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Abstract

We define stacky Lie groups to be group objects in the 2-category of differentiable stacks. We show that every connected and étale stacky Lie group is equivalent to a crossed module of the form (Γ, G) where Γ is the fundamental group of the given stacky Lie group and G is the connected and simply connected Lie group integrating the Lie algebra of the stacky group. Our result is closely related to a strictification result of Baez and Lauda.

1. Introduction

Over the last few years there has been a lot of interest in the so-called *higher groups* [BL04, Blo08, Hen08]. As the name suggests, a higher group should be regarded as a 'generalized group' in some suitable sense. In practice, the precise definition one adopts depends very much on the applications one has in mind. A first example of higher group is provided by the string group String(n) [BM96, Sch11, ST04] in mathematical physics. Historically, this object first arose as the 3-connected cover of Spin(n); one of the possible models for String(n) is given by a crossed module of (infinite-dimensional) Lie groups [BSCS07], which is a type of higher group that will also play a role in the present paper. Another example, which can be regarded as a generalization of the previous one, comes from the integration theory of L_∞-algebras (also known as homotopy Lie algebras) [Get09, Hen08]. In this case the appropriate definition for the higher group integrating an L_∞-algebra is given in terms of Kan simplicial manifolds (compare § 3 below). Yet another example (and yet another way of defining higher groups) originates in the integration theory of Poisson manifolds or Lie algebroids [CF01, CF03]. It turns out that not all Lie algebroids are integrable (luckily enough, the integrability conditions can be made explicit). That is not so bad, however, as one can take a way around this difficulty by working with the notion of *Weinstein groupoid* [TZ06a, TZ06b]. The study of the isotropy groups of Weinstein groupoids was the original motivation for the present paper, as we will explain in more detail below.

There is unfortunately neither general agreement on what the standard definition of a stacky Lie group(oid) should be nor on the corresponding terminology; one sometimes refers to stacky Lie groups as *Lie 2-groups.* One possibility is to define stacky Lie groups as group objects in the *Hilsum–Skandalis* bicategory $\mathfrak{HS}$, that is, the bicategory (i.e., weak 2-category) which has Lie groupoids as objects, right principal bibundles as 1-morphisms and smooth biequivariant maps between bibundles as 2-morphisms [Blo08]. This approach has some advantages in that many standard constructions, such as fibred products for instance, can be given a rather explicit

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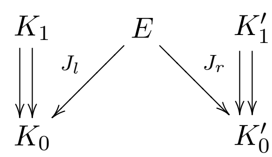
description [Sch11]. However, from a conceptual point of view this is very much like working all the time with a fixed choice of local coordinates (or atlas) when doing differential geometry. In this paper we prefer to adopt a more intrinsic foundational framework. Namely, we define stacky Lie groups as group objects in the 2-category $\underline{\mathfrak{DSta}}$ which has differentiable stacks [BX06, Met03] as objects, and maps of stacks over the smooth site, respectively isomorphisms between them, as 1-morphisms, respectively 2-morphisms (Definition 2.4). This coincides exactly with the notion of Lie 2-groups via Kan complexes by Getzler and Henriques [Get09, Hen08] (see § 3).

For the reader's convenience and because of its relevance to the present work, we analyse the relation between these two notions of stacky Lie group in greater detail. Recall that a *Hilsum–Skandalis morphism* (*HS-morphism*) from a Lie groupoid $K = \{K_1 \rightrightarrows K_0\}$ to another one, $K' = \{K'_1 \rightrightarrows K'_0\}$, is given by a *right-principal bibundle*,



that is to say, by a manifold E on which both K and K' act in a compatible way along the *moment maps* J_l and J_r with respective smooth actions

$$\Phi_l : K_1 \times_{\mathbf{s},K_0,J_l} E \to E \quad \text{and} \quad \Phi_r : E \times_{J_r,K'_0,\mathbf{t}} K'_1 \to E,$$

in such a way that the right action of K' on E is *principal:* in other words, J_l is a surjective submersion and $\mathrm{id} \times \Phi_r : E \times_{J_r,K'_0,\mathbf{t}} K'_1 \xrightarrow{\simeq} E \times_{J_l,K_0,J_l} E$ a diffeomorphism. If E is also left-principal (in the obvious sense) then it is called a *Morita bibundle,* and it is said to yield a *Morita equivalence* between the Lie groupoids K and K'. Now, any differentiable stack can be presented by a Lie groupoid, uniquely up to Morita equivalence [BX06]. Namely, given a differentiable stack $\mathcal{X}$, a groupoid presentation $X = \{X_1 \rightrightarrows X_0\}$ of $\mathcal{X}$ can be obtained from any representable surjective submersion $X_0 \twoheadrightarrow \mathcal{X}$ (this is called a *chart* or *atlas* for $\mathcal{X}$) by taking the fibred product $X_1 := X_0 \times_{\mathcal{X}} X_0$. In more rigorous categorical terms, there is a canonical equivalence of bicategories between the 2-category $\underline{\mathfrak{DSta}}$ and the bicategory $\underline{\mathfrak{HS}}$ defined above. Consequently, a stacky Lie group can always be presented by a stacky Lie group in the former sense, i.e., by a group object in the bicategory $\underline{\mathfrak{HS}}$, very much like a smooth manifold can always be defined by giving a particular atlas for it.

The stacky Lie groups which prove to be really important for our applications are the *étale* ones, which are defined to be those whose underlying differentiable stack can be presented by an étale Lie groupoid; a generic presentation for such a differentiable stack will be a *foliation* groupoid [MM03]. The isotropy groups of Weinstein groupoids constitute, for us, the fundamental example of a stacky Lie group; compare § 6.2. The purpose of this paper is precisely to understand better the structure of these isotropy groups.

Recall that a *crossed module* (of Lie groups) is a pair of Lie groups (H, G) given with a homomorphism $\partial : H \to G$ and a smooth left action $(g, h) \mapsto g * h$ of G on H by automorphisms of H such that the following two axioms are satisfied:

(Eq) $\partial(g * h) = g\partial(h)g^{-1}$ (equivariance);

(Pf) $\partial(h) * h' = hh'h^{-1}$ (Pfeiffer identity).

It is our goal here to establish the following result.

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Theorem 5.13. *Every connected stacky Lie group $\mathcal{G}$ is equivalent to a crossed module of the form $(\pi_1(\mathcal{G}), G)$, where $\pi_1(\mathcal{G})$ denotes the fundamental group of $\mathcal{G}$ (viewed as a discrete Lie group), and G is a connected and simply connected Lie group.*

(For a stacky Lie group, connectedness just means path-connectedness of the underlying differentiable stack; compare Definition 2.16. The connectedness assumption is natural from the point of view of our applications. The notion of equivalence of stacky Lie groups is defined in § 2.1.)

It is always possible to strictify a discrete 2-group. This is a well-known result of Baez and Lauda [BL04], who provide a proof via group cohomology. However, if a 2-group carries a topology or a smooth structure, then it is not clear how one can achieve the strictification result by the same methods. As an example of these difficulties, the string Lie 2-group (unfortunately it is not one of the étale stacky Lie groups we consider) sometimes has a strict but infinite-dimensional model [BSCS07], and sometimes has a finite-dimensional but nonstrict model [Sch11]. Thus, we can see from this example that the strictification procedure is in general highly nontrivial. Moreover, the strictification method provided in [BL04] is far from being constructive. By contrast, our method is completely constructive and solves the problem within the étale, finite-dimensional world. On the other hand, our result relies very much on the fact that $\pi_2(G)=0$ for a finite-dimensional Lie group G. Thus the same proof will not work for infinite-dimensional setting, and one should not expect a similar result for n-groups in general.

