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$L(n)$-HYPONORMALITY: A MISSING BRIDGE BETWEEN SUBNORMALITY AND PARANORMALITY

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Abstract

A new notion of $L(n)$-hyponormality is introduced in order to provide a bridge between subnormality and paranormality, two concepts which have received considerable attention from operator theorists since the 1950s. Criteria for $L(n)$-hyponormality are given. Relationships to other notions of hyponormality are discussed in the context of weighted shift and composition operators.

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1. Towards $L(n)$-hyponormality

Let $\mathcal{H}$ be a complex Hilbert space and let $\boldsymbol{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Denote by I the identity operator on $\mathcal{H}$. We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the kernel and the range of $T \in \boldsymbol{B}(\mathcal{H})$. Given two operators $A, B \in \boldsymbol{B}(\mathcal{H})$, we denote by $[A, B]$ their commutator, that is, $[A, B] := AB - BA$. Recall that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be *subnormal* if there exists a complex Hilbert space $\mathcal{K}$ and a normal operator $N \in \boldsymbol{B}(\mathcal{K})$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Th = Nh$ for all $h \in \mathcal{H}$. The celebrated Halmos–Bram characterization of subnormality (see [5, 19]) states that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j=0}^{n} \langle T^i f_j, T^j f_i \rangle \geq 0 \tag{1.1}$$

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for all finite sequences $f_0, \dots, f_n \in \mathcal{H}$. To build a bridge between subnormality and hyponormality, McCullough and Paulsen introduced the notion of strong n-hyponormality ([23]; see also [4, 8, 9, 11]): T is said to be (strongly) *n-hyponormal* ($n \geq 1$) if inequality (1.1) holds for all $f_0, \dots, f_n \in \mathcal{H}$, or, equivalently, the operator matrix $(T^{*j}T^i)_{i,j=0}^n$ is positive; this turns out to be equivalent to the positivity of the operator matrix $([T^{*j}, T^i])_{i,j=1}^n$ (see [23, 24]). Hence, 1-hyponormality coincides with hyponormality. Two decades later a more subtle characterization of subnormality was described by Embry (see [12]). It states that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j=0}^{n} \langle T^{i+j} f_j, T^{i+j} f_i \rangle \geq 0 \tag{1.2}$$

for all finite sequences $f_0, \dots, f_n \in \mathcal{H}$. Based on Embry's characterization, McCullough and Paulsen introduced in [24] a new class of operators which, following [17], will be called $E(n)$-hyponormal: T is said to be *$E(n)$-hyponormal* ($n \geq 1$) if inequality (1.2) holds for all $f_0, \dots, f_n \in \mathcal{H}$, or, equivalently, the operator matrix $(T^{*i}(T^{*j}T^i)T^j)_{i,j=0}^n$ is positive. As shown in [24], $E(1)$-hyponormality is essentially weaker than 1-hyponormality. Moreover, in view of [17], T is $E(1)$-hyponormal if and only if $|T|^4 \leq |T^2|^2$, and so, by the Heinz inequality, such T must be an *A-class operator*, that is, $|T|^2 \leq |T^2|$ (see [14, p. 166]). Hence, $E(n)$-hyponormality can be thought of as a bridge between subnormal operators and A-class operators. The class of $E(n)$-hyponormal composition operators on L^2-spaces was completely characterized in terms of Radon–Nikodym derivatives in [17].

Let us recall the well-known characterization of positivity of a two by two operator matrix $\left(\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right)$, where $A\colon \mathcal{H} \to \mathcal{H}$, $B\colon \mathcal{K} \to \mathcal{H}$ and $C\colon \mathcal{K} \to \mathcal{K}$ are bounded linear operators, $\mathcal{K}$ is a complex Hilbert space and $A \geq 0$ (see [26]):

$$\text{if } A \text{ is invertible, then } \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \text{ is positive if and only if } B^*A^{-1}B \leq C. \tag{1.3}$$

Applying this to $A = I$, $B = (T^*T, \dots, T^{*n}T^n)$ and $C = (T^{*i+j}T^{i+j})_{i,j=1}^n$, we get the following proposition.

PROPOSITION 1.1. *An operator $T \in \boldsymbol{B}(\mathcal{H})$ is $E(n)$-hyponormal if and only if the operator matrix $(T^{*i}[T^{*j}, T^i]T^j)_{i,j=1}^n$ is positive.*

The following fact (which was mentioned in [17, p. 3956]) can be deduced either from the original definition of $E(n)$-hyponormality or from Proposition 1.1 by using the fact that powers of an operator with dense range have dense range.

