \leqslant |z_1|$. Then*

$$\omega(z_1, F_0, \Omega) \geqslant c\rho^2 (|z_1|^{-1} - |z_1|) \sin(\arg z_1), \tag{2.2}$$

in which c is a positive constant, independent of ρ and z_1.

Proof. Let d_j denote positive constants, independent of ρ and z_1, and set $w = \phi(z) = 2\rho(z + 1/z)$. Then $|z| = \rho$ gives $|w| > 3/2$, so that $\phi(\Omega)$ contains the semi-disc $D_1 = \{w : |w| < 1,\ \mathrm{Im}(w) < 0\}$. Also $\phi(F_0) = G_0$ is a subset of $[-4\rho, 4\rho]$ of measure $d_1\rho$, and $w_1 = \phi(z_1)$ has $|w_1| \leqslant 1/50$.

Let ψ map D_1 to the unit disc, with $\psi(-\mathrm{i}/2) = 0$. Then the Schwarz reflection principle (or elementary calculation) gives $d_2 \leqslant |\psi'(w)| \leqslant 1/d_2$ for w in $D_1 \cap B(0, \frac{1}{4})$, and so Poisson's formula leads to (2.2), since

$$\omega(z_1, F_0, \Omega) \geqslant \omega(w_1, G_0, D_1) \geqslant d_3\rho \ \mathrm{dist}\{w_1, \partial D_1\} = d_3\rho|\mathrm{Im}(w_1)|.$$

□

Next we recall that for $0 < L < \infty$ and a subset E of $(0, \infty)$ the upper logarithmic density of E satisfies

$$\overline{\mathrm{logdens}}\, E = \limsup_{r\to\infty} \frac{\int_{[1,r]\cap E} \mathrm{d}t/t}{\log r} = \overline{\mathrm{logdens}}\, \{t : Lt \in E\}. \tag{2.3}$$

Lemma 2.5. *Let $S(r)$ be an unbounded positive non-decreasing function on $[r_0, \infty)$, continuous from the right, of finite order ρ. Let $A > 1$, $B > 1$. Then*

$$\overline{\mathrm{logdens}}\, G \leqslant \rho\left(\frac{\log A}{\log B}\right), \quad G = \{r \geqslant r_0 : S(Ar) \geqslant BS(r)\}.$$

Lemma 2.5 is stated in [**10**] for a characteristic function $T(r, F)$, but the proof goes through for $S(r)$. Finally, we require some standard facts from the Wiman–Valiron theory [**11**, **31**]. Let F be a transcendental entire function. Provided r is *normal* for F, that is provided r lies outside an exceptional set E of finite logarithmic measure, we have, for z_0 with $|z_0| = r$ and $|F(z_0)| > (1 - o(1))M(r, F)$,

$$\frac{F'(z_0)}{F(z_0)} = \nu(r)z_0^{-1}(1 + o(1)), \tag{2.4}$$

in which $\nu(r) = \nu(r, F)$ is the non-decreasing central index of F. Suppose now that G is transcendental and meromorphic in the plane, with finitely many poles $b_1, \dots, b_q$, repeated according to multiplicity. Then $F(z) = G(z)\prod_{j=1}^{q}(z - b_j)$ is entire and the estimate (2.4) holds with F replaced by G. Thus, with a slight abuse of notation, we may regard $\nu(r, F)$ as the central index of G.

3. Preliminaries

Suppose that h is transcendental and meromorphic in the plane, and that $h(z)$ tends to the finite complex number a as z tends to infinity along a path γ. Then the inverse function h^{-1} is said to have a transcendental singularity over a [**3**, **25**]. For each positive t, a domain $C(t)$ is uniquely determined as that component of the set $C'(t) = \{z : |h(z) - a| < t\}$ which contains an unbounded component of the intersection of $C'(t)$ with the path γ. Here $C(t) \subseteq C(s)$ if $0 < t < s$, and the intersection of all the $C(t)$, $t > 0$, is empty.

The singularity of h^{-1} over a corresponding to γ is said to be direct if $C(t)$, for some positive t, contains finitely many zeros of $h(z)-a$, and indirect otherwise. If the singularity is direct, then $C(t)$, for sufficiently small t, contains no zeros of $h(z)-a$. Singularities over ∞ are classified analogously.

Theorem 3.1 (see [3]). *If the transcendental meromorphic function h has finite order and the inverse function h^{-1} has an indirect transcendental singularity over a, then a is a limit point of critical values of h, that is, values taken by h at multiple points of h.*

Consequently, if h is meromorphic of finite order in the plane with finitely many critical values, then all transcendental singularities of h^{-1} are direct and, by the Denjoy–Carleman–Ahlfors Theorem [**3**, **25**], the number of direct transcendental singularities of h^{-1} is at most $2\rho(h)$.

Next we need a modification of some standard facts discussed in [**25**]. Suppose that F is a transcendental meromorphic function with finitely many asymptotic values a_n, and with finitely many critical values b_n. Suppose that F has no asymptotic values in $c_0 \leqslant |w| < \infty$ and no critical values in $c_1 \leqslant |w| < \infty$, where $c_0 \leqslant c_1$. Let V_0 be the domain obtained by deleting from the annulus $A(0, c_0, \infty)$, the half-open line segment

$$w = \rho e^{i \arg b_n}, \quad c_0 < \rho \leqslant c_1,$$

for each finite non-zero critical value b_n of F.

Consider a component C_0 of the set $F^{-1}(V_0)$, and choose $z_0 \in C_0$ and v_0 such that $e^{v_0} = w_0 = F(z_0)$. Then

$$\phi(v) = \psi(e^v) = F^{-1}(e^v),$$

with $\psi = F^{-1}$ the branch of the inverse function mapping w_0 to z_0, extends by continuation to an analytic function on the simply connected domain $U_0 = \{v : e^v \in V_0\}$.

Further, ϕ maps U_0 into C_0. Indeed, $\phi(U_0) = C_0$, for if $z^* \in C_0$ we may join z_0 to z^* by a path γ_1 in C_0 and choose a path γ_2 in U_0 starting at v_0 such that $e^{\gamma_2} = F(\gamma_1)$. Then $F(\phi(\gamma_2)) = F(\gamma_1)$ and so $\phi(\gamma_2) = \gamma_1$ by uniqueness of lifts, since both paths start at z_0.

There are now two possibilities. The first is that the function ϕ is univalent on C_0, so that the image under ϕ of $\mathrm{Re}(v) = 1 + \log c_1$ is a simple curve tending to infinity in both directions. Thus, by a standard argument, such as the Phragmén–Lindelöf principle, $\phi(u) \to \infty$ as $u \to \infty$ with $\mathrm{Re}(u) > 1 + \log c_1$, and C_0 is an unbounded simply connected domain containing a path tending to infinity on which $F(z) \to \infty$.

On the other hand, if ϕ is not univalent in U_0, then the open mapping theorem shows that ϕ has period $k2\pi i$, for some minimal positive integer k. In this case $\psi_1(\zeta) = \psi(\zeta^k) = \phi(k \log \zeta)$ extends to be analytic in $W_0 = \{\zeta : \zeta^k \in V_0\}$, mapping W_0 univalently onto C_0. Also, $\psi_1(\zeta)$ has a limit as $\zeta \to \infty$, which must be finite, and so a pole z_1 of F, since F is transcendental, and $F^{1/k} : C_0 \cup \{z_1\} \to W_0 \cup \{\infty\}$ is univalent.

The same two possibilities occur for any component C_1 of the set $\{z : c_0 < |F(z)| < \infty\}$ such that C_1 contains no critical point of F.

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4. An estimate on components where the derivative is small

Lemma 4.1. *Suppose that G is a transcendental meromorphic function of finite order ρ and that G' has no asymptotic values in $0 < |w| \leqslant d_1 < \infty$, and no critical values in $0 < |w| \leqslant d_2 \leqslant d_1$, and that G' has finitely many critical points z with $|G'(z)| \leqslant d_1$.*

Form the domain V_0 by deleting from the annulus $A(0, 0, d_1)$ the half-open line segment

$$w = s\mathrm{e}^{\mathrm{i}\arg b_n}, \quad d_2 \leqslant s < d_1,$$

for each non-zero finite critical value b_n of G'. Let D be a component of the set $(G')^{-1}(V_0)$ containing a path γ on which $G'(z) \to 0$ as $z \to \infty$.