As we have already said, étale stacky Lie groups make their appearance as the isotropy groups of Weinstein groupoids in Tseng–Zhu's approach to Lie algebroid integration [TZ06a]. The problem of integrating Poisson manifolds into symplectic groupoids falls under the scope of Lie algebroid integration [TZ06b], so the same stacky Lie groups become relevant to Poisson geometry. In fact, we expect our result to be an effective tool in studying the precise significance of the assumptions under which the normal form result for Poisson manifolds around symplectic leaves recently obtained by Crainic and Mărcuţ [CM09] holds. We postpone a more satisfactory discussion of the last topic to § 6.3.

Our result is likely to have consequences for the study of noneffective orbifold groups. For example, since every orbifold is an étale differentiable stack, as soon as the orbifold carries a group structure, an immediate corollary of our result is that one can find a global quotient compatible with the group structure.

Finally, we also expect the main result of this paper to be relevant to the task of obtaining results analogous to those of [Tre10a, Tre10b, Tre11] in the more general context of stacky Lie groupoids and representations up to homotopy.

Structure of the paper. In § 2, we recall the essential background on group objects in 2-categories and introduce the basic notions of *stacky Lie group* and of (*strict*) *coherent Lie 2-group*. We also discuss (coherent) homomorphisms and equivalences. In § 3, we show that every connected stacky Lie group is equivalent to a base connected, semistrict, coherent Lie 2-group. This is the first, key step in the proof of our strictification theorem. In § 4, we operate the second fundamental step in our strictification procedure, which consists in showing that, thanks to étaleness, every connected, semistrict coherent Lie 2-group is equivalent to a strict Lie 2-group. The remaining arguments in this section, which show that a strict Lie 2-group is the same thing as a crossed module, are quite standard. The following section explicitly identifies the group Γ figuring in the crossed module presentation $\partial:\Gamma\to G_0$ of a stacky Lie group as the fundamental group of the stacky Lie group itself. In the final § 6, we discuss in detail a couple of fundamental motivating examples of stacky Lie groups, and use them to give an idea of how our strictification result works in practical terms. We conclude with some speculations about the potential applications of our strictification theorem to normal form theorems in Poisson geometry.

2. From stacky Lie groups to semistrict Lie 2-groups

All the material in this section, with the exception of the last statement (Theorem 2.17), is completely standard, and can be found for instance in [BL04]. We work throughout in a smooth, étale context.

2.1 Background on group objects in 2-categories

Our 2-categories are assumed to always have finite products. In particular, there is always a terminal object $\star$.

Definition 2.1. A *group object in a 2-category* $\mathcal{C}$ (or $\mathcal{C}$-*group,* for brevity) consists of the following data:

- an object $A \in \mathcal{C}$;
- a list of 1-morphisms,

  $\mu : A \times A \to A$ (the *multiplication*),
  $\eta : \star \to A$ (the *unit*),
  $\iota : A \to A$ (the *inverse*);
- a list of 2-morphisms,

$$a : \mu \circ (\mu \times 1_A) \Rightarrow \mu \circ (1_A \times \mu) \quad \text{(the } \textit{associator}\text{)},$$

$$\left.\begin{aligned} \ell : \mu \circ (\eta \times 1_A) \Rightarrow \mathrm{pr}_A \\ r : \mu \circ (1_A \times \eta) \Rightarrow \mathrm{pr}_A \end{aligned}\right\} \quad \text{(the } \textit{left,} \text{ respectively } \textit{right, unit constraint}\text{)},$$

$$\left.\begin{aligned} d : \eta \circ \tau_A \Rightarrow \mu \circ (1_A \times \iota) \circ \delta_A \\ e : \mu \circ (\iota \times 1_A) \circ \delta_A \Rightarrow \eta \circ \tau_A \end{aligned}\right\} \quad \text{(the } \textit{adjunction constraints}\text{)}$$

  (where $\tau_A : A \to \star$ denotes the unique 1-morphism from A to the terminal object, and $\delta_A : A \to A \times A$ denotes the diagonal),

subject to the requirement that certain coherence conditions hold, for which we refer the reader to [BL04, p. 37].

Remark 2.2. The notion of $\mathcal{C}$-*monoid* is obtained from the previous one by neglecting the inversion 1-morphism ι and the adjunction constraints d, e.

Recall that an *étale atlas* for a differentiable stack $\mathcal{X}$ is a representable surjective submersion $X \twoheadrightarrow \mathcal{X}$ such that the associated Lie groupoid $\{X \times_{\mathcal{X}} X \rightrightarrows X\}$ is étale.

Definition 2.3. Let $\underline{\mathfrak{DSta}}$ denote the 2-category whose objects are the differentiable stacks admitting an étale atlas, whose 1-morphisms are the maps of differentiable stacks, and whose 2-morphisms are the 2-isomorphisms between maps of differentiable stacks.

Definition 2.4. A *stacky Lie group* is a group object in the 2-category $\underline{\mathfrak{DSta}}$.

Definition 2.5. We denote by $\underline{\mathfrak{LGpd}}$ the 2-category whose objects are the étale Lie groupoids, whose 1-morphisms are the homomorphisms of Lie groupoids (smooth functors), and whose

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2-morphisms are the (smooth) natural transformations between homomorphisms of Lie groupoids.

Definition 2.6. A *homomorphism* $\Phi : A \to B$ between two group objects A, B in a 2-category $\mathcal{C}$ consists of the following data:

- a 1-morphism $\phi : A \to B$;
- a pair of 2-morphisms,
  $t : \mu^{(B)} \circ (\phi \times \phi) \Rightarrow \phi \circ \mu^{(A)}$,
  $u : \eta^{(B)} \Rightarrow \phi \circ \eta^{(A)}$,

making the appropriate coherence diagrams commute [BL04, p. 41].

Definition 2.7. A *transformation* $a : \Phi \Rightarrow \Psi$ between two homomorphisms of $\mathcal{C}$-groups $\Phi, \Psi : A \to B$ is a 2-morphism $a : \phi \to \psi$ compatible with the rest of the structure in the obvious sense [BL04, p. 42].

Definition 2.8. An *equivalence* $A \simeq B$ between two $\mathcal{C}$-groups A, B is a homomorphism $\Phi : A \to B$ such that there exists a homomorphism $\Psi : B \to A$ together with transformations $\Psi \circ \Phi \Rightarrow 1_A$ and $\Phi \circ \Psi \Rightarrow 1_B$.

One has a canonical 2-functor from the 2-category $\underline{\mathfrak{LGpd}}$ into the 2-category $\underline{\mathfrak{DSta}}$; for this well-known construction, we refer the reader to [Met03, Noo05]. Hence we have the following lemma.