COROLLARY 1.2. *An operator $T \in \boldsymbol{B}(\mathcal{H})$ with dense range is n-hyponormal if and only if it is $E(n)$-hyponormal.*

Embry's characterization of subnormality was essentially simplified by Lambert in [21]. The original characterization by Lambert was proved only for injective operators.

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The version formulated below gets rid of this unnecessary restriction (see [27, Theorem 7]): an operator $T \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if

$$\sum_{i,j=0}^{n} \|T^{i+j} f\|^2 \lambda_i \bar{\lambda}_j \geq 0 \tag{1.4}$$

for every vector $f \in \mathcal{H}$ and for all finite sequences $\lambda_0, \ldots, \lambda_n \in \mathbb{C}$. By analogy with previous definitions, we give the following one.

DEFINITION 1.3. An operator $T \in \boldsymbol{B}(\mathcal{H})$ is *$L(n)$-hyponormal* ($n \geq 1$) if inequality (1.4) holds for all $\lambda_0, \ldots, \lambda_n \in \mathbb{C}$ and for every $f \in \mathcal{H}$.

Clearly, the class of $L(n)$-hyponormal operators is closed under the operations of taking (finite or infinite) orthogonal sums and multiplication by scalars. However, it is not closed under addition and multiplication in $\boldsymbol{B}(\mathcal{H})$ (this disadvantage is shared by other types of hyponormality including subnormality; see [11, 22]). Replacing f by $T^{-2n} f$ in (1.4), we see that the following proposition holds.

PROPOSITION 1.4. *If an operator $T \in \boldsymbol{B}(\mathcal{H})$ is $L(n)$-hyponormal and T is invertible in $\boldsymbol{B}(\mathcal{H})$, then T^{-1} is $L(n)$-hyponormal.*

We now show that an inductive-type limit procedure preserves $L(n)$-hyponormality. Since the same property is valid for other kinds of hyponormality, we formulate the result for all of them.

PROPOSITION 1.5. *Let $\{\mathcal{H}_\sigma\}_{\sigma \in \Sigma}$ be a monotonically increasing net of (closed linear) subspaces of $\mathcal{H}$ such that $\mathcal{H} = \bigvee_{\sigma \in \Sigma} \mathcal{H}_\sigma$, let $\{T_\sigma\}_{\sigma \in \Sigma}$ be a net of operators $T_\sigma \in \boldsymbol{B}(\mathcal{H}_\sigma)$ and let $T \in \boldsymbol{B}(\mathcal{H})$ be an operator such that*

$$\sup_{\tau \in \Sigma} \|T_\tau\| < \infty \quad \textit{and} \quad Tf = \lim_{\tau \in \Sigma} T_\tau f \quad \forall f \in \bigcup_{\sigma \in \Sigma} \mathcal{H}_\sigma. \tag{1.5}$$

If for every $\sigma \in \Sigma$ the operator T_σ is $L(n)$-hyponormal ($E(n)$-hyponormal, n-hyponormal, subnormal), then so is T.

PROOF. A standard approximation argument reduces the proof to showing that

$$T^m f = \lim_{\tau \in \Sigma} T_\tau^m f \quad \forall m \geq 0,\ f \in \bigcup_{\sigma \in \Sigma} \mathcal{H}_\sigma. \tag{1.6}$$

We do so by induction on m. The cases when $m = 0, 1$ are obvious due to the equality in (1.5). Suppose that (1.6) holds for a fixed $m \geq 1$. It follows from the inequality in (1.5) that for all $\tau, \tau_0, \sigma \in \Sigma$ such that $\tau \geq \tau_0 \geq \sigma$ and for every $f \in \mathcal{H}_\sigma$,

$$\begin{aligned} \|T_\tau^{m+1} f - T^{m+1} f\| &\leq \|T_\tau(T_\tau^m) f - T_\tau(T_{\tau_0}^m f)\| \\ &\quad + \|T_\tau(T_{\tau_0}^m f) - T(T_{\tau_0}^m f)\| + \|T(T_{\tau_0}^m f) - T(T^m f)\| \\ &\leq \Big(\sup_{\rho \in \Sigma} \|T_\rho\|\Big) \|T_\tau^m f - T_{\tau_0}^m f\| \\ &\quad + \|T_\tau(T_{\tau_0}^m f) - T(T_{\tau_0}^m f)\| + \|T\| \|T_{\tau_0}^m f - T^m f\|, \end{aligned}$$

which by the equality in (1.5) and the induction hypothesis completes the proof. □

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COROLLARY 1.6. *Let $\{\mathcal{H}_k\}_{k=1}^{\infty}$ be a monotonically increasing sequence of invariant subspaces for an operator $T \in \boldsymbol{B}(\mathcal{H})$ such that $\bigvee_{k=1}^{\infty} \mathcal{H}_k = \mathcal{H}$. Then T is $L(n)$-hyponormal ($E(n)$-hyponormal, n-hyponormal, subnormal) if and only if $T|_{\mathcal{H}_k}$ is $L(n)$-hyponormal ($E(n)$-hyponormal, n-hyponormal, subnormal) for every $k \geq 1$.*

Below, we characterize $L(n)$-hyponormality by means of square matrices.