Let N be an integer with $N > 2+\rho$. Choose d_3 with $0 < d_3 < d_2$ such that $|G'(z)| > d_3$ on some circle $S(0, \sigma)$ with $1 \leqslant \sigma \leqslant 2$, and let $D_1 = \{z \in D : |z| > \sigma,\ |G'(z)| < d_3\}$. Choose d as in Lemma 2.2, with $0 < d < d_3$, such that the length of the level curves $|z^N G'(z)| = d$ lying in $|z| \leqslant r$ is $O(r^{2+\rho})$ for all sufficiently large r. Define

$$u(z) = \begin{cases} \log^+\left|\dfrac{d}{z^N G'(z)}\right|, & \text{if } z \in D_1; \\ 0, & \text{otherwise.} \end{cases}$$

Then $u(z)$ is subharmonic in the plane, and D contains finitely many components W_j of the set $\{z : u(z) > 0\}$, and these are simply connected. Let $z^* \in W_j$. Then there exists $M^* > 0$ such that to each $z \in W_j$ corresponds a path γ_z from z^* to z, lying in the closure of W_j, with

$$\int_{\gamma_z} |t^\mu G'(t)|\,|\mathrm{d}t| \leqslant M^* \tag{4.1}$$

for each non-negative integer μ with $N - \mu > 2 + \rho$.

Finally, there exist positive constants S_0, S_1 depending on D such that for z in D with $|z| > S_0$ and $|G'(z)| < \mathrm{e}^{-1} d_1$ we have

$$|G(z)| \leqslant S_1 + \frac{C|zG'(z)|}{\log|d_1/G'(z)|}, \tag{4.2}$$

in which C is a positive absolute constant, in particular not depending on d_1, d_2, G or D.

Proof. The W_j are simply connected by the maximum principle, since $z^N G'(z) \neq 0, \infty$ on D_1, by the discussion in § 3. Since G' has finite order and

$$B_0(r, u) \leqslant 3m(2r, u) \leqslant 3m(2r, 1/G') + O(\log r), \quad r \to \infty,$$

in which $B_0(r, u) = \sup\{u(r\mathrm{e}^{\mathrm{i}t}) : 0 \leqslant t \leqslant 2\pi\}$, the number of W_j is finite [**12**, Chapter 8].

Next, if z is in W_j, then we join z^* to z by a path γ_z in the closure of W_j consisting of part of the ray $\arg t = \arg z^*$, part of the circle $|t| = |z|$, and part of the boundary ∂W_j of W_j. Dividing ∂W_j into its intersections with annuli $\{z : 2^{q-1} < |t| \leqslant 2^q\}$ we have

$$\int_{\partial W_j} |t^\mu G'(t)|\,|\mathrm{d}t| \leqslant \sum_{q=q_0}^{\infty} d2^{(\mu-N)(q-1)} O(2^{q(2+\rho)}) + O(1) \leqslant M^*_\mu,$$

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and

$$\int_{\gamma_z} |t^{\mu} G'(t)|\,|\mathrm{d}t| \leqslant M_{\mu}^{*} + \int_{|z^*|}^{\infty} dt^{\mu-N}\,\mathrm{d}t + 2\pi d|z|^{1+\mu-N} = O(1),$$

which proves (4.1).

To prove (4.2), fix $z_0 \in D$. Let $g = G'$ and let $\psi = g^{-1}$ be that branch of the inverse function mapping $w_0 = g(z_0)$ to z_0. Choose v_0 such that $\mathrm{e}^{-v_0} = w_0$ and set

$$\phi(v) = \psi(\mathrm{e}^{-v}) = g^{-1}(\mathrm{e}^{-v}), \qquad H = \{v : \mathrm{e}^{-v} \in V_0\}.$$

Then H is the half-plane $\{v : \mathrm{Re}(v) > \log(1/d_1)\}$ with the half-open line segments

$$L_{n,q} = \{s + q2\pi\mathrm{i} - \mathrm{i}\arg b_n : \log(1/d_1) < s \leqslant \log(1/d_2)\}, \quad q \in \boldsymbol{Z},$$

deleted. Further, as in § 3, ϕ is analytic and univalent on H and $\phi(H) = D$, and D is simply connected.

Now suppose that we attempt to analytically continue ϕ along one of the line segments $L_{n,q}$. This continuation can only fail if $\phi(v)$ hits a critical point of g and, since ϕ is univalent on H, this can only happen for finitely many $L_{n,q}$. Hence there exists a constant $R_0 > 0$ (depending on D) such that ϕ extends analytically and univalently to the set

$$H_1 = \{v : \mathrm{Re}(v) > c_0,\ |v - c_0| > R_0\}, \quad c_0 = \log(1/d_1),$$

with $\phi(v) \neq 0$ on H_1. Set

$$H_2 = \{v : \mathrm{Re}(v) > c_0,\ |v - c_0| > 100R_0\}.$$

Then $\phi(H \setminus H_2)$ is bounded, since G' has no asymptotic value in $0 < |w| \leqslant d_1$. Further, for v_1 in H_2, ϕ is univalent on the disc $B(v_1, \frac{1}{2}(\mathrm{Re}(v_1) - c_0))$.

We apply a logarithmic change of variables as used in [**1**, **2**, **4**] and elsewhere. Since $\phi \neq 0$ on H_1, we may define an analytic and univalent branch of $\zeta = \log \phi(v)$ on H_2. By Koebe's one-quarter theorem [**27**], we thus have

$$\left|\frac{\mathrm{d}\zeta}{\mathrm{d}v}\right| = \left|\frac{\phi'(v)}{\phi(v)}\right| \leqslant \frac{8\pi}{\mathrm{Re}(v) - c_0} < \frac{32}{\mathrm{Re}(v) - c_0} \tag{4.3}$$

for v in H_2. Let v_1 be in H_2 with

$$z_1 = \phi(v_1), \qquad v_1 = Q + \mathrm{i}y, \qquad Q = \log\left|\frac{1}{G'(z_1)}\right| > c_0 + 1. \tag{4.4}$$

Let L be the line given by $v = s + \mathrm{i}y$, $s \geqslant Q$. For $s \geqslant Q$, by (4.3),

$$\left|\frac{\phi'(s+\mathrm{i}y)}{\phi(s+\mathrm{i}y)}\right| \leqslant \frac{32}{s - c_0},$$

and so

$$|\phi(s+\mathrm{i}y)| \leqslant |\phi(Q+\mathrm{i}y)| \exp\left(\int_Q^s 32(t-c_0)^{-1}\,\mathrm{d}t\right) = |\phi(Q+\mathrm{i}y)|(s-c_0)^{32}(Q-c_0)^{-32}, \tag{4.5}$$

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and, recalling (4.3) and (4.4),

$$|\phi'(s+\mathrm{i}y)| \leqslant |z_1| 32 (s-c_0)^{31} (Q-c_0)^{-32}. \tag{4.6}$$

Now $\phi(L)$ is unbounded, and we have

$$\int_{\phi(L)} |G'(z)|\,|\mathrm{d}z| = \int_L \exp(-\operatorname{Re}(v))|\phi'(v)|\,|\mathrm{d}v| = \int_Q^\infty \mathrm{e}^{-s}|\phi'(s+\mathrm{i}y)|\,\mathrm{d}s.$$

Thus, (4.6) and integration by parts give

$$\int_{\phi(L)} |G'(z)|\,|\mathrm{d}z| \leqslant |z_1| \int_Q^\infty \mathrm{e}^{-s} 32 (s-c_0)^{31}(Q-c_0)^{-32}\,\mathrm{d}s \leqslant C_1 |z_1| \mathrm{e}^{-Q}(Q-c_0)^{-1}, \tag{4.7}$$

in which C_1 is a positive absolute constant.

Now we assert that for large s we have $z=\phi(s+\mathrm{i}y)\in W_j$, for some j. By (4.4) and (4.5),

$$|z| \leqslant |z_1|(s-c_0)^{32}(Q-c_0)^{-32}$$

and so

$$s = \log\left|\frac{1}{G'(z)}\right| \geqslant c_0 + (Q-c_0)\left|\frac{z}{z_1}\right|^{1/32}$$

so that

$$\log|z| = o\left(\log\left|\frac{1}{G'(z)}\right|\right)$$

as $s\to+\infty$. It follows that a sub-path of $\phi(L)$ joins z_1 to a point in one of the finitely many W_j. But $G(z)=O(1)$ on W_j, so that using (4.7) we deduce (4.2), and Lemma 4.1 is proved. □

5. Critical points and asymptotic values

Suppose now that F is meromorphic of finite order in the plane, such that F has infinitely many poles, but F' has finitely many zeros. Then, by Theorem 3.1, F has finitely many asymptotic values, and each corresponds to finitely many direct transcendental singularities [**3**, **25**] of the inverse function.