Lemma 2.9. *Any $\underline{\mathfrak{LGpd}}$-group $\mathbb{G}$ canonically determines a corresponding stacky Lie group, and any equivalence of $\underline{\mathfrak{LGpd}}$-groups induces, canonically, one of the corresponding stacky Lie groups.*

Definition 2.10. We denote by $\mathsf{Stack}(\mathbb{G})$ the stacky Lie group corresponding to $\mathbb{G}$ in the above statement.

2.2 Coherent Lie 2-groups

We refer to monoid objects in the 2-category $\underline{\mathfrak{LGpd}}$ (in short, $\underline{\mathfrak{LGpd}}$-monoids) also as *smooth monoidal groupoids*. For them we adopt the standard notation for monoidal categories: $\mathbf{1}$ for the unit object; $(x, y) \mapsto x \otimes y$ for the monoidal bifunctor; $a_{x,y,z} : x \otimes (y \otimes z) \to (x \otimes y) \otimes z$ for the associator; and $\ell_x : x \otimes \mathbf{1} \to x$, $r_x : \mathbf{1} \otimes x \to x$ for the unit constraints.

Definition 2.11. A *coherent Lie 2-group* $\mathbb{G}$ is a smooth monoidal groupoid $\{G_1 \rightrightarrows G_0, \otimes, \mathbf{1}\}$ supplied with the extra structure of:

- a smooth map $\{x \mapsto \overline{x}\} : G_0 \to G_0$;
- two smooth maps $\{x \mapsto d_x\}, \{x \mapsto e_x\} : G_0 \to G_1$ with

$$d_x : \mathbf{1} \to x \otimes \overline{x}, \quad e_x : \overline{x} \otimes x \to \mathbf{1} \tag{1}$$

so that the usual *adjunction properties* hold [BL04, p. 10].

Definition 2.12. A *coherent homomorphism* $\mathbb{G} \to \mathbb{H}$ between coherent Lie 2-groups $\mathbb{G}, \mathbb{H}$ consists of:

(1) a homomorphism Φ between the underlying Lie groupoids;

(2) a monoidal functor structure for Φ, namely, the data of a natural transformation $t_{x,y} : \Phi(x) \otimes' \Phi(y) \to \Phi(x \otimes y)$ between Lie groupoid homomorphisms and of an arrow $u : \mathbf{1}' \to \Phi(\mathbf{1})$ satisfying the standard coherence conditions as in the classical definition of a monoidal functor [Mac71].

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A *coherent equivalence* $\mathbb{G} \xrightarrow{\sim} \mathbb{H}$ is a coherent homomorphism $\mathbb{G} \to \mathbb{H}$ which is also a *strong equivalence* of the underlying Lie groupoids [MM03, § 5.4].

Lemma 2.13. *A coherent Lie 2-group is exactly the same thing as a $\underline{\mathfrak{LGpd}}$-group, i.e., a group object in the 2-category $\underline{\mathfrak{LGpd}}$. To any coherent equivalence between coherent Lie 2-groups, there remains canonically associated an equivalence of $\underline{\mathfrak{LGpd}}$-groups.*

Proof. Although the proof is completely standard, we will briefly recall the construction of the inversion homomorphism $i = i_{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$ (compare also the proof of Proposition 4.6 below), and make a few additional clarifying remarks.

We set $i_0(x) = \overline{x}$ on objects. Let $g : x \to y$ be any arrow. Define ${}^t g : \overline{y} \to \overline{x}$, the *transpose* of g, as the following composition:

$$\begin{aligned} \overline{y} \xrightarrow{\ell_{\overline{y}}^{-1}} \overline{y} \otimes \mathbf{1} \xrightarrow{\overline{y} \otimes d_x} \overline{y} \otimes (x \otimes \overline{x}) \xrightarrow{\overline{y} \otimes (g \otimes \overline{x})} \overline{y} \otimes (y \otimes \overline{x}) \\ \xrightarrow{a_{\overline{y},y,\overline{x}}} (\overline{y} \otimes y) \otimes \overline{x} \xrightarrow{e_y \otimes \overline{x}} \mathbf{1} \otimes \overline{x} \xrightarrow{r_{\overline{x}}} \overline{x}. \end{aligned} \tag{2}$$

One can check that ${}^t(h \circ g) = {}^t g \circ {}^t h$; this follows easily from a characterization of the transpose of any arrow $g : x \to y$ as the unique arrow $h : \overline{y} \to \overline{x}$ such that the following diagram commutes.

$$\begin{array}{ccc} \overline{y} \otimes x & \xrightarrow{h \otimes x} & \overline{x} \otimes x \\ \downarrow{\scriptstyle \overline{y} \otimes g} & & \downarrow{\scriptstyle e_x} \\ \overline{y} \otimes y & \xrightarrow{e_y} & \mathbf{1} \end{array} \tag{3}$$

Then, if we put $i_1(g) = {}^t(g^{-1})$ on arrows, we get a functor, and hence a homomorphism of Lie groupoids. Note, conversely, that the above characterization of transposition (3) implies that the inversion functor $i : \mathbb{G} \to \mathbb{G}$ of any $\underline{\mathfrak{LGpd}}$-group is uniquely determined by the associated data on objects (i.e., by the adjunction data $x \mapsto i_0(x)$, $x \mapsto d_x$, $x \mapsto e_x$). Indeed, by the naturality of e [BL04, p. 37], we must have

$$\begin{aligned} e_x \circ (i_1(g^{-1}) \otimes x) &= e_x \circ [(i_1(g^{-1}) \circ \mathrm{id}_{i_0(y)}) \otimes (g^{-1} \circ g)] \\ &= e_x \circ [i_1(g^{-1}) \otimes g^{-1}] \circ [\mathrm{id}_{i_0(y)} \otimes g] \\ &= e_y \circ [i_0(y) \otimes g] \end{aligned}$$

and therefore, by (3), $i_1(g^{-1}) = {}^t g$.

As to the claim about equivalences, we observe that for each coherent equivalence $\Phi : \mathbb{G} \xrightarrow{\sim} \mathbb{H}$ between coherent Lie 2-groups one can find a *coherent quasi inverse,* namely, a coherent homomorphism $\Psi : \mathbb{H} \xrightarrow{\sim} \mathbb{G}$ so that there exist *monoidal* natural transformations $\Psi \circ \Phi \simeq \mathrm{id}_{\mathbb{G}}$ and $\Phi \circ \Psi \simeq \mathrm{id}_{\mathbb{H}}$. Then, as explained for example in [BL04], one can define natural transformations $i_{\mathbb{H}} \circ \Phi \to \Phi \circ i_{\mathbb{G}}$ and $i_{\mathbb{G}} \circ \Psi \to \Psi \circ i_{\mathbb{H}}$ in such a way as to obtain a pair of $\underline{\mathfrak{LGpd}}$-group homomorphisms forming an equivalence of $\underline{\mathfrak{LGpd}}$-groups. □

Definition 2.14. We call a coherent Lie 2-group $\{G_1 \rightrightarrows G_0, \otimes, \mathbf{1}\}$ *semistrict,* if the monoidal bifunctor $\otimes$ makes the manifold of objects G_0 into a Lie group with unit $\mathbf{1}$, and if the constraints (1) are trivial (that is, $d_x = \mathrm{id}_{\mathbf{1}} = e_x$ for all $x \in G_0$).