PROPOSITION 1.7. *If $T \in \boldsymbol{B}(\mathcal{H})$, then the following conditions are equivalent.*

(i) *T is $L(n)$-hyponormal.*
(ii) *For every $f \in \mathcal{H}$,*

$$\left|\sum_{i=1}^{n} \|T^i f\|^2 \lambda_i\right|^2 \leq \|f\|^2 \sum_{i,j=1}^{n} \|T^{i+j} f\|^2 \lambda_i \bar{\lambda}_j \quad \forall \lambda_1, \ldots, \lambda_n \in \mathbb{C}. \tag{1.7}$$

(iii) *For every $f \in \mathcal{H}$, the matrix $M_f := (\|T^{i+j} f\|^2)_{i,j=1}^{n}$ is positive, and there exists $\boldsymbol{x} \in \mathbb{C}^n$ such that $M_f^{1/2}\boldsymbol{x} = (\|Tf\|^2, \ldots, \|T^n f\|^2)$ and*[1] $\|\boldsymbol{x}\| \leq \|f\|$.

PROOF. Set $\boldsymbol{y} = (\|Tf\|^2, \ldots, \|T^n f\|^2)$.

To show that (i) and (ii) are equivalent, we apply (1.3) to $A = \|f\|^2$, the row matrix $B = (\|Tf\|^2, \ldots, \|T^n f\|^2)$ and the square matrix $C = (\|T^{i+j} f\|^2)_{i,j=1}^{n}$.

Suppose (ii) holds. To show (iii) also holds, use (1.7) to show that the matrix M_f is positive and

$$|\langle \boldsymbol{\lambda}, \boldsymbol{y}\rangle|^2 \leq \|f\|^2 \|M_f^{1/2}\boldsymbol{\lambda}\|^2 \quad \forall \boldsymbol{\lambda} \in \mathbb{C}^n.$$

This implies that there exists a linear functional $\varphi\colon \mathcal{R}(M_f^{1/2}) \to \mathbb{C}$ such that

$$\varphi(M_f^{1/2}\boldsymbol{\lambda}) = \langle \boldsymbol{\lambda}, \boldsymbol{y}\rangle \quad \forall \boldsymbol{\lambda} \in \mathbb{C}^n, \ \|\varphi\| \leq \|f\|. \tag{1.8}$$

As a consequence, there exists $\boldsymbol{x} \in \mathcal{R}(M_f^{1/2})$ such that

$$\varphi(M_f^{1/2}\boldsymbol{\lambda}) = \langle M_f^{1/2}\boldsymbol{\lambda}, \boldsymbol{x}\rangle \quad \forall \boldsymbol{\lambda} \in \mathbb{C}^n, \ \|\boldsymbol{x}\| = \|\varphi\|. \tag{1.9}$$

Combining (1.8) with (1.9), we obtain

$$\langle \boldsymbol{\lambda}, \boldsymbol{y}\rangle = \varphi(M_f^{1/2}\boldsymbol{\lambda}) = \langle M_f^{1/2}\boldsymbol{\lambda}, \boldsymbol{x}\rangle = \langle \boldsymbol{\lambda}, M_f^{1/2}\boldsymbol{x}\rangle \quad \forall \boldsymbol{\lambda} \in \mathbb{C}^n,$$

which gives $M_f^{1/2}\boldsymbol{x} = \boldsymbol{y}$ and $\|\boldsymbol{x}\| \leq \|f\|$.

To show that (iii) implies (ii), we use the Cauchy–Schwarz inequality, and get

$$|\langle \boldsymbol{\lambda}, \boldsymbol{y}\rangle|^2 = |\langle \boldsymbol{\lambda}, M_f^{1/2}\boldsymbol{x}\rangle|^2 = |\langle M_f^{1/2}\boldsymbol{\lambda}, \boldsymbol{x}\rangle|^2 \leq \|\boldsymbol{x}\|^2 \|M_f^{1/2}\boldsymbol{\lambda}\|^2 \leq \|f\|^2 \langle M_f \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle$$

for all $\boldsymbol{\lambda} \in \mathbb{C}^n$. This is just the inequality in (1.7). □

[1] Here $\|\boldsymbol{x}\|^2 = |x_1|^2 + \cdots + |x_n|^2$ for $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n$; the inner product induced by this norm is denoted, as usual, by $\langle \cdot, \cdot \rangle$.