Let J be a circle or a simple closed polygonal path, such that every finite asymptotic value of F lies on J, but is not a vertex of J. Then J divides its complement in $\boldsymbol{C}^* = \boldsymbol{C}\cup\{\infty\}$ into two simply connected domains B_1 and B_2, such that B_1 is bounded, while $\infty\in B_2$. Fix conformal mappings

$$h_m : B_m \to \Delta = B(0,1), \quad m=1,2, \quad h_2(\infty)=0.$$

By the Schwarz reflection principle, if I is a line segment contained in J and not meeting any vertex of J, then for $m=1,2$ there are positive constants d_m such that

$$d_m \leqslant |h_m'(w)| \leqslant \frac{1}{d_m}, \quad w\in I. \tag{5.1}$$

Let J' be the set of vertices of J and asymptotic values of F, and let $J'' = J \setminus J'$. For each component J^* of J'' we choose an arc I_q of J^* whose closure does not meet J', and for each such I_q there are constants d_m as in (5.1).

We consider the components of the sets $F^{-1}(B_m)$. This is more complicated than in [**22**] because of the different way that J was chosen. It is convenient to take a quasiconformal homeomorphism ψ_1 of the extended plane onto itself such that $\psi_1(\infty) = \infty$ and $\psi_1(B_1) = \Delta$. There exist a function g meromorphic in the plane and a quasiconformal mapping ψ such that

$$\psi(\infty) = \infty, \qquad \psi_1 \circ F = g \circ \psi.$$

This g has finitely many asymptotic values, all of modulus 1, and g' has finitely many zeros. By choosing ψ_1 appropriately, we may assume that 0 is not a critical value of g and that the distinct finite asymptotic and critical values of g have pairwise distinct principal arguments.

Since g may have finite critical values off the unit circle, we choose $c_1 \in (0,1)$ and $c_2 > 1$ such that g has no critical values in $|w| \leqslant c_1$ nor in $c_2 \leqslant |w| < \infty$. Let M be the union of the line segments

$$w = s\mathrm{e}^{\mathrm{i} \arg \zeta_n}, \qquad c_1 \leqslant s \leqslant c_2,$$

in which the ζ_n are the finite critical values of g, and let

$$A_1 = B(0,1) \setminus M, \quad A_2 = \{w : 1 < |w| \leqslant \infty\} \setminus M.$$

Then, as in § 3, all components of the sets $g^{-1}(A_j)$ are simply connected. Further, for each component T of $g^{-1}(A_2)$, either T contains just one pole of g, or T contains no pole of g, but instead a path tending to infinity on which $g(z)$ tends to infinity. Because the inverse function g^{-1} has finitely many singularities, there are only finitely many components T of this second type.

Consider now a pole z_1 of g. Then z_1 lies in a component T_1 of the set $g^{-1}(A_2)$. We assert first that if z_1 is large enough, then T_1 is unbounded, and to prove this we assume the contrary. Since g' has finitely many zeros the closure T_2 of T_1 is a bounded component of the set $\{z : |g(z)| \geqslant 1\}$ and, by analytic continuation, T_1 is a subset of a bounded component T_3 of the set $\{z : g(z) \in \boldsymbol{C}^* \setminus A_1\}$, such that $g'(z) \neq 0$ on T_3. Hence the set $g^{-1}(A_1)$ has a multiply connected component, which is impossible.

Consider next an unbounded component S of $\{z : |g(z)| < 1\}$ having no zero of g' in its closure in the finite plane. By § 3, S is simply connected and conformally equivalent under g to the unit disc. There must be at least one path tending to infinity in S on which $g(z)$ tends to one of its finitely many asymptotic values: we call S type I if there is only one such asymptotic value of g approached along a path tending to infinity in S, and type II if there are at least two distinct such values. Clearly a type I component S with no zero of g' on its boundary ∂S is such that ∂S consists of just one simple analytic curve going to infinity in both directions, and such an S cannot separate the plane. We shall call an unbounded component S' of the set $F^{-1}(B_1)$ type I or II if $S = \psi(S')$ is a type I or II component of $\{z : |g(z)| < 1\}$.

We return now to a pole z_1 of g, of multiplicity p, with $|z_1|$ large, lying in a component T_1 of $g^{-1}(A_2)$. Then T_1 is unbounded and cannot share a finite boundary point with another component of $g^{-1}(A_2)$. Thus T_1 is a subset of a component T_4 of $\{z : |g(z)| > 1\}$, such that $\partial T_4 \subseteq \partial T_1$ and such that g' has no zeros in the closure of T_4 in the finite plane. By § 3, T_4 is simply connected and $v(z) = g(z)^{-1/p}$ is conformal on T_4. Each boundary point of T_4 is a boundary point of a component of $g^{-1}(A_1)$. Indeed, the boundary of T_4 consists of finitely many simple level curves L^* of g on which $\arg g(z)$ is monotone, each mapped by g onto an open arc of $|w| = 1$. Each such arc must form a boundary curve of a type I or type II component of the set $\{z : |g(z)| < 1\}$, with type II for at least one L^*. In particular, g must have at least two distinct finite asymptotic values and so must F.

Lemma 5.1. *Let $M_1 > 0$ and let $\phi : [0,\infty) \to [0,\infty)$ be such that $\phi(r) \to \infty$ as $r \to \infty$, and let*

$$A(k) = \{z : r^{1/k} \leqslant |z| \leqslant r^k\} \tag{5.2}$$

for large r and for positive integer k. Suppose that $A(2)$ contains N_1 distinct poles $z_1, \dots, z_{N_1}$ of F, with $N_1 \geqslant \phi(r)$. Then provided r is large enough, there exist $N \geqslant c_0 N_1$ distinct type II components E_j of the set $F^{-1}(B_1)$, each with the property that

$$L_j = \{z \in E_j : |V(z)| < 1 - r^{-M_1}\} \subseteq A(8), \quad V = h_1 \circ F. \tag{5.3}$$

Here c_0 is a positive constant depending only on the finite asymptotic values of F.

Proof. Let D_j be the component of $F^{-1}(B_2)$ in which z_j lies, and denote by $\theta_j(t)$ the angular measure of the intersection of D_j with the circle $S(0,t)$. Since r is assumed large the D_j are simply connected.

We shall use in this proof c to denote positive constants, not necessarily the same at each occurrence, but depending only on the asymptotic values of F, and in particular not on r or N_1. By the discussion above, we may assume that at least $256N$ of these D_j, say $D_1, \dots, D_{256N}$, with N an integer satisfying

$$N \geqslant cN_1 \geqslant c\phi(r), \tag{5.4}$$

are such that the following is true. There are distinct finite asymptotic values a_1, a_2 of F such that to each D_j corresponds a type II component E_j of $F^{-1}(B_1)$, the boundaries of D_j and E_j sharing a component K_j. Here K_j is a simple piecewise smooth curve going to infinity in both directions and mapped by F onto a fixed sub-path J_1 of the curve J, the closure of J_1 joining a_1 to a_2. Since F is univalent on each E_j, we have $E_j \neq E_k$ for $1 \leqslant j < k \leqslant 256N$.

Now each D_j meets $|z| > S_1$, and at least $64N$ of the D_j, $1 \leqslant j \leqslant 256N$, are such that

$$\int_{2r^2}^{(1/2)r^4} \frac{\mathrm{d}t}{t\theta_j(t)} > cN \log r, \tag{5.5}$$

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since if (5.5) fails for $D_1, \dots, D_M$ we have

$$M^2 \leqslant \biggl(\sum_{j=1}^{M} \theta_j(t)\biggr)\biggl(\sum_{j=1}^{M} 1/\theta_j(t)\biggr),$$

$$cNM \log r \geqslant \sum_{j=1}^{M} \int_{2r^2}^{(1/2)r^4} \frac{\mathrm{d}t}{t\theta_j(t)} \geqslant \frac{M^2}{\pi} \log(r/2).$$

Of these $64N$ domains D_j, at least $16N$ of them, say $D_1, \dots, D_{16N}$, have

$$\int_{2r^{1/4}}^{(1/2)r^{1/2}} \frac{\mathrm{d}t}{t\theta_j(t)} > cN \log r. \tag{5.6}$$

If the closures of at least $16N$ of the D_j satisfying (5.5) fail to meet $\{z : |z| \leqslant 2r^{1/4}\}$, then we choose $16N$ of these domains, and (5.6) is obvious, while otherwise we use the same argument as in (5.5).