In a semistrict (coherent) Lie 2-group, the inverse for each object x is precisely given by $\overline{x}$.

Definition 2.15. We say that a coherent Lie 2-group is *base connected,* when the base manifold of its underlying Lie groupoid is connected.

DEFINITION 2.16. We call a stacky Lie group $\mathcal{G}$ *connected,* when for any pair of points $x, y : \star \to \mathcal{G}$ there exists a path $\mathbb{R} \to \mathcal{G}$ which restricts to x at zero and to y at one (of course, up to 2-isomorphism).

We say that a stacky Lie group $\mathcal{G}$ can be *presented* by a coherent Lie 2-group $\mathbb{G}$, if $\mathcal{G}$ is equivalent, as a stacky Lie group, to $\mathsf{Stack}(\mathbb{G})$ (Definition 2.10). Then we claim the following.

THEOREM 2.17. *Every connected stacky Lie group can be presented by a base connected, semistrict, coherent Lie 2-group.*

The next section will be devoted to proving this theorem.

3. The universal cover of a stacky Lie group

Let $\mathcal{G}$ be an arbitrary connected stacky Lie group, and choose a presentation of its underlying differentiable stack by some Lie groupoid $K_\bullet = \{K_1 \rightrightarrows K_0\}$. Both $\mathcal{G}$ and $K_\bullet$ shall be regarded as fixed once and for all throughout the present section.

By Lie II Theorem [Zhu07], the infinitesimal counterpart of $\mathcal{G}$ is a Lie algebra $\mathfrak{g}$, and the simply connected Lie group G which integrates $\mathfrak{g}$ has a canonical projection onto $\mathcal{G}$

$$p : G \to \mathcal{G}. \tag{4}$$

We are going to establish a few fundamental properties of this map. In order to do this, we first need to review the precise construction of p, which involves some technicalities. We shall limit ourselves to the strictly indispensable notions without going into details; the interested reader is referred to [Zhu07, § 4] for a complete discussion.

To begin with, we need to introduce yet another point of view on Lie 2-groups, according to which these objects should be defined in terms of simplicial manifolds [Hen08, Zhu09b]. Even though this approach via simplicial manifolds is very effective, as it allows us to give quick proofs of the results we need, and even though it probably reflects much better the nature of higher groups in general, it has the disadvantage of being not very explicit. For these reasons, and in order not to confuse the reader with too many definitions, no mention of this alternative viewpoint was made within the previous section.

Recall that a *simplicial manifold* X consists of a sequence of *manifolds* X_n, $n \in \mathbb{Z}^{\geqslant 0}$ and a collection of *smooth* maps (faces and degeneracies) for each n

$$\left.\begin{array}{ll} d_i^n : X_n \to X_{n-1} & \text{(face maps)} \\ s_i^n : X_n \to X_{n+1} & \text{(degeneracy maps)} \end{array}\right\} \quad \text{for } i \in \{0, 1, 2, \dots, n\} \tag{5}$$

satisfying the standard axioms in the definition of a simplicial set (see for example [Fri82]).

DEFINITION 3.1. An *n-Kan complex* X ($n \in \mathbb{N} \cup \infty$) is a simplicial manifold that satisfies the following analogs of the familiar Kan conditions.

(1) For all $m \geqslant 1$ and $0 \leqslant j \leqslant m$, the restriction map

$$\hom(\Delta[m], X) \to \hom(\Lambda[m, j], X) \tag{6}$$

is a surjective submersion.

(2) For each $m > n$ and each $0 \leqslant j \leqslant m$, the same map (6) is a diffeomorphism.

Here, as usual, $\Delta[m]$ and $\Lambda[m, j]$ denote the fundamental m-simplex and its jth horn, respectively. Note that '∞-Kan complex' is usually abbreviated to 'Kan complex'.

Clearly, a 1-Kan complex is the same thing as the nerve of a Lie groupoid. This suggests viewing an n-Kan complex as the nerve of a Lie n-groupoid. (In fact, n-Kan complexes are sometimes themselves referred to as 'Lie n-groupoids' in the literature. However, since this usage of the term contrasts with the definitions we adopted in the preceding section, we prefer to stick to the more traditional terminology.) In particular, when $n = 2$ and the Kan complex is pointed, namely $X_0 = \star$, we obtain the nerve of an $\underline{\mathfrak{HS}}$-group. We briefly recall the explicit correspondence [Zhu09b, § 4]. Given a Lie groupoid $G_\bullet = \{G_1 \rightrightarrows G_0\}$ endowed with an $\underline{\mathfrak{HS}}$-group structure, the corresponding 2-Kan complex, which is completely determined by its first three layers and by some structure maps, is given by

$$X_0 := \star, \quad X_1 := G_1, \quad X_2 := E_m,$$

where E_m is the bibundle presenting the multiplication. We call this associated 2-Kan complex the *nerve of* $G_\bullet$, and we denote it by $NG_\bullet$. The axioms satisfied by the given $\underline{\mathfrak{HS}}$-group structure on the groupoid $G_\bullet = \{G_1 \rightrightarrows G_0\}$ then imply the Kan conditions (Definition 3.1) on the simplicial manifold $NG_\bullet$. Conversely, given a 2-Kan complex X, take $G_0 := X_1$ and $G_1 := d_2^{-1}(s_0(X_0)) \subset X_2$. Then $\{d_0, d_1 : G_1 \rightrightarrows G_0\}$ is a Lie groupoid, which can be endowed with an $\underline{\mathfrak{HS}}$-group structure such that the multiplication bibundle is given by X_2.

A *local Lie group* is more or less like a Lie group, the difference being in that its multiplication is defined only locally near the identity. More precisely, a local Lie group G^{loc} is given by two open neighborhoods $V \subset U$ of the origin in $\mathbb{R}^n$, by a smooth multiplication $m : V \times V \to U$, and by a smooth inversion mapping $i : V \to V$, subject to the condition that the usual algebraic axioms should hold whenever they make sense. To any local Lie group G^{loc} one can still associate a simplicial manifold, the *nerve* NG^{loc} of G^{loc}, exactly like one does for groups. However, NG^{loc} is evidently not a 1-Kan complex anymore:

$$(NG^{\mathrm{loc}})_0 = \star, \quad (NG^{\mathrm{loc}})_1 = V, \quad (NG^{\mathrm{loc}})_2 = m^{-1}(V) \subset V \times V,$$

and in general $(NG^{\mathrm{loc}})_n$ is the following open set of $\mathbb{R}^n$

$$(NG^{\mathrm{loc}})_n = \{(g_1, \ldots, g_n) \in \mathbb{R}^n : g_i \cdot g_{i+1} \cdots g_{i+j} \in V, \text{for all possible } 1 \leqslant i \leqslant n, 0 \leqslant j\}.$$

The face and degeneracy maps are exactly like for nerves of groups. Two local Lie groups are *isomorphic* if they agree on an open neighborhood of the identity. Local Lie groupoids and their nerves are similarly defined. We refer the reader to [Zhu07, § 2.1] for details.