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Recall that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be *paranormal* (see [13, 16]) if

$$\|Tf\|^2 \le \|f\| \|T^2 f\| \quad \forall f \in \mathcal{H}. \tag{1.10}$$

The following is an immediate consequence of Proposition 1.7.

COROLLARY 1.8. *An operator $T \in \boldsymbol{B}(\mathcal{H})$ is $L(1)$-hyponormal if and only if it is paranormal.*

It is known that every A-class operator is paranormal but not conversely (see [15, Example 8(2)]). Therefore, if T is a paranormal operator which is not an A-class operator, then T is $L(1)$-hyponormal but not $E(1)$-hyponormal. It may happen that a nonzero translate $T + \alpha I$ of an $L(1)$-hyponormal operator T is not $L(1)$-hyponormal ([3, pp. 174–175]; see also [7, Theorem 4] for an example concerning other types of hyponormality).

It follows from Corollary 1.8 that every $L(n)$-hyponormal operator is automatically paranormal and, as such, shares all properties of the latter. In particular, every $L(n)$-hyponormal operator is normaloid (see [14] for more information on the subject). Moreover, by the celebrated theorem of Ando (see [3, Theorem 5]), the following characterization of normal operators turns out to be true.

COROLLARY 1.9. *Let n be a positive integer. An operator $T \in \boldsymbol{B}(\mathcal{H})$ is normal if and only if T and T^* are $L(n)$-hyponormal and $\mathcal{N}(T) = \mathcal{N}(T^*)$.*

In this paper we show that the notions of $L(n)$-hyponormality and $E(n)$-hyponormality coincide for weighted shifts (see Section 2) and composition operators (see Section 3). In Section 3 we characterize $L(n)$-hyponormal composition operators in terms of Radon–Nikodym derivatives. As a byproduct, we obtain a simpler proof of [17, Theorem 2.3]. In Section 4 we discuss $L(n)$-hyponormality and $E(n)$-hyponormality in the framework of Agler's functional model.

2. Weighted shifts

Given a unilateral weighted shift T on ℓ^2 with a positive weight sequence $\{\alpha_k\}_{k=0}^{\infty}$, we set $\gamma_0 = 1$ and $\gamma_k = \alpha_0^2 \cdots \alpha_{k-1}^2$ for $k \ge 1$. It was shown in [24, Theorem 2.2] that the weighted shift T is $E(n)$-hyponormal if and only if it is n-hyponormal. The latter turns out to be equivalent to positivity of all $(n+1) \times (n+1)$ Hankel matrices $(\gamma_{k+i+j})_{i,j=0}^{n}$, where $k \ge 0$ (see [8, 9]). Below we show that there is no distinction among the notions of n-hyponormality, $E(n)$-hyponormality and $L(n)$-hyponormality as far as unilateral and bilateral weighted shifts are concerned.

PROPOSITION 2.1. *If T is either a unilateral weighted shift or a bilateral weighted shift, then T is n-hyponormal if and only if it is $L(n)$-hyponormal.*

PROOF. We only have to prove that if T is $L(n)$-hyponormal, then T is n-hyponormal. First, we consider the case where T is a unilateral weighted shift on ℓ^2 with a positive

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weight sequence $\{\alpha_n\}_{n=0}^{\infty}$. If $\{e_l\}_{l=0}^{\infty}$ is the standard orthonormal basis of ℓ^2 and $\boldsymbol{x} = \{x_l\}_{l=0}^{\infty} \in \ell^2$, then

$$0 \le \sum_{i,j=0}^{n} \left\| T^{i+j}\left(\sum_{l=0}^{\infty} x_l e_l\right)\right\|^2 \lambda_i \bar{\lambda}_j = \sum_{l=0}^{\infty} |x_l|^2 \sum_{i,j=0}^{n} \frac{\gamma_{l+i+j}}{\gamma_l} \lambda_i \bar{\lambda}_j \quad \forall \lambda_0, \ldots, \lambda_n \in \mathbb{C},$$

which, after substituting $\boldsymbol{x} = e_k$ into the above inequality, implies that the matrix $(\gamma_{k+i+j})_{i,j=0}^{n}$ is positive for all integers $k \ge 0$. By [8, Theorem 4], the weighted shift T is n-hyponormal.