We now fix a sub-arc J_0 of J_1, one of the arcs I_q chosen following (5.1). We write p_j for the multiplicity of the pole of F at z_j, and for $1 \leqslant j \leqslant 16N$ we define $v_j = (h_2 \circ F)^{1/p_j}$, so that v_j maps D_j conformally onto $\varDelta$, with $v_j(z_j) = 0$. The path K_j forming the boundary between D_j and E_j has a sub-path λ_j mapped onto J_0 by F. As z describes the arc λ_j, the image $(h_2 \circ F)(z)$ describes an arc of the unit circle of length at least c, using (5.1), so that $v_j(z)$ describes an arc of the unit circle of length at least $c/p_j \geqslant cr^{-\rho(F)-1}$. This gives

$$\omega(z_j, \lambda_j, D_j) \geqslant c/p_j \geqslant cr^{-\rho(F)-1}. \tag{5.7}$$

Set $\sigma_j = \lambda_j \setminus A(4)$. Since z_j lies in $A(2)$, Lemma 2.3, (5.5) and (5.6) imply that

$$\begin{aligned}\omega(z_j, \sigma_j, D_j) &\leqslant c \exp\biggl(-\pi \int_{2r^2}^{(1/2)r^4} \frac{\mathrm{d}t}{t\theta_j(t)}\biggr) + c \exp\biggl(-\pi \int_{2r^{1/4}}^{(1/2)r^{1/2}} \frac{\mathrm{d}t}{t\theta_j(t)}\biggr) \\ &\leqslant c \exp(-cN \log r).\end{aligned}$$

Thus (5.4) and (5.7) give, provided r is large enough,

$$\omega(z_j, \lambda_j^*, D_j) \geqslant c/p_j \geqslant cr^{-\rho(F)-1}, \quad \lambda_j^* = \lambda_j \cap A(4). \tag{5.8}$$

By (5.8), λ_j^* is mapped by v_j into a finite union of sub-arcs of the unit circle of total length at least c/p_j and so is mapped by F into a union of sub-arcs of J_0 of total length at least c, using (5.1) again. Let $\phi_j(t)$ be the angular measure of the intersection of E_j with the circle $S(0,t)$. The above reasoning gives at least N of the E_j, say $E_1, \dots, E_N$, each having

$$\int_{2r^4}^{(1/2)r^8} \frac{\mathrm{d}t}{t\phi_j(t)} > cN \log r, \qquad \int_{2r^{1/8}}^{(1/2)r^{1/4}} \frac{\mathrm{d}t}{t\phi_j(t)} > cN \log r. \tag{5.9}$$

We know that V maps E_j univalently onto $\varDelta$, with λ_j^* mapped onto a union μ_j of sub-arcs of the unit circle of total length at least c. Hence

$$\omega(w, \mu_j, \varDelta) \geqslant c(1 - |w|) \tag{5.10}$$

for $|w| < 1$. If z lies in $E_j \setminus A(8)$, then, because λ_j^* lies in $A(4)$, Lemma 2.3 and (5.9) imply that

$$\begin{aligned}\omega(V(z), \mu_j, \Delta) &= \omega(z, \lambda_j^*, E_j) \\ &\leqslant c \exp\biggl(-\pi \int_{2r^4}^{(1/2)r^8} \frac{\mathrm{d}t}{t\phi_j(t)}\biggr) + c \exp\biggl(-\pi \int_{2r^{1/8}}^{(1/2)r^{1/4}} \frac{\mathrm{d}t}{t\phi_j(t)}\biggr) \\ &\leqslant c \exp(-cN \log r).\end{aligned}$$

(5.3) now follows using (5.10). □

6. Proof of Theorem 1.4

We assume that f is meromorphic of finite order $\rho(f)$, and that f has infinitely many poles, while f'' has finitely many zeros. We apply the reasoning of § 5, with $F = f'$, and retain the notation there. Let the finite asymptotic values of f' be a_n, repeated according to how often they occur as direct transcendental singularities of $(f')^{-1}$. Choose a path Γ, starting at 0 and tending to infinity, such that $f'(z) \to a_{n_0}$ as z tends to infinity on Γ. Next choose d_1, d_2 with $0 < d_2 \leqslant d_1$ such that:

(i) for each n, there are no asymptotic values of f' in $0 < |w - a_n| \leqslant d_1$; and

(ii) for each n, there are no critical values of f' in $0 < |w - a_n| \leqslant d_2$.

Obviously, d_1 depends only on the a_n, while d_2 depends also on f.

For each n, define a domain V_n as follows. From the annulus $A(a_n, 0, d_1)$ delete, for each finite critical value $b_m \neq a_n$ of f', the half-open line segment

$$w = a_n + s\mathrm{e}^{\mathrm{i}\arg(b_m - a_n)}, \quad d_2 \leqslant s < d_1.$$

The following lemma is an immediate consequence of Lemma 4.1 and the discussion preceding it.

Lemma 6.1. *Choose $\varepsilon_0 > 0$ such that $|a_n - a_m| > 4\varepsilon_0$ for $a_n \neq a_m$. There exist a positive constant ε_1 and, for each n, an unbounded simply connected domain U_n, a component of the set $(f')^{-1}(V_n)$, such that U_n contains a path tending to infinity on which $f'(z)$ tends to a_n. Further, $f'(z) \neq a_n$ on U_n and $|f(z) - a_n z| < \varepsilon_0 |z|$ for all large z in U_n with $|f'(z) - a_n| < \varepsilon_1$. The constant ε_1 depends only on the asymptotic values of f'.*

Now let ε_2 be such that, for each n, if $|h_1(w) - h_1(a_n)| \leqslant \varepsilon_2$, then $|w - a_n| < \varepsilon_1$, in which ε_1 is as determined in Lemma 6.1. Next, let ε_3 be positive but so small that $|w - a_n| < \varepsilon_3$ implies that $|h_1(w) - h_1(a_n)| < \frac{1}{4}\varepsilon_2$, for $n = 1, 2$. Both ε_2 and ε_3 depend only on the a_n. Let p, q be such that $a_p \neq a_q$ and, for $n = p, q$, let W_n be a component of the set $\{z \in U_n : |f'(z) - a_n| < \varepsilon_3\}$. For $r \geqslant r_0$, with r_0 large, let $\psi(r)$ be the angular measure of the intersection of $S(0, r)$ with the complement of $W_p \cup W_q$.

Lemma 6.2. *There exists a positive constant C, depending only on the asymptotic values of f', such that for large r the number of distinct poles of f in the annulus $A(2)$, as defined by (5.2), is at most*

$$C + C\int_{r_0}^{r^8} \frac{\psi(t)}{t}\,\mathrm{d}t.$$

Proof. Suppose that r is large and that $A(2)$ contains N_1 distinct poles of f, where

$$1 + \int_{r_0}^{r^8} \frac{\psi(t)}{t}\,\mathrm{d}t = o(N_1). \tag{6.1}$$

Applying Lemma 5.1 we obtain $N \geqslant c_0 N_1$ distinct type II components E_j of the set $(f')^{-1}(B_1)$, each satisfying (5.3). Since there are finitely many a_n, we may assume that $a_1 \neq a_2$ and that a_1, a_2 are each asymptotic values of f' in each E_j. For $n = 1, 2$, as w tends to a_n along a path in B_1, the pre-image in E_j tends to infinity in U_n. Provided r is large enough, (5.3) shows that $A(8)$ contains the pre-image H_j under $V = h_1 \circ f'$ of the disc $B(0, 1 - \frac{1}{2}\varepsilon_2)$, for $1 \leqslant j \leqslant N$.