Let us go back to the simply connected Lie group G of (4). We have a local Lie group G^{loc} defined by any choice of suitably small open sets $V \subset U$ about the identity of G (any two such choices will yield the same result up to isomorphism of local Lie groups). Then, by [Zhu09b, Lemma 3.7], we can assume that U embeds as an open subset of the manifold K_0 of objects of the Lie groupoid $K_\bullet$. Hence we have a Lie groupoid homomorphism, induced by the identity structural embedding $K_0 \to K_1$, from the trivial Lie groupoid $V \rightrightarrows V$ into $K_\bullet$. This morphism preserves the group-like structure; for example, the multiplication bibundle E_m, restricted to $V \times V$, is simply the multiplication map $V \times V \to U$ of G^{loc} (see [TZ06a, § 5] for details). Thus, we obtain a simplicial morphism on the level of nerves

$$NG^{\mathrm{loc}} \to NK_\bullet \text{ (see [Zhu07, § 4]).} \tag{7}$$

Then, by applying the operations 'Kan replacement' (Kan) and '2-truncation' (τ_2) [Zhu09a, Proposition–Definition 2.3] to this morphism, we obtain a generalized morphism between 2-Kan complexes

$$NG \sim \tau_2(\mathrm{Kan}(NG^{\mathrm{loc}})) \to \tau_2(\mathrm{Kan}(NK_\bullet)) \sim NK_\bullet, \tag{8}$$

which is a composition of two Morita equivalences (denoted by $\sim$) and of a strict morphism. (Here NG is the nerve of the Lie group G.) By using the correspondence between 2-Kan complexes and $\underline{\mathfrak{HS}}$-groups, we obtain an $\underline{\mathfrak{HS}}$-morphism $G \to K_\bullet$ compatible with the $\underline{\mathfrak{HS}}$-group structures. Using the correspondence between differentiable stacks and Lie groupoids mentioned in the introduction, from this $\underline{\mathfrak{HS}}$-morphism we finally obtain the desired morphism of stacky Lie groups (4).

A brief digression is perhaps in order at this point to explain where the Morita equivalences in (8) come from. The existence of the first Morita equivalence, $NG \sim \tau_2(\mathrm{Kan}(G^{\mathrm{loc}}))$, is essentially a consequence of the fact that $\pi_2(G) = 1$. The details are as follows. To begin with, recall that in general to any Lie algebroid A over a manifold M one can associate a certain infinite-dimensional manifold, P_aA, called the *A-path space* [CF03, § 1], and, on P_aA, a canonical finite-codimensional foliation, $\mathcal{F} \equiv \mathcal{F}(A)$ (see [CF03, Proposition 4.7]). This foliated manifold determines [MM03, § 5.2] a corresponding monodromy groupoid $\mathrm{Mon}_{\mathcal{F}}(P_aA)$ over P_aA, which represents a certain differentiable stack, $\mathcal{G}(A)$. There is a canonical stacky groupoid structure over M on $\mathcal{G}(A)$, which makes the latter into the stacky Lie groupoid integrating A (see [TZ06a]). As in [Zhu09b], one can form the nerve of the differentiable groupoid $\mathrm{Mon}_{\mathcal{F}}(P_aA) \rightrightarrows P_aA$, which will be a 2-Kan complex. We then have the following lemma.

LEMMA 3.2. *Given a Lie algebroid A, let $G^{\mathrm{loc}}(A)$ be its local Lie groupoid. The 2-Kan complex $\tau_2(\mathrm{Kan}(NG^{\mathrm{loc}}(A)))$ and the nerve of $\mathrm{Mon}_{\mathcal{F}}(P_aA) \rightrightarrows P_aA$ are Morita equivalent.*

Proof. This follows from the first part of the proof of [TZ06a, Theorem 4.9] and from [Zhu07, Theorem 3.8, Proposition 4.1]. □

Now, when $A = \mathfrak{g}$ is a Lie algebra, one has that $\mathrm{Mon}_{\mathcal{F}}(P_a\mathfrak{g}) \rightrightarrows P_a\mathfrak{g}$ is Morita equivalent to the trivial groupoid $G \rightrightarrows G$ associated to the Lie group G integrating $\mathfrak{g}$ because, in this case, $\pi_2(G) = 1$; compare [Zhu07, Remark 5.3]. This equivalence being also an equivalence of $\underline{\mathfrak{HS}}$-groups, it follows that $NG \sim \tau_2(\mathrm{Kan}(NG^{\mathrm{loc}}))$ as 2-Kan complexes. This accounts for the first Morita equivalence appearing in (8). The other Morita equivalence there follows from [Zhu09a, Theorem 3.6], which says that if X is already a 2-Kan complex then the 2-truncation of the Kan replacement will not change the Morita equivalence class of X.

Having recalled the necessary technical background about the construction of the map (4), we can now proceed to study its basic properties.

We make an elementary observation.

LEMMA 3.3. *Given a Lie group H and a smooth map $\varphi : X \to H$, the equation*

$$\Phi(x, y) := \varphi(x)\varphi(y)^{-1}$$

defines a Lie groupoid homomorphism Φ from the pair groupoid $P_X = \{X \times X \rightrightarrows X\}$ into H. Conversely, given a Lie groupoid homomorphism $\Phi : P_X \to H$ and a prescribed value $\varphi(x_0) \in H$, one recovers φ by setting $\varphi(x) := \Phi(x, x_0)\varphi(x_0)$.

An analogous statement holds for maps of differentiable stacks $\varphi : \mathcal{X} \to \mathcal{H}$ into a stacky Lie group $\mathcal{H}$.

These constructions are *natural*. For instance, in the stacky case, given another map $\varphi' : \mathcal{X}' \to \mathcal{H}'$, a map $a : \mathcal{X} \to \mathcal{X}'$, and a stacky Lie group homomorphism $\theta : \mathcal{H} \to \mathcal{H}'$, commutativity of the

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first diagram below implies commutativity of the second.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\varphi} & \mathcal{H} \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \theta} \\ \mathcal{X}' & \xrightarrow{\varphi'} & \mathcal{H}' \end{array} \qquad \begin{array}{ccc} P_{\mathcal{X}} & \xrightarrow{\Phi} & \{\mathcal{H} \Rightarrow \star\} \\ \downarrow{\scriptstyle P_a} & & \downarrow{\scriptstyle \theta} \\ P_{\mathcal{X}'} & \xrightarrow{\Phi'} & \{\mathcal{H}' \Rightarrow \star\} \end{array}$$

The commutativity of the first diagram follows from that of the second one so long as we have $\theta(\varphi(x_0)) \sim \varphi'(a(x_0))$ at some point $x_0 : \star \to \mathcal{X}$.

The first basic property of the map p is *surjectivity*. Precisely, we have the following lemma.