Consider now the case where T is a bilateral weighted shift on $\ell^2(\mathbb{Z})$, where $\mathbb{Z}$ is the set of all integers. If $\{\varepsilon_l\}_{l=-\infty}^{\infty}$ is the standard orthonormal basis of $\ell^2(\mathbb{Z})$, then for every integer $k \ge 1$, the space $\mathcal{H}_k = \bigvee_{l=-k}^{\infty} \varepsilon_l$ is invariant for T and $T|_{\mathcal{H}_k}$ is a unilateral weighted shift. Applying what was proved in the previous paragraph and Corollary 1.6, we complete the proof. □

REMARK 2.2. Let us note that if T is a unilateral weighted shift, then for every integer $n \ge 1$ the adjoint of T is never $L(n)$-hyponormal. Indeed, otherwise by Corollary 1.8 the operator T^* is paranormal, and so $\|T^* e_1\|^2 \le \|T^{*2} e_1\| = 0$, which is impossible ($e_1$ is as in the proof of Proposition 2.1). On the other hand, since the adjoint of a bilateral weighted shift is unitarily equivalent to a bilateral weighted shift, we can apply Proposition 2.1 in this case as well.

Using weighted shift operators we show that the classes of $L(n)$-hyponormal operators are distinct from one another. Let W_α be a subnormal weighted shift on ℓ^2 with a positive weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. Set

$$\mathsf{Lh}(n) = \{x \in (0, \infty) \mid W_{\alpha(x)} \text{ is } L(n)\text{-hyponormal}\} \quad \text{and} \quad \mathsf{Lh}(\infty) = \bigcap_{n=1}^{\infty} \mathsf{Lh}(n),$$

where $\alpha(x) := (x, \alpha_1, \alpha_2, \ldots)$ for $x > 0$. Then $\mathsf{Lh}(\infty)$ is the set of all $x \in (0, \infty)$ such that the weighted shift $W_{\alpha(x)}$ is subnormal. By Proposition 2.1, the $L(n)$-hyponormality of $W_{\alpha(x)}$ is equivalent to its n-hyponormality. Hence, [8, Proposition 7] (see also [18, Example 3.1]) can be interpreted as follows.

EXAMPLE 2.3. Assume that the corresponding Berger measure of W_α (that is, a representing measure of the Stieltjes moment sequence $\{\gamma_k\}_{k=0}^{\infty}$) has infinite support. Then, by [18, Corollary 2.3], $\mathsf{Lh}(n) \backslash \mathsf{Lh}(n+1) \ne \varnothing$ for all $n = 1, 2, \ldots$. In particular, if W_α is the Bergman shift, that is, the weighted shift on ℓ^2 with the weight sequence $\alpha = \{\sqrt{(n+1)/(n+2)}\}_{n=0}^{\infty}$, then

$$\mathsf{Lh}(1) = (0, \sqrt{2/3}], \quad \mathsf{Lh}(2) = (0, 3/4],$$
$$\mathsf{Lh}(3) = (0, \sqrt{8/15}], \quad \mathsf{Lh}(4) = (0, \sqrt{25/48}],$$

and so on, and $\mathsf{Lh}(\infty) = (0, \sqrt{1/2}]$.

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3. Composition operators

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $\phi\colon X \to X$ be a measurable transformation, that is, $\phi^{-1}\mathcal{A} \subseteq \mathcal{A}$. The mapping $C_\phi\colon L^2(\mu) \ni f \mapsto f \circ \phi \in L^2(\mu)$ is called the *composition* operator. If it is well defined, then, by the closed graph theorem, it is a bounded linear operator, and consequently $\mu \circ \phi^{-1} \ll \mu$ and $h_k := d\mu \circ \phi^{-k}/d\mu \in L^\infty(\mu)$ for every integer $k \geq 0$ (see [25] for more details).

THEOREM 3.1. *Let C_ϕ be a bounded composition operator on $L^2(\mu)$. Then the following three assertions are equivalent.*

(i) *C_ϕ is $E(n)$-hyponormal.*
(ii) *C_ϕ is $L(n)$-hyponormal.*
(iii) *The $(n+1) \times (n+1)$ matrix $(h_{i+j}(x))_{i,j=0}^n$ is positive for μ-almost every $x \in X$.*

If, additionally, C_ϕ has dense range, then C_ϕ is $L(n)$-hyponormal if and only if C_ϕ is n-hyponormal.

PROOF. It is obvious that (i) implies (ii).