We may also assume that r is so large that none of the H_j meet the path Γ chosen prior to Lemma 6.1, on which $f'(z) \to a_{n_0}$ as $z \to \infty$. Defining an analytic and univalent branch of $\zeta = \log z$ on the complement of the path Γ, the regions $\zeta(H_j)$ are disjoint and, since the H_j all lie in the intersection of $A(8)$ with the complement of $W_p \cup W_q$, (6.1) shows that at least one of the $\zeta(H_j)$, say $\zeta(H_1)$, has area $o(1)$. Using Lemma 2.1, the pre-image in $\zeta(H_1)$ under $V \circ \exp$ of the line segment $w = th_1(a_n)$, $0 \leqslant t \leqslant 1 - \frac{3}{4}\varepsilon_2$, has length $o(1)$. This allows us to choose a path γ^* in $\zeta(H_1)$, of length $o(1)$, such that the path $\gamma = \exp(\gamma^*)$ in H_1 joins η_1 to η_2, and such that

$$|V(\eta_n) - h_1(a_n)| \leqslant \tfrac{3}{4}\varepsilon_2, \quad n = 1, 2.$$

By the choice of ε_2, there are points η_n^* arbitrarily close to η_n, with $f'(\eta_n^*) \in V_n$. By the choice of J and V_n, there exists a path σ_0 in $V_n \cap B_1$ which starts at $f'(\eta_n^*)$ and tends to a_n. Thus $\eta_n^* \in U_n$, and Lemma 6.1 gives

$$|f(\eta_n) - a_n\eta_n| \leqslant \varepsilon_0|\eta_n|, \quad n = 1, 2. \tag{6.2}$$

We estimate the length of γ. Since γ^* has length $o(1)$, we have $z = (1 + o(1))\eta_1$ for all z on γ and

$$\int_\gamma |\mathrm{d}z| = \int_{\gamma^*} |z|\,|\mathrm{d}\zeta| = o(|\eta_1|).$$

But f' maps γ into the bounded domain B_1, and so

$$f(\eta_2) - f(\eta_1) = \int_\gamma f'(z)\,\mathrm{d}z = o(|\eta_1|).$$

Since $a_1 \neq a_2$, this contradicts (6.2), and Lemma 6.2 is proved. □

We now complete the proof of Theorem 1.4. By Lemma 6.2 and the fact that $\psi(t) \leqslant 2\pi$, there exist positive C_j depending only on the a_n such that, for all large r,

$$\bar{n}(r^2, f) - \bar{n}(r^{1/2}, f) \leqslant C_1 + C_1 \int_{r_0}^{r^8} \frac{\psi(t)}{t} \, \mathrm{d}t \leqslant C_2 \log r \leqslant C_3(\log r^2 - \log r^{1/2}).$$

Thus $\bar{n}(r, f) \leqslant C_4 \log r$, and Theorem 1.4 is proved. □

7. Proof of Theorem 1.5

We assume that f is meromorphic in the plane of order less than $1+\varepsilon$, where $0 < \varepsilon < \frac{1}{2}$, and that f'' has finitely many zeros but f has infinitely many poles. We retain the notation of the previous section. By the discussion in § 5, f' has at least two distinct finite asymptotic values a_1, a_2. By the Denjoy–Carleman–Ahlfors Theorem [**3**, **25**], these are the only asymptotic values of f'. Hence we may assume that $a_p = a_1 = 1$, $a_q = a_2 = -1$.

Lemma 7.1. *We have*

$$\int_{r_0}^{r} \frac{\psi(t)}{t} \, \mathrm{d}t \leqslant 2\pi\varepsilon \log r, \quad \bar{n}(r, f) \leqslant C\varepsilon \log r, \quad r \to \infty, \tag{7.1}$$

in which C is a positive absolute constant, in particular not depending on ε.

Proof. For $n = 1, 2$, define the following. For $r \geqslant r_0$, let $\psi_n(r)$ be the angular measure of the intersection of W_n with the circle $|z| = r$. Let $u_n(z)$ be defined by $u_n(z) = \log|\varepsilon_3/(f'(z) - a_n)|$ for z in W_n, with $u_n(z) = 0$ for z outside W_n. Then u_1 and u_2 are subharmonic in the plane and Lemma 2.3 gives

$$\int_{r_0}^{r} \frac{\pi}{t\psi_n(t)} \, \mathrm{d}t \leqslant \log B_0(2r, u_n) + O(1) \leqslant (1+\varepsilon) \log r,$$

as $r \to \infty$, for $n = 1, 2$. But, for $t \geqslant r_0$,

$$\frac{\pi}{\psi_1(t)} + \frac{\pi}{\psi_2(t)} \geqslant \frac{4\pi}{\psi_1(t) + \psi_2(t)} = \frac{4\pi}{2\pi - \psi(t)} \geqslant 2 + \frac{\psi(t)}{\pi}.$$

This proves the first assertion of Lemma 7.1, and the second follows as in the previous section. The following is a simple consequence of Lemma 2.3. □

Lemma 7.2. *There exists $L_0 > 1$ such that the following is true. Let $r > 0$, $L > 1$ and let γ_r be a simple piecewise smooth path which, apart from its endpoints, lies in $r < |z| < Lr$ and which joins $|z| = r$ to $|z| = Lr$. Let $U_r = \{z : r < |z| < Lr,\ z \notin \gamma_r\}$. Then if $L \geqslant L_0$ we have*

$$\omega(z, S(0, r), U_r) + \omega(z, S(0, Lr), U_r) < \tfrac{1}{2}, \quad z \in U_r, \quad |z| = L^{1/2} r.$$

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Let $L \geqslant L_0$, with L_0 as in Lemma 7.2. By (2.3) and (7.1), the sets

$$K_1 = \{r \geqslant r_0 : \psi(r) \geqslant \varepsilon^{1/2}\}, \qquad K_2 = \{r \geqslant r_0 : \psi(Lr) \geqslant \varepsilon^{1/2}\} \tag{7.2}$$

each have upper logarithmic density at most $2\pi\varepsilon^{1/2}$. Next we note that by Lemma 2.5 the set

$$K_3 = \{r \geqslant 1 : T(L^2 r, f'') \geqslant L^6 T(r, f'')\} \tag{7.3}$$

has upper logarithmic density at most $\frac{2}{3}$. Further, (7.1) gives

$$h(r) = \exp(\bar{n}(r, f)) = O(r^{C\varepsilon}), \quad r \to \infty,$$

and so by Lemma 2.5 again and (2.3) the set

$$K_4 = \{r \geqslant 1 : \bar{n}(L^2 r, f) > \bar{n}(r/L, f)\} = \{r \geqslant 1 : h(L^2 r) \geqslant e h(r/L)\} \tag{7.4}$$

has upper logarithmic density at most $3C\varepsilon \log L$.

Provided ε is small enough we may choose arbitrarily large r, not in any of the exceptional sets K_1, K_2, K_3, K_4, and such that

$$\left|\frac{f''(z)}{f'(z) - a_1}\right| + \left|\frac{f''(z)}{f'(z) - a_2}\right| \leqslant r^{c_0}, \quad |z| = r, Lr, \tag{7.5}$$

denoting by c_j positive constants which do not depend on ε. By (7.4), f has no poles in $r/L < |z| < L^2 r$. Hence, by (7.3) and a standard application of the Poisson–Jensen formula we have

$$\log |f''(z)| < c_1 T(r, f''), \quad r \leqslant |z| \leqslant Lr, \tag{7.6}$$

since L does not depend on ε. Further, by (7.2) and (7.5), we have

$$\log |f''(z)| < c_2 \log r, \quad z \in (S(0,r) \setminus T_r) \cup (S(0,Lr) \setminus T_{Lr}), \tag{7.7}$$

in which $T_r \subseteq S(0,r)$ and $T_{Lr} \subseteq S(0,Lr)$, each having angular measure at most $\varepsilon^{1/2}$.

Choose a simple piecewise smooth curve γ_r on which

$$\log |f''(z)| < -(\tfrac{1}{2}) T(r, f''), \tag{7.8}$$

such that γ_r joins $|z| = r$ to $|z| = Lr$ and, apart from its endpoints, lies in $r < |z| < Lr$. Such a curve exists by the maximum principle applied to $1/f''$. Define U_r as in Lemma 7.2, so that

$$\omega(z, \gamma_r, U_r) > \tfrac{1}{2}, \quad z \in U_r, \quad |z| = L^{1/2} r. \tag{7.9}$$

For z in U_r with $|z| = L^{1/2} r$,

$$\omega(z, T_{Lr}, U_r) \leqslant \omega(z, T_{Lr}, B(0, Lr)) < c_3 \varepsilon^{1/2},$$

and the change of variables $\zeta = 1/z$ shows that the same estimate holds for $\omega(z, T_r, U_r)$. Hence, (7.6), (7.7), (7.8) and (7.9) give

$$\log |f''(z)| < (-\tfrac{1}{4} + c_4 \varepsilon^{1/2}) T(r, f''),$$

so that $f''(z)$ is small on the whole circle $|z| = L^{1/2} r$, provided ε is small enough. This contradicts the existence of the distinct asymptotic values ± 1 of f' and Theorem 1.5 is proved. □

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8. Proof of Theorem 1.6

Assume that f satisfies the hypotheses of Theorem 1.6, but has infinitely many poles. By Theorem 1.4 we have (1.2). Let the finite asymptotic values of $f^{(k-1)}$ be a_n, repeated according to how often they occur as direct transcendental singularities of the inverse function of $f^{(k-1)}$. By § 5 there are at least two distinct a_n. If the positive constant ε_0 is small enough, then to each a_n corresponds, as in § 4, an unbounded simply connected component U_n of the set $\{z : |f^{(k-1)}(z) - a_n| < \varepsilon_0\}$, lying in $\{z : |z| > 2\}$, such that $f^{(k-1)}(z) \neq a_n$ on U_n and

$$|f^{(k-2)}(z) - a_n z| \leqslant C_1 |z|\, |f^{(k-1)}(z) - a_n| + O(1), \quad z \in U_n, \tag{8.1}$$

in which C_1 is a positive constant not depending on a_n or f.