Lemma 3.4. *The map $p : G \to \mathcal{G}$ is* surjective, *in the sense that for any point $x : \star \to \mathcal{G}$ one can lift x to some point $\tilde{x} : \star \to G$ making the following diagram 2-commute.*

$$\begin{array}{ccc} & & G \\ & {\scriptstyle \tilde{x}} \nearrow & \downarrow{\scriptstyle p} \\ \star & \xrightarrow{x} & \mathcal{G} \end{array}$$

Proof. Let $\eta : \star \to \mathcal{G}$ denote the group unit. Since $\mathcal{G}$ is connected, we can find a map $a : \mathbb{R} \to \mathcal{G}$ which, up to 2-isomorphism, restricts to η at zero and to x at one (recall Definition 2.16). By Lemma 3.3, a yields a groupoid morphism

$$P_{\mathbb{R}} \xrightarrow{a_{gpd}} \mathcal{G}$$

(where $P_{\mathbb{R}}$ is the pair groupoid $\mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$), which differentiates to a Lie algebroid morphism $a_{algd} : T\mathbb{R} \to \mathfrak{g}$ and then, by Lie II theorem, integrates back to a Lie groupoid morphism

$$P_{\mathbb{R}} \xrightarrow{\tilde{a}_{gpd}} G$$

such that the following diagram of Lie groupoids commutes.

$$\begin{array}{ccc} & & G \\ & {\scriptstyle \tilde{a}_{gpd}} \nearrow & \downarrow{\scriptstyle p} \\ P_{\mathbb{R}} & \xrightarrow{a_{gpd}} & \mathcal{G} \end{array}$$

Since p is a stacky group morphism, η lifts to $e : \star \to G$, the identity of G. Then, again by Lemma 3.3, we obtain a map $\tilde{a} : \mathbb{R} \to G$ lifting $a : \mathbb{R} \to \mathcal{G}$, since we can choose $\tilde{a}(0) = e$ as initial value for $\tilde{a}$: in fact, $\tilde{a} = \tilde{a}_{gpd}|_{\mathbb{R} \times 0} \cdot e$. Hence $\tilde{x} := \tilde{a}|_1 : \star \to G$ will lift $x : \star \to \mathcal{G}$. $\square$

Our next lemma says that p is, in a sense, a 'Serre fibration'.

Lemma 3.5. *Suppose that the outer square 2-commutes in the diagram below. Then there exists a unique smooth map $\tilde{f}$ such that both triangles in the diagram 2-commute.*

$$\begin{array}{ccc} \mathbb{R}^n \times 0 & \longrightarrow & G \\ {\scriptstyle i_0} \downarrow & {\scriptstyle \tilde{f}} \nearrow & \downarrow{\scriptstyle p} \\ \mathbb{R}^n \times \mathbb{R}^k & \xrightarrow{f} & \mathcal{G} \end{array} \tag{9}$$

(Of course, the upper triangle will then be strictly commutative.)

Proof. Let us put $U := \mathbb{R}^n$ and $V := \mathbb{R}^n \times \mathbb{R}^k$, so that the left vertical map in the diagram reads $i_0 : U \hookrightarrow V$.

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(Part I. Existence of a lift.) By Lemma 3.3, we obtain from (9) a commutative diagram of stacky Lie groupoids,

$$\begin{array}{ccc} P_U & \longrightarrow & G \\ \big\downarrow{\scriptstyle P_{i_0}} & & \big\downarrow{\scriptstyle p} \\ P_V & \xrightarrow{f_{gpd}} & \mathcal{G} \end{array} \qquad (10)$$

where $P_\square$ denotes the pair groupoid $\square \times \square \rightrightarrows \square$. By differentiation of (10), we get the following commutative square of Lie algebroid homomorphisms

$$\begin{array}{ccc} TU & \longrightarrow & \mathfrak{g} \\ \big\downarrow & {\scriptstyle f_{algd}} \nearrow & \big\downarrow{\scriptstyle \mathrm{id}} \\ TV & \xrightarrow{f_{algd}} & \mathfrak{g} \end{array} \qquad (11)$$

in which a lift of f_{algd} exists uniquely: take f_{algd} itself. Next, we integrate everything back as we did in the proof of the last lemma. The local exponential map gives a commutative diagram of local Lie groupoids.

$$\begin{array}{ccc} P_U^{\mathrm{loc}} & \longrightarrow & G^{\mathrm{loc}} \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle \mathrm{id}} \\ P_V^{\mathrm{loc}} & \longrightarrow & G^{\mathrm{loc}} \end{array} \qquad (12)$$

By applying the nerve functor and by composing in front with the simplicial morphism $NG^{\mathrm{loc}} \to NK_\bullet$ of equation (7), the last diagram is turned into a commutative diagram (of simplicial manifolds) of the form

$$\begin{array}{ccc} N(P_U^{\mathrm{loc}}) & \longrightarrow & NG^{\mathrm{loc}} \\ \big\downarrow & \nearrow & \big\downarrow \\ N(P_V^{\mathrm{loc}}) & \longrightarrow & NK_\bullet \end{array}$$

to which we then apply $\tau_2(\mathrm{Kan}(-))$. By Lemma 3.2 and the fact that the stacky Lie groupoid $\mathcal{G}(TU)$ integrating TU equals P_U if U is 2-connected [Zhu07, § 6], we obtain a 2-commutative diagram of stacky Lie groupoids.[1]

$$\begin{array}{ccc} P_U & \longrightarrow & G \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ P_V & \longrightarrow & \mathcal{G} \end{array} \qquad (13)$$

Since each one of the morphisms indicated by a solid arrow in this diagram induces the same infinitesimal morphism as its counterpart in the diagram (10), it follows (according to Lie II Theorem, which says that any two morphisms integrating the same infinitesimal morphism can at most differ by a 2-morphism) that the diagonal map in (13) is a lift of f_{gpd} in (10), which we call $\tilde{f}_{gpd}$. Now to obtain a lift of f, we use again the remarks following Lemma 3.3. Namely, choose any point $a_0 : \star \to V$, let $x := f(a_0, 0) : \star \to \mathcal{G}$, and let $\tilde{x} : \star \to G$ be the point to which a_0 is mapped by the upper horizontal arrow in (9), so that, in particular, $p(\tilde{x}) = x$ (compare 3.4). Choosing precisely $\tilde{x}$ as prescribed value at $(a_0, 0)$, it follows that $\tilde{f} := \tilde{f}_{gpd}|_{V \times a_0} \cdot \tilde{x}$ is the required lift of f in (9). The proof of existence is finished.

[1] For the notion of morphism of stacky Lie groupoids that we are using, see [Zhu07, § 4].

(Part II. Uniqueness.) Let two lifts $\tilde{f}$ and $\tilde{f}'$ as in (9) be given. By the naturality statement in Lemma 3.3, they both give rise to maps lifting f_{gpd} in (10). These maps must coincide up to a 2-morphism, by Lie II, because at the infinitesimal level the lift is unique. In other words, $\tilde{f}_{gpd}$ and $\tilde{f}'_{gpd}$ might differ by a 2-morphism; however, since P_V and G happen to be Lie groupoids, the two maps actually coincide. Moreover, since $\tilde{f}$ and $\tilde{f}'$ have the same prescribed value at any point $(a_0, 0) \in \mathbb{R}^n \times \{0\}$ (by assumption, they are both lifts in (9)!), it follows (once again from Lemma 3.3) that $\tilde{f} = \tilde{f}'$. ∎

Corollary 3.6. *The map $p : G \to \mathcal{G}$ is a covering map, in the sense that given any map f from $\mathbb{R}^k$ to $\mathcal{G}$ and any point $g_0 : \star \to G$, there is a unique map $\tilde{f} : \mathbb{R}^k \to G$ such that $\tilde{f}(0) = g_0$; more exactly, $\tilde{f}$ makes the following diagram 2-commute.*

$$\begin{array}{ccc} \star & \xrightarrow{g_0} & G \\ {\scriptstyle i_0}\downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathbb{R}^k & \xrightarrow{f} & \mathcal{G} \end{array}$$