Suppose (ii) holds. To show (iii) also holds, take $f \in L^2(\mu)$ and $\lambda_0, \ldots, \lambda_n \in \mathbb{C}$. Using the measure transport theorem (see [20, Theorem C, p. 163]), we obtain

$$\begin{aligned} 0 \leq \sum_{i,j=0}^n \|C_\phi^{i+j} f\|^2 \lambda_i \bar{\lambda}_j &= \sum_{i,j=0}^n \lambda_i \bar{\lambda}_j \int_X |f \circ \phi^{i+j}|^2 \, d\mu \\ &= \sum_{i,j=0}^n \lambda_i \bar{\lambda}_j \int_X |f|^2 \, d\mu \circ \phi^{-(i+j)} \\ &= \int_X \bigg(\sum_{i,j=0}^n h_{i+j} \lambda_i \bar{\lambda}_j \bigg) |f|^2 \, d\mu. \end{aligned}$$

Substituting $f = \chi_\sigma$ with $\sigma \in \mathcal{A}$ such that $\mu(\sigma) < \infty$, we get

$$\int_\sigma \bigg(\sum_{i,j=0}^n h_{i+j}(x) \lambda_i \bar{\lambda}_j \bigg) \, d\mu(x) \geq 0 \tag{3.1}$$

for all $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \mathbb{C}^{n+1}$. By assumption μ is σ-finite, so we may write $X = \bigcup_{k=1}^\infty X_k$ with $X_k \in \mathcal{A}$ such that $\mu(X_k) < \infty$. For $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \mathbb{C}^{n+1}$, we set $\Omega_{\boldsymbol{\lambda}} = \{x \in X : H_{\boldsymbol{\lambda}}(x) \geq 0\}$, where $H_{\boldsymbol{\lambda}}(x) = \sum_{i,j=0}^n h_{i+j}(x) \lambda_i \bar{\lambda}_j$. Since (3.1) holds for all $\sigma \in \mathcal{A}$ such that $\sigma \subseteq X_k$, we deduce that $H_{\boldsymbol{\lambda}}(x) \geq 0$ for μ-almost every $x \in X_k$, that is, $\mu(X_k \backslash \Omega_{\boldsymbol{\lambda}}) = 0$. As k is an arbitrary positive integer, we see that $\mu(X \backslash \Omega_{\boldsymbol{\lambda}}) = 0$ for all $\boldsymbol{\lambda} \in \mathbb{C}^{n+1}$. Consider now any countable dense subset $\mathcal{Z}$ of $\mathbb{C}^{n+1}$ and define $\Omega_{\mathcal{Z}} = \bigcap_{\boldsymbol{\lambda} \in \mathcal{Z}} \Omega_{\boldsymbol{\lambda}}$. Then $\Omega_{\mathcal{Z}} \in \mathcal{A}$ and $\mu(X \backslash \Omega_{\mathcal{Z}}) = 0$. For every $\boldsymbol{\lambda} \in \mathbb{C}^{n+1}$, there exists a sequence $\{\boldsymbol{\lambda}^{(l)}\}_{l=1}^\infty \subseteq \mathcal{Z}$ which converges to $\boldsymbol{\lambda}$. Since $H_{\boldsymbol{\lambda}^{(l)}}(x) \geq 0$ for all $x \in \Omega_{\mathcal{Z}}$ and $l \geq 1$, we deduce that $H_{\boldsymbol{\lambda}}(x) \geq 0$ for all $x \in \Omega_{\mathcal{Z}}$. Hence the matrix $(h_{i+j}(x))_{i,j=0}^n$ is positive for all $x \in \Omega_{\mathcal{Z}}$, which together with $\mu(X \backslash \Omega_{\mathcal{Z}}) = 0$ gives condition (iii).

To show that (iii) implies (i), we apply the equality

$$\langle C_\phi^k f, C_\phi^k g\rangle = \int_X f\bar{g}h_k\, d\mu \quad \forall f, g \in L^2(\mu),\ k = 0, 1, 2, \ldots, \tag{3.2}$$

which is a direct consequence of the measure transport theorem.

The last part of the conclusion follows from the above and Corollary 1.2. □

According to [6, Theorem 2.3], a composition operator is of A-class if and only if it is paranormal. This fact also follows from Corollary 1.8 and Theorem 3.1. As shown in [6, Example 3.1], there are paranormal (read: $E(1)$-hyponormal) composition operators which are not hyponormal.

We now prove that the notions of $L(n)$-hyponormality and $E(n)$-hyponormality coincide for adjoints of composition operators with dense range. As a byproduct, we show that the assumption $h_1 > 0$ of [17, Proposition 2.6] (which is equivalent to $\overline{\mathcal{R}(C_\phi^*)} = L^2(\mu)$) can be dropped without affecting the result.