Lemma 8.1. *Choose a large positive integer N and for each n let the subharmonic function u_n be defined as in Lemma 4.1 by $u_n(z) = \log^+ |d_n/(z^N(f^{(k-1)}(z) - a_n))|$ for z in U_n, with $u_n(z) = 0$ otherwise, and with d_n a small positive constant.*

Then U_n contains finitely many components $W_{j,n}$ of the set $\{z : u_n(z) > 0\}$, each simply connected, and we have

$$f^{(\nu)}(z) = O(|z|^{k-\nu-1}), \quad z \in W_{j,n}, \quad \nu = 0, \dots, k-2. \tag{8.2}$$

Each u_n has lower order at least $1/(2 - 1/\rho)$.

Proof. The estimate (8.2) will be proved by applying (4.1) to $f^{(k-2)}(z) - a_n z$. Fixing z^* in $W_{j,n}$, choose a polynomial P_n of degree at most $k-1$ such that

$$f^{(\nu)}(z) = P_n^{(\nu)}(z) + \int_{z^*}^{z} \frac{(z-t)^{k-\nu-2}}{(k-\nu-2)!}(f^{(k-1)}(t) - a_n)\,\mathrm{d}t, \quad z \in W_{j,n}, \quad 0 \leqslant \nu \leqslant k-2. \tag{8.3}$$

Expanding out the $(z-t)^{k-\nu-2}$ term in (8.3), and using (4.1), we obtain (8.2).

To prove that each u_n has lower order at least $1/(2 - 1/\rho)$, assume without loss of generality that $a_1 \neq a_2$ and, for $n = 1, 2$ and $t > 0$, let $\theta_n^*(t)$ be the angular measure of the intersection of U_n with the circle $S(0, t)$. Proceeding as in [**28**, Lemma 3], the Cauchy–Schwarz inequality gives

$$\left(\int_1^r \frac{\pi}{t\theta_n^*(t)}\,\mathrm{d}t\right)\left(\int_1^r \frac{\theta_n^*(t)}{t\pi}\,\mathrm{d}t\right) \geqslant (\log r)^2, \quad r \to \infty, \quad n = 1, 2.$$

But, by Lemma 2.3, for large r,

$$(\rho + o(1))\log r \geqslant \log B_0(2r, u_2) + O(1) \geqslant \int_1^r \frac{\pi}{t\theta_2^*(t)}\,\mathrm{d}t.$$

Thus

$$\int_1^r \frac{\theta_2^*(t)}{t\pi}\,\mathrm{d}t \geqslant (\rho + o(1))^{-1}\log r, \qquad \int_1^r \frac{\theta_1^*(t)}{t\pi}\,\mathrm{d}t \leqslant (2 - (\rho + o(1))^{-1})\log r,$$

so that

$$\log B_0(2r, u_1) + O(1) \geqslant \int_1^r \frac{\pi}{t\theta_1^*(t)}\,\mathrm{d}t \geqslant \left(\frac{1}{2-1/\rho} - o(1)\right)\log r.$$

□

Lemma 8.2. *Choose ρ_j with $\rho_0 < \rho_1 < \cdots < \rho_9 < 1/(2-1/\rho)$, and let δ_1 be a small positive constant. Then there exists $\delta_2 > 0$ such that the following is true. If H_0 is a subset of $[1,\infty)$ of finite measure, then for each sufficiently large r and each n there exists $s \notin H_0$ such that*

$$r^{1+\delta_1} \leqslant s \leqslant r^{1+2\delta_1}, \quad u_n(z) > r^{\rho_8}, \quad z \in H_n(r), \tag{8.4}$$

in which $H_n(r)$ is a subset of the circle $|z| = s$, of angular measure at least δ_2.

Proof. Using Lemma 8.1, take ζ_0 with $|\zeta_0| = r$ and $u_n(\zeta_0) > r^{\rho_9}$, and let D_0 be the component of the set $\{z \in U_n : u_n(z) > r^{\rho_8}\}$ in which ζ_0 lies. Let $\theta(t)$ be the angular measure of the intersection of D_0 with the circle $|z| = t$. Since u_n has order at most ρ, Lemma 2.3 gives

$$r^{\rho_9} \leqslant u_n(\zeta_0) \leqslant r^{\rho_8} + r^{(1+\rho)(1+2\delta_1)} \exp\left(-\pi \int_{2r^{1+\delta_1}}^{(1/2)r^{1+2\delta_1}} \frac{\mathrm{d}t}{t\theta(t)}\right),$$

and Lemma 8.2 follows. □

Lemma 8.3. *Let $a_n \neq 0$ and let K be a large positive constant. Let the positive function $\eta(r)$ tend to 0 slowly as $r \to \infty$. Then for all sufficiently large r, at least one of the sets*

$$\Omega_r^+ = \{z : r/K \leqslant |z| \leqslant Kr,\ \eta(r) \leqslant \arg z \leqslant \pi - \eta(r)\},$$
$$\Omega_r^- = \{z : r/K \leqslant |z| \leqslant Kr,\ \pi + \eta(r) \leqslant \arg z \leqslant 2\pi - \eta(r)\}$$

is a subset of one of the $W_{j,n}$.

Proof. Using (1.2) and (1.3), write

$$\frac{f^{(k)}(z)}{f^{(m)}(z)} = \frac{h_1(z)}{h_2(z)}, \tag{8.5}$$

in which h_1 is analytic outside the region $|\arg z^2| \leqslant \phi(|z|)$, and h_2 is entire of order less than ρ_0. Choose a family of discs B_ν, with finite sum of radii, and a positive constant M_1, such that for all z not in the union H^* of the B_ν we have

$$\left.\begin{aligned} \left|\frac{f^{(m)}(z)}{f(z)}\right| + \left|\frac{f^{(k-1)}(z)}{f^{(m)}(z)}\right| + \left|\frac{f^{(k)}(z)}{f^{(m)}(z)}\right| + \left|\frac{f^{(k)}(z)}{f^{(k-1)}(z) - a_n}\right| &\leqslant |z|^{M_1}, \\ |\log|h_2(z)|| &\leqslant |z|^{\rho_0}. \end{aligned}\right\} \tag{8.6}$$

Further, choose a small positive δ_1 and $s = s_n$ satisfying (8.4). We may assume without loss of generality that the part $H_n^*(r)$ of $H_n(r)$ lying in $\{z : \eta(r) \leqslant \arg z \leqslant \pi - \eta(r)\}$ has angular measure at least $\delta_3 \geqslant \delta_2/4$, in which δ_2 is as in Lemma 8.2. We may choose s, as well as r_1 and λ with

$$K^{-6}r \leqslant r_1 \leqslant K^{-4}r, \qquad \eta(r)/8 < \lambda < \eta(r)/4,$$

such that

$$\partial\Omega(s, 4s, \lambda) \cap H^* = \partial\Omega(r_1, 2s, 2\lambda) \cap H^* = \partial\Omega(K^2 r_1, s, 4\lambda) \cap H^* = \emptyset,$$

in which

$$\Omega(t_1, t_2, t_3) = \{z : t_1 < |z| < t_2,\ t_3 < \arg z < \pi - t_3\}.$$

Since δ_1 is small, we have, by (8.5) and (8.6),

$$\log|h_1(z)| \leqslant r^{\rho_1}, \quad z \in \partial\Omega(s, 4s, \lambda) \cup \partial\Omega(r_1, 2s, 2\lambda). \tag{8.7}$$

We apply the two-constants theorem to $\log|h_1(z)|$ on the region $\Omega(s, 4s, \lambda)$. Since

$$\frac{f^{(k)}}{f^{(m)}} = \left(\frac{f^{(k)}}{f^{(k-1)} - a_n}\right)\left(\frac{f^{(k-1)} - a_n}{f^{(k-1)}}\right)\left(\frac{f^{(k-1)}}{f^{(m)}}\right),$$

(8.4) and (8.6) give

$$\log|h_1(z)| < -r^{\rho_7}, \quad z \in H_n^*(r).$$

Thus, (8.7) and a standard estimate for harmonic measure lead to

$$\log|h_1(z)| < -r^{\rho_6}, \quad z \in E_0 = \{z : |z| = 2s,\ \pi/8 \leqslant \arg z \leqslant 7\pi/8\}. \tag{8.8}$$