Corollary 3.7. *The map $p : G \to \mathcal{G}$ is an epimorphism of stacks.*

Proof. Take any object $x : V \to \mathcal{G}$ of the stack $\mathcal{G}$. Cover V by contractible open sets $V_i \cong \mathbb{R}^k$. We take $n = 0$ and f to be the composition $\mathbb{R}^k \cong V_i \to \mathcal{G}$ in Lemma 3.5. By Lemma 3.4, we can build a diagram of the form (9). Then we have a lift $\tilde{x}_i : V_i \to G$, which maps to $x|_{V_i} : V_i \to \mathcal{G}$ upon composing with p. Hence p is an epimorphism. ∎

Recall that a map $f : \mathcal{X} \to \mathcal{Y}$ between differentiable stacks is said to be a *submersion* if and only if there is a chart X for $\mathcal{X}$ and a chart Y for $\mathcal{Y}$ such that the map $X \times_{\mathcal{Y}} Y \to Y$ in the diagram below is an ordinary submersion of smooth manifolds. Similarly, f is said to be *étale* if and only if there are as above étale charts X and Y such that the same map $X \times_{\mathcal{Y}} Y \to Y$ is étale (a local diffeomorphism). These definitions do not depend on the choice of charts, namely if the condition is satisfied in one pair of charts then it is satisfied in any pair of charts. For any f and any choice of charts X and Y, the pullback diagram

$$\begin{array}{ccccc} X \times_{\mathcal{Y}} Y & & \longrightarrow & & Y \\ \downarrow & & & & \downarrow \\ X & \twoheadrightarrow & \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \tag{14}$$

is in fact an HS-morphism, from the Lie groupoid $X \times_{\mathcal{X}} X \rightrightarrows X$ to the Lie groupoid $Y \times_{\mathcal{Y}} Y \rightrightarrows Y$, presenting the map of differentiable stacks $f : \mathcal{X} \to \mathcal{Y}$; the manifold $X \times_{\mathcal{Y}} Y$ is the HS-bibundle, and the two maps from $X \times_{\mathcal{Y}} Y$ to X and Y are the moment maps. A well-known argument in the theory of stacks gives the following two lemmas.

Lemma 3.8. *A map $\mathcal{X} \to \mathcal{Y}$ between differentiable stacks is a submersion if and only if for any given groupoid presentations $X = \{X_1 \rightrightarrows X_0\}$ of $\mathcal{X}$ and $Y = \{Y_1 \rightrightarrows Y_0\}$ of $\mathcal{Y}$ the moment map $E \to Y_0$ of any presenting HS-bibundle E is a submersion. In particular, if $\mathcal{X} \to \mathcal{Y}$ can be presented by a strict Lie groupoid homomorphism $X \to Y$ then we can take $E := X_0 \times_{Y_0} Y_1$, and thus $\mathcal{X} \to \mathcal{Y}$ is a submersion if and only if $X_0 \to Y_0$ is a submersion.*

Lemma 3.9. *A map $\mathcal{X} \to \mathcal{Y}$ between étale differentiable stacks is étale if and only if for any given étale groupoid presentations X of $\mathcal{X}$ and Y of $\mathcal{Y}$ as before the moment map $E \to Y_0$ of any presenting HS-bibundle E is étale. In particular, if $\mathcal{X} \to \mathcal{Y}$ can be presented by a strict*

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homomorphism $X \to Y$ then we can take $E := X_0 \times_{Y_0} Y_1$ and thus $\mathcal{X} \to \mathcal{Y}$ is étale if and only if so is $X_0 \to Y_0$.

Lemma 3.10. *The map $p : G \to \mathcal{G}$ is a submersion.*

Proof. Take the groupoid presentations of G and $\mathcal{G}$ coming from $\tau_2(\mathrm{Kan}(NG^{\mathrm{loc}}))$ and $\tau_2(\mathrm{Kan}(NK_\bullet))$ respectively. The map p is induced by a strict morphisms of simplicial manifolds $\tau_2(\mathrm{Kan}(NG^{\mathrm{loc}})) \to \tau_2(\mathrm{Kan}(NK_\bullet))$, which is in turn induced by the inclusion $V \to K_\bullet$. Hence p is represented by a strict morphism of groupoids with respect to these presentations. On the level of objects, the map is simply a disjoint union of iterated copies of the inclusion $V \to K_0$:

$$\tau_2(\mathrm{Kan}(NG^{\mathrm{loc}}))_1 = V \sqcup V \times V \sqcup \cdots \longrightarrow \tau_2(\mathrm{Kan}(NK_\bullet))_1 = K_0 \sqcup K_0 \times K_0 \sqcup \cdots$$

(see [Zhu09a, § 2] for the complete formula). Since the inclusion $V \to K_0$ is a submersion, the induced map $\tau_2(\mathrm{Kan}(NG^{\mathrm{loc}}))_1 \to \tau_2(\mathrm{Kan}(NK_\bullet))_1$ will be a submersion as well. Hence, by Lemma 3.8, the map p must be a submersion. □

Lemma 3.11. *The map $p : G \to \mathcal{G}$ is étale.*

Proof. In the étale groupoid presentations G of G and $K_\bullet$ of $\mathcal{G}$, p is represented by an HS-bibundle E_p. By Lemmas 3.10 and 3.8, the moment map $E_p \to K_0$ is a submersion. However, since E_p is a $K_\bullet$-principal bundle over G, E_p has the same dimension as G and therefore as K_0. Hence the moment map $E_p \to K_0$ is étale. By Lemma 3.9, the map p is itself étale. □

Lemma 3.12. *The map $p : G \to \mathcal{G}$ is a representable surjective submersion.*

Proof. Since p is an epimorphism by Corollary 3.7, we only need to show that p is a representable submersion. Since $\mathcal{G}$ is a differentiable stack, we have a chart $\varphi : U \to \mathcal{G}$. By [BX06, Lemma 2.11], in order to show that p is a representable submersion it is enough to show that $U \times_{\varphi,\mathcal{G},p} G$ is representable and that the map $U \times_{\varphi,\mathcal{G},p} G \to U$ below is a submersion.

$$\begin{array}{ccc} U \times_{\varphi,\mathcal{G},p} G & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\varphi} & \mathcal{G} \end{array}$$

However, the fact that $U \times_{\varphi,\mathcal{G},p} G$ is representable is clear, because φ is representable, and $U \times_{\varphi,\mathcal{G},p} G \to U$ is a submersion since by Lemma 3.10 p is a submersion. □

Consider the following pullback diagram.

$$\begin{array}{ccc} G_1 := G \times_{\mathcal{G}} G & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{p} & \mathcal{G} \end{array} \tag{15}$$

Then $\{G_1 \rightrightarrows G\}$ is an étale Lie groupoid presenting the stack $\mathcal{G}$.