PROPOSITION 3.2. *Let C_ϕ be a bounded composition operator on $L^2(\mu)$ with dense range. Then the following three assertions are equivalent.*

(i) *C_ϕ^* is $E(n)$-hyponormal.*

(ii) *C_ϕ^* is $L(n)$-hyponormal.*

(iii) *The $(n+1)\times(n+1)$ matrix $(h_{i+j}\circ\phi^{i+j}(x))_{i,j=0}^n$ is positive for μ-almost every $x \in X$.*

If, additionally, C_ϕ is injective, then C_ϕ^* is $L(n)$-hyponormal if and only if C_ϕ^* is n-hyponormal.

PROOF. Fix a nonnegative integer k. By (3.2), $C_\phi^{*k}C_\phi^k f = h_k \cdot f$ for all $f \in L^2(\mu)$. Hence $C_\phi^k C_\phi^{*k}(C_\phi^k f) = M_{h_k\circ\phi^k}(C_\phi^k f)$ for all $f \in L^2(\mu)$, where $M_{h_k\circ\phi^k}$ is the operator of multiplication by $h_k\circ\phi^k$. Since $h_k\circ\phi^k \in L^\infty(\mu)$, the operator $M_{h_k\circ\phi^k}$ is bounded. Therefore

$$C_\phi^k C_\phi^{*k}(g) = M_{h_k\circ\phi^k}(g) \quad \forall g \in \overline{\mathcal{R}(C_\phi^k)}.$$

As $\overline{\mathcal{R}(C_\phi^k)} = L^2(\mu)$, we get

$$C_\phi^k C_\phi^{*k} = M_{h_k\circ\phi^k} \quad \forall k = 0, 1, 2, \ldots. \tag{3.3}$$

It is obvious that (i) implies (ii).

To show that (ii) implies (iii), we apply (3.3) and obtain

$$0 \le \sum_{i,j=0}^n \|C_\phi^{*i+j}f\|^2\lambda_i\bar{\lambda}_j = \int_X\bigg(\sum_{i,j=0}^n h_{i+j}\circ\phi^{i+j}(x)\lambda_i\bar{\lambda}_j\bigg)|f(x)|^2\, d\mu(x)$$

for all $f \in L^2(\mu)$ and $\lambda_0, \ldots, \lambda_n \in \mathbb{C}$. Next, arguing as in the proof that (ii) implies (iii) in Theorem 3.1, we derive (iii).

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Suppose that (iii) holds. To show (i), take $f_0, \ldots, f_n \in L^2(\mu)$, then; by (3.3),

$$\sum_{i,j=0}^{n} \langle C_\phi^{*i+j} f_j, C_\phi^{*i+j} f_i \rangle = \int_X \bigg(\sum_{i,j=0}^{n} h_{i+j} \circ \phi^{i+j}(x) \overline{f_i(x)} f_j(x) \bigg) \, d\mu(x) \geq 0.$$

The last part of the conclusion follows from the above and Corollary 1.2. □

Proposition 3.2 can be applied to (unilateral and bilateral) weighted shift operators because the adjoint of a weighted shift is a composition operator with dense range.

4. Connections with Agler's functional model

Let $\mathbb{C}[z]$ stand for the ring of all complex polynomials in complex variable z. For every integer $n \geq 0$, we define the linear subspace $\mathbb{C}_n[z]$ of $\mathbb{C}[z]$ via

$$\mathbb{C}_n[z] = \{p \in \mathbb{C}[z] : \deg p \leq n\} \quad \forall n = 0, 1, 2, \ldots.$$

We denote by $\mathbb{C}[z, \bar{z}]$ the ring of all complex polynomials in z and $\bar{z}$. It is well known that the ring $\mathbb{C}[z, \bar{z}]$ can be identified with that of all complex functions of the form $\mathbb{C} \ni z \to p(z, \bar{z}) \in \mathbb{C}$, where p is a complex polynomial in two complex variables; such a representation is unique. For every integer $n \geq 1$, we define the following four convex cones in $\mathbb{C}[z, \bar{z}]$:

$$\begin{aligned} \mathcal{C} &= \operatorname{conv}\{(1 - |z|^2)|p(z)|^2 : p \in \mathbb{C}[z]\}, \\ \widetilde{\mathcal{L}}^n &= \operatorname{conv}\{|p(z)q(|z|^2)|^2 : p \in \mathbb{C}[z], q \in \mathbb{C}_n[z]\}, \\ \widetilde{\mathcal{E}}^n &= \operatorname{conv}\bigg\{\bigg|\sum_{i=0}^{n} p_i(z)|z|^{2i}\bigg|^2 : p_0, \ldots, p_n \in \mathbb{C}[z]\bigg\}, \qquad (4.1) \\ \widetilde{\mathcal{S}}^n &= \operatorname{conv}\bigg\{\bigg|\sum_{i=0}^{n} p_i(z)\bar{z}^i\bigg|^2 : p_0, \ldots, p_n \in \mathbb{C}[z]\bigg\}, \qquad (4.2) \end{aligned}$$