By Lemma 2.4 and a simple change of variables,

$$\omega(z, E_0, \Omega(r_1, 2s, 2\lambda)) \geqslant r^{-6\delta_1}, \quad z \in \partial\Omega(K^2 r_1, s, 4\lambda).$$

Hence, using (8.6), (8.7) and (8.8) we have

$$\log|h_1(z)| < -r^{\rho_5}, \quad \left|\frac{f^{(k)}(z)}{f^{(m)}(z)}\right| \leqslant \exp(-r^{\rho_4}), \quad z \in \partial\Omega(K^2 r_1, s, 4\lambda). \tag{8.9}$$

We estimate f on $\partial\Omega(K^2 r_1, s, 4\lambda)$. Choose z_1 in $H_n^*(r)$ and so in $\partial\Omega(K^2 r_1, s, 4\lambda) \cap W_{j,n}$, and a polynomial P_1 such that $P_1^{(\nu)}(z_1) = f^{(\nu)}(z_1)$ for $0 \leqslant \nu \leqslant k-1$. Then we may write

$$f(z) = P_1(z) + \int_{z_1}^{z} \frac{(z-t)^{k-1}}{(k-1)!} f^{(k)}(t)\,\mathrm{d}t = P_1(z) + \int_{z_1}^{z} \eta(t) f(t)\,\mathrm{d}t,$$

in which, using (8.2), (8.6) and (8.9), for some $M_2 > 0$ independent of r and K,

$$|P_1(z)| \leqslant r^{M_2}, \quad |\eta(t)| \leqslant \exp(-r^{\rho_3}), \quad z \in \partial\Omega(K^2 r_1, s, 4\lambda).$$

A standard application of Gronwall's Lemma [**14**] and (8.6) and (8.9) give

$$\left.\begin{aligned} \log^+ |f(z)| &= O(\log r), \\ \log^+ |f^{(m)}(z)| &= O(\log r), \end{aligned}\right\} \quad |f^{(k)}(z)| \leqslant \exp(-r^{\rho_2}), \quad \text{for } z \in \partial\Omega(K^2 r_1, s, 4\lambda).$$

Since z_1 is in $H_n(r)$, a further integration shows that $\partial\Omega(K^2 r_1, s, 4\lambda)$ is a subset of $W_{j,n}$, and so is $\Omega(K^2 r_1, s, 4\lambda)$, since $W_{j,n}$ is simply connected. This proves Lemma 8.3. □

Lemma 8.4. *We have* $\bar{N}(r, f) \neq o(\log r)^2$ *as* $r \to \infty$.

Proof. Suppose on the contrary that $\bar{N}(r, f) = o(\log r)^2$ as $r \to \infty$. Then

$$T(r, f^{(k+1)}/f^{(k)}) \leqslant \bar{N}(r, f) + O(\log r) = o(\log r)^2.$$

It follows from Lemma 2 of [**20**] that there exist sequences $R_\mu \to \infty$ and $S_\mu \to \infty$ such that

$$\frac{f^{(k+1)}(z)}{f^{(k)}(z)} = \beta_\mu z^{\tau_\mu}(1 + o(1)), \qquad S_\mu^{-2} R_\mu \leqslant |z| \leqslant S_\mu^2 R_\mu, \tag{8.10}$$

in which each τ_μ is an integer and each β_μ is a non-zero complex number. There is no loss of generality in assuming that both R_μ and $2R_\mu$ are normal for the Wiman–Valiron theory [**11, 31**] applied to $1/f^{(k)}$, for otherwise we may adjust R_μ and make S_μ slightly smaller. Since the central index $\sigma(r)$ of $1/f^{(k)}$ is non-decreasing, (2.4) gives $\tau_\mu \geqslant -1$ for each μ. We may also assume that

$$\frac{f^{(k)}(z)}{f^{(k-1)}(z) - a_n} = O(R_\mu^M), \quad |z| = R_\mu, \tag{8.11}$$

for all finite asymptotic values a_n of $f^{(k-1)}$ and for some fixed M, independent of μ.

Case 1. Suppose that $\tau_\mu = -1$.

In this case, (2.4) shows that we may assume without loss of generality that $\beta_\mu = -N_1 = -\sigma(R_\mu)$. Integration of (8.10) gives, with C a non-zero constant,

$$1/f^{(k)}(z) = C(z/R_\mu)^{N_1} \mathrm{e}^{o(N_1)}, \quad 2R_\mu \leqslant |z| \leqslant 3R_\mu.$$

Since $M(2R_\mu, 1/f^{(k)})$ is large, this implies that $C(\frac{5}{2})^{N_1}$ must be large. Thus $f^{(k)}(z) = O(R_\mu^{-2})$ on $|z| = 3R_\mu$ and a further integration leads to a contradiction to the established fact that $f^{(k-1)}$ has at least two asymptotic values.

Case 2. Suppose that $\tau_\mu \geqslant 0$.

Choose z_1, z_2 with

$$|z_1| = R_\mu, \qquad |z_2| = R_\mu S_\mu^{-1}, \qquad |1/f^{(k)}(z_j)| = M(|z_j|, 1/f^{(k)}).$$

Next, choose a branch of $\log f^{(k)}(z)$ with

$$|\mathrm{Im}(\log f^{(k)}(z_2))| \leqslant \pi. \tag{8.12}$$

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For z with
$$R_\mu S_\mu^{-1} \leqslant |z| \leqslant R_\mu S_\mu, \quad -\pi < \arg(z/z_1) \leqslant \pi, \tag{8.13}$$
we integrate by parts along the straight line from z_2 to $z_2|z/z_2|$ and then around an arc of the circle $|\zeta| = |z|$ to obtain

$$\log f^{(k)}(z) = \log f^{(k)}(z_2) + \frac{\beta_\mu z^{\tau_\mu+1}}{\tau_\mu+1}(1+o(1)) - \frac{\beta_\mu z_2^{\tau_\mu+1}}{\tau_\mu+1}(1+o(1)) + \int_{z_2}^{z} \frac{\beta_\mu \zeta^{\tau_\mu+1}}{\tau_\mu+1} o(|\zeta^{-1}|)\,\mathrm{d}\zeta.$$

Thus, for z satisfying (8.13),
$$\log f^{(k)}(z) = D + Az^N(1+o(1)), \tag{8.14}$$
in which
$$D = \log f^{(k)}(z_2) - \frac{\beta_\mu z_2^{\tau_\mu+1}}{\tau_\mu+1}, \qquad A = \frac{\beta_\mu}{\tau_\mu+1}, \qquad N = \tau_\mu + 1. \tag{8.15}$$

We set $T_\mu = R_\mu S_\mu^{-3/4}$ and distinguish two subcases.

Case 2(a). Suppose that $|AT_\mu^N| < |\log f^{(k)}(z_2)|$.

Then using (8.12) and the fact that $\tau_\mu \geqslant 0$,
$$\log f^{(k)}(z) = D(1+o(1)) = (1+o(1)) \log |f^{(k)}(z_2)|, \quad |z| = R_\mu S_\mu^{-1}, \quad -\pi < \arg(z/z_1) \leqslant \pi,$$
and $f^{(k)}(z) = O(|z|^{-2})$ on $|z| = R_\mu S_\mu^{-1}$, a contradiction arising as in Case 1.

Case 2(b). Suppose that $|AT_\mu^N| \geqslant |\log f^{(k)}(z_2)|$.

Then $|AT_\mu^N|$ is large and (8.14) becomes
$$\log f^{(k)}(z) = Az^N(1+o(1)), \quad R_\mu S_\mu^{-1/2} \leqslant |z| \leqslant R_\mu S_\mu^{1/2}, \quad -\pi < \arg(z/z_1) \leqslant \pi. \tag{8.16}$$
But $f^{(k)}(z)$ is small on an arc of $|z| = R_\mu$ of angular measure at least $\pi - o(1)$, by Lemma 8.3 and (8.11), so that (8.16) gives $N = 1$. However,
$$-(1+o(1))\frac{\sigma(R_\mu)}{z_1} = \frac{f^{(k+1)}(z_1)}{f^{(k)}(z_1)} = A(1+o(1)),$$
by (8.10) and (8.15), since R_μ is normal for the Wiman–Valiron theory applied to $1/f^{(k)}$. Thus
$$\arg Az_1 = \pi + o(1). \tag{8.17}$$
Writing (8.16) in the form
$$u = -\log f^{(k)}(z) = -Az_1(z/z_1)(1+o(1)), \tag{8.18}$$

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it follows that u is univalent with

$$\frac{\mathrm{d}u}{\mathrm{d}z} = -A(1 + o(1)) \tag{8.19}$$

on

$$\Omega = \{z : R_\mu/16 \leqslant |z| \leqslant 16R_\mu,\ |\arg(z/z_1)| \leqslant 5\pi/8\},$$

and $u(\Omega)$ contains the region

$$\Omega_1 = \{u : |A|R_\mu/8 \leqslant |u| \leqslant 8|A|R_\mu,\ |\arg u| \leqslant 9\pi/16\}.$$

Let

$$\Omega_2 = \{z : R_\mu/4 \leqslant |z| \leqslant 4R_\mu,\ \log|f^{(k)}(z)| < -2M\log R_\mu\}.$$

Then (8.17) and (8.18) imply that Ω_2 is a subset of Ω, and

$$u(\Omega_2) \subseteq \Omega_3 = \{u : |A|R_\mu/8 \leqslant |u| \leqslant 8|A|R_\mu,\ \mathrm{Re}(u) > 2M\log R_\mu\} \subseteq \Omega_1.$$

Using Lemma 8.1, choose distinct asymptotic values a_1, a_2 of $f^{(k-1)}$, and ψ_1 and ψ_2 with

$$|\psi_n| = R_\mu, \quad \log|f^{(k-1)}(\psi_n) - a_n| < -4M\log R_\mu, \quad n = 1, 2.$$

Then ψ_1, ψ_2 lie in Ω_2 by (8.11), and their images under u lie in Ω_3. Thus $u(\psi_1)$ and $u(\psi_2)$ may be joined by a path λ in Ω_3, of length $O(|A|R_\mu)$. Now the pre-image $\Lambda = u^{-1}(\lambda)$ joining ψ_1 to ψ_2 has length $O(R_\mu)$, by (8.19), and is such that

$$\log|f^{(k)}(z)| < -2M\log R_\mu, \quad z \in \Lambda.$$

Thus $f^{(k-1)}(\psi_1) - f^{(k-1)}(\psi_2) = o(1)$, which contradicts the choice of the ψ_n. Lemma 8.4 is proved. □

As in § 5, choose a closed path J on which each finite asymptotic value a_n of $f^{(k-1)}$ lies. If there are just two distinct a_n, say a_1, a_2, then J is the circle of centre $(a_1 + a_2)/2$ and diameter $|a_1 - a_2|$. Let B_1 be the interior domain of J, and let B_2, h_1, h_2 be defined as in § 5. In particular, if J is a circle, then h_1 is simply a linear transformation.

Lemma 8.5. *For each type II component E_j of the set $\{z : f^{(k-1)}(z) \in B_1\}$, choose $\zeta_j \in E_j$ such that $h_1(f^{(k-1)}(\zeta_j)) = 0$. Let $n_0(r)$ be the number of ζ_j in $|z| \leqslant r$. Then $n_0(r) \neq o(\log r)$ as $r \to \infty$.*

This follows at once from Lemmas 5.1 and 8.4.

Choose a large positive L such that for arbitrarily large r there are at least 64 distinct ζ_j in $A(0, r/L, Lr)$. Since $w = h_1(f^{(k-1)}(z))$ maps E_j univalently onto $B(0, 1)$, we may choose G_j to be the inverse function mapping $B(0, 1)$ onto E_j.

Lemma 8.6. *Denote by c_j positive constants independent of r and L. Then*

$$c_1 r \leqslant |G_j'(0)| \leqslant c_2 r. \tag{8.20}$$

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Proof. The right-hand estimate of (8.20) follows from the Koebe one-quarter theorem, since 0 is not in E_j. To prove the left-hand estimate, suppose that $G_j'(0) = o(r)$. Let a_1, a_2 be distinct finite asymptotic values of $f^{(k-1)}$ in E_j. Koebe's distortion theorem gives a path γ, of length $o(r)$, joining $z_p \in U_p$ to $z_q \in U_q$, with $a_p \neq a_q$ and

$$\int_\gamma f^{(k-1)}(z)\,\mathrm{d}z = o(r),$$

which contradicts (8.1) if ε_0 was chosen small enough.

Lemma 8.7. *$f^{(k-1)}$ has precisely one finite non-zero asymptotic value.*

Proof. Suppose that $f^{(k-1)}$ has more than one finite non-zero asymptotic value. Then Lemma 8.3 and the Koebe one-quarter theorem applied to G_j on $B(0, \frac{1}{2})$ give $G_j'(0) = o(r)$. On the other hand, $f^{(k-1)}$ has at least two finite asymptotic values, and this proves Lemma 8.7. □

We may assume henceforth that the finite asymptotic values of $f^{(k-1)}$ are 0 and 1. Thus J is the circle $S(\frac{1}{2}, \frac{1}{2})$, while B_1 is the disc $B(\frac{1}{2}, \frac{1}{2})$, and $h_1(w) = 2(w - \frac{1}{2})$. Set

$$g(z) = 2(f^{(k-1)}(z) - \tfrac{1}{2}) = h_1(f^{(k-1)}(z)).$$

Let $\theta_j(t)$ be the angular measure of the intersection of E_j with the circle $S(0, t)$. Recall that $w = g(z)$ maps E_j univalently onto $B(0, 1)$, with $g(\zeta_j) = 0$ and inverse function $z = G_j(w)$. Since there are 64 of the E_j, at least one of them must be such that

$$\int_{Lr}^{L^2 r} \frac{\mathrm{d}t}{t\theta_j(t)} \geqslant 4\log L, \qquad \int_{r/L^2}^{r/L} \frac{\mathrm{d}t}{t\theta_j(t)} \geqslant 4\log L. \tag{8.21}$$

Suppose that $Z \in E_j \setminus A(0, r/L^2, L^2 r)$ and $W = g(Z)$. Then

$$\log\left(\frac{1+|W|}{1-|W|}\right) = 2\int_{[0,W]} \frac{|\mathrm{d}w|}{1-|w|^2} = 2\int_{G_j([0,W])} \frac{|\mathrm{d}z|}{(1-|w|^2)|G_j'(w)|}$$

and so Koebe's one-quarter theorem and (8.21) give

$$\log\left(\frac{1+|W|}{1-|W|}\right) \geqslant \frac{1}{2}\int_{G_j([0,W])} \frac{|\mathrm{d}z|}{\operatorname{dist}\{z, \partial E_j\}} \geqslant \int_{G_j([0,W])} \frac{|\mathrm{d}z|}{|z|\theta_j(|z|)} \geqslant 4\log L, \tag{8.22}$$

since $\zeta_j = G_j(0) \in A(0, r/L, Lr)$. Define v_1, v_2 by

$$v_\mu = G_j(t_\mu), \qquad t_1 = -1 + L^{-3}, \qquad t_2 = 1 - L^{-3}. \tag{8.23}$$

Then (8.22) gives

$$H_0 = G_j([t_1, t_2]) \subseteq A(0, r/L^2, L^2 r). \tag{8.24}$$

Let

$$h(z) = 2f^{(k-2)}(z) - z, \qquad h'(z) = g(z).$$

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Using Lemma 4.1, we obtain

$$|h(v_1) + v_1| \leqslant c_3|v_1|L^{-3}, \qquad |h(v_2) - v_2| \leqslant c_3|v_2|L^{-3}. \tag{8.25}$$

Integration by parts gives

$$h(v_2) - h(v_1) = \int_{H_0} g(z)\,\mathrm{d}z = v_2 g(v_2) - v_1 g(v_1) - \int_{H_0} z g'(z)\,\mathrm{d}z.$$

Thus, using (8.23), (8.24) and (8.25),

$$\left|\int_{[t_1,t_2]} z\,\mathrm{d}w\right| = \left|\int_{H_0} z g'(z)\,\mathrm{d}z\right| \leqslant |h(v_1) - v_1 g(v_1)| + |h(v_2) - v_2 g(v_2)| \leqslant c_4 r L^{-1}. \tag{8.26}$$

But Lemmas 8.3 and 8.6 and the Koebe Theorems give, without loss of generality, $\mathrm{Im}(\zeta_j) > c_5 r$, and $\mathrm{Im}(G_j(t)) > c_6 r$ for $-c_7 \leqslant t \leqslant c_7$, while $\mathrm{Im}(G_j(t)) > -o(r)$ for $t_1 \leqslant t \leqslant t_2$. This contradicts (8.26) and Theorem 1.6 is proved. □

Acknowledgements. The author acknowledges valuable correspondence with Jim Clunie.

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