where 'conv' denotes the convex hull. Denote by $\mathcal{L}^n$, $\mathcal{E}^n$ and $\mathcal{S}^n$ the convex cones generated by $\mathcal{C} \cup \widetilde{\mathcal{L}}^n$, $\mathcal{C} \cup \widetilde{\mathcal{E}}^n$ and $\mathcal{C} \cup \widetilde{\mathcal{S}}^n$, respectively. Substituting $p_i(z) = a_i p(z)$ into (4.1), where $p(z) = \sum_{i=0}^{n} a_i z^i$, we see that $\widetilde{\mathcal{L}}^n \subseteq \widetilde{\mathcal{E}}^n$. In turn, substituting $p_i(z) z^i$ into (4.2) in place of $p_i(z)$, we get $\widetilde{\mathcal{E}}^n \subseteq \widetilde{\mathcal{S}}^n$. Hence $\mathcal{L}^n \subseteq \mathcal{E}^n \subseteq \mathcal{S}^n$.

Let us recall Agler's functional model ([1, 2]; see also [23]). If $T \in \boldsymbol{B}(\mathcal{H})$ is a cyclic contraction with a cyclic vector γ, then we can associate with T a unique linear functional $\Lambda_T : \mathbb{C}[z, \bar{z}] \to \mathbb{C}$ such that

$$\Lambda_T(z^m \bar{z}^n) = \langle T^m \gamma, T^n \gamma \rangle = \langle T^{*n} T^m \gamma, \gamma \rangle \quad \forall m, n = 0, 1, 2, \ldots. \qquad (4.3)$$

In terms of the functional Λ_T, the $L(n)$-hyponormality of cyclic contractions can be characterized as follows (the proof, being standard, is omitted; consult [23]).

PROPOSITION 4.1. *The mapping* $(T, \gamma) \mapsto \Lambda_T$ *given by (4.3) is a surjection between the set of all* $L(n)$*-hyponormal cyclic contractions and the set of all linear functionals* $\Lambda\colon \mathbb{C}[z, \bar{z}] \to \mathbb{C}$ *which are nonnegative on the convex cone* $\mathcal{L}^n$*. Moreover, if* T_1 *and* T_2 *are cyclic contractions on complex Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *with cyclic vectors* γ_1 *and* γ_2 *respectively, then* $\Lambda_{T_1} = \Lambda_{T_2}$ *if and only if there exists a unitary isomorphism* $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ *such that* $U\gamma_1 = \gamma_2$ *and* $UT_1 = T_2U$*.*

The case of $E(n)$-hyponormality can be described in a similar manner.

PROPOSITION 4.2. *The mapping* $(T, \gamma) \mapsto \Lambda_T$ *given by (4.3) is a surjection between the set of all* $E(n)$*-hyponormal cyclic contractions and the set of all linear functionals* $\Lambda\colon \mathbb{C}[z, \bar{z}] \to \mathbb{C}$ *which are nonnegative on the convex cone* $\mathcal{E}^n$*.*

In view of [23, Propositions 2.2 and 2.3], a cyclic contraction T is n-hyponormal if and only if Λ_T is nonnegative on $\mathcal{S}^n$, while T is weakly n-hyponormal if and only if Λ_T is nonnegative on the convex cone $\mathcal{W}^n$ generated by $\mathcal{C} \cup \widetilde{\mathcal{W}}^n$, where

$$\widetilde{\mathcal{W}}^n = \operatorname{conv}\big\{|r(z) + p(z)\overline{q(z)}|^2 : p, r \in \mathbb{C}[z], q \in \mathbb{C}_n[z]\big\}.$$

One can verify that $\widetilde{\mathcal{L}}^1 \subseteq \widetilde{\mathcal{W}}^1$ and so $\mathcal{L}^1 \subseteq \mathcal{W}^1$. If $n \geq 2$, then there is no nice relationship between convex cones $\mathcal{L}^n$ and $\mathcal{W}^n$. The reason for this is that according to Proposition 2.1 and [23, Theorem 4.1] there are weakly 2-hyponormal weighted shifts which are not $L(2)$-hyponormal (see also [10] and references therein for recent examples of this sort). Nevertheless, it is tempting to continue investigations along these lines.

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