Lemma 3.13. *Suppose that a morphism $\Phi : \mathcal{X} \to \mathcal{Y}$ of differentiable stacks and a morphism $\phi : X_0 \to Y_0$ of their charts fit into a 2-commutative diagram of the form*

$$\begin{array}{ccc} X_0 & \xrightarrow{\phi} & Y_0 \\ {\scriptstyle x}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle y} \\ \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \end{array}$$

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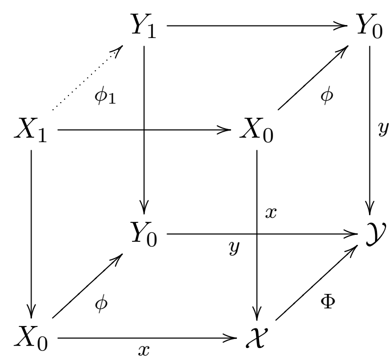
where $x : X_0 \to \mathcal{X}$ and $y : Y_0 \to \mathcal{Y}$ are the chart projection maps. Then the HS-morphism presenting Φ (from the Lie groupoid $X := \{X_1 = X_0 \times_{\mathcal{X}} X_0 \Rightarrow X_0\}$ to the Lie groupoid $Y := \{Y_1 = Y_0 \times_{\mathcal{Y}} Y_0 \Rightarrow Y_0\}$) is strict.

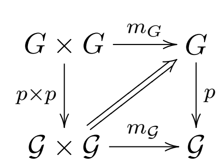
Proof. This is proved in detail in [Zhu07, § 4]; here we only recall the idea. Consider the following 2-commutative diagram.



There are two composite maps of the form $X_1 \to X_0 \to Y_0 \to \mathcal{Y}$, the one going through the upper Y_0 and the other one going through the lower Y_0. These two maps are the same up to a 2-morphism, because the front, back, right, and bottom faces of the diagram are 2-commutative. Thus, by the universal property of the pullback, there exists a morphism $\phi_1 : X_1 \to Y_1$ making all the faces of the diagram 2-commute. Then ϕ_1 and ϕ together form a groupoid morphism, because Φ is a morphism of categories fibred in groupoids. □

Corollary 3.14. *With respect to the groupoid presentation $G_1 \Rightarrow G$, the multiplication law $m_{\mathcal{G}}$ of $\mathcal{G}$ can be presented by a strict morphism of Lie groupoids $(m_1, m_G) : \{G_1 \times G_1 \Rightarrow G \times G\} \to \{G_1 \Rightarrow G\}$, where m_G is the multiplication of G.*

Proof. Since the chart projection $p : G \to \mathcal{G}$ is a morphism of stacky groups, we have a 2-commutative diagram



so the result follows from Lemma 3.13. □

Similarly, one can prove that the unit and the inverse of $\mathcal{G}$ are presentable by strict groupoid morphisms under the groupoid presentation $G_1 \Rightarrow G$. Thus $G_1 \Rightarrow G$ is a $\underline{\mathfrak{LGpd}}$-group. Hence, by the correspondence between coherent Lie 2-groups and $\underline{\mathfrak{LGpd}}$-groups mentioned in Lemma 2.13, we have finally proved Theorem 2.17.

4. From semistrict Lie 2-groups to crossed modules

In this section we carry out the second step of our strictification procedure. Connectedness and étaleness of the objects involved play an essential role in our proofs. We begin by recalling a well-known property of monoidal categories.

Lemma 4.1. *For any semistrict Lie 2-group $\{G_1 \Rightarrow G_0, \otimes, \mathbf{1}\}$, the isotropy group $H = t^{-1}(\mathbf{1}) \cap s^{-1}(\mathbf{1})$ is abelian.*

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Proof. Let $h, h' \in H$. Since $h' = {r_{\mathbf{1}}}^{-1} \circ (h' \otimes \mathbf{1}) \circ r_{\mathbf{1}}$ and $h = {\ell_{\mathbf{1}}}^{-1} \circ (\mathbf{1} \otimes h) \circ \ell_{\mathbf{1}} = {r_{\mathbf{1}}}^{-1} \circ (\mathbf{1} \otimes h) \circ r_{\mathbf{1}}$, the claim $h' \circ h = h \circ h'$ follows at once from the equation

$$(h' \otimes \mathbf{1}) \circ (\mathbf{1} \otimes h) = h' \otimes h = (\mathbf{1} \otimes h) \circ (h' \otimes \mathbf{1}). \qquad \square$$

Lemma 4.2. *For any base connected, semistrict Lie 2-group $\{G_1 \rightrightarrows G_0, \otimes, \mathbf{1}\}$, the associator is trivial. In particular, $g_1 \otimes (g_2 \otimes g_3) = (g_1 \otimes g_2) \otimes g_3$ for all arrows $g_1, g_2, g_3 \in G_1$.*

Proof. Since $\otimes$ is strictly associative on objects, the associator is an automorphism

$$a_{x,y,z} : x \otimes y \otimes z \to x \otimes y \otimes z$$

for all objects $x, y, z \in G_0$. Define a map $h : G_0 \times G_0 \times G_0 \to t^{-1}(\mathbf{1}) \cap s^{-1}(\mathbf{1}) = H$ by

$$h(x, y, z) = a_{x,y,z} \otimes \overline{x \otimes y \otimes z}, \tag{16}$$

where $\overline{x \otimes y \otimes z}$ denotes the (identity arrow corresponding to the) inverse of the object $x \otimes y \otimes z$ in the Lie group $(G_0, \otimes, \mathbf{1})$. Since h is continuous, since G_0 is connected, and since H is discrete, h is a constant map of value $h_0 \in H$.

We contend that $h_0 = \mathrm{id}_{\mathbf{1}}$. To begin with, observe that, by (16) and by Lemma 4.1,

$$h_0 = a_{\mathbf{1},\mathbf{1},\mathbf{1}} \otimes \mathbf{1} = r_{\mathbf{1}} \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}} \circ {r_{\mathbf{1}}}^{-1} = a_{\mathbf{1},\mathbf{1},\mathbf{1}} = \ell_{\mathbf{1}} \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}} \circ {\ell_{\mathbf{1}}}^{-1} = \mathbf{1} \otimes a_{\mathbf{1},\mathbf{1},\mathbf{1}}. \tag{17}$$

Now, by the pentagon coherence condition for the associator,

$$a_{\mathbf{1},\mathbf{1}\otimes\mathbf{1},\mathbf{1}} \circ (a_{\mathbf{1},\mathbf{1},\mathbf{1}} \otimes \mathbf{1}) = (\mathbf{1} \otimes a_{\mathbf{1},\mathbf{1},\mathbf{1}}) \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}\otimes\mathbf{1}} \circ a_{\mathbf{1}\otimes\mathbf{1},\mathbf{1},\mathbf{1}}.$$

Hence, by (17),

$${h_0}^2 = {h_0}^3,$$

from which our claim follows.

Summarizing, we have shown that $a_{x,y,z} \otimes \overline{x \otimes y \otimes z} = \mathrm{id}_{\mathbf{1}}$ for all $x, y, z \in G_0$. We proceed to show that $a_{x,y,z} = \mathrm{id}_{x\otimes y\otimes z}$. Put $u = x \otimes y \otimes z$. Then

$$\begin{aligned}
\mathrm{id}_{x\otimes y\otimes z} &= \mathrm{id}_{\mathbf{1}} \otimes u \\
&= (a_{x,y,z} \otimes \overline{x \otimes y \otimes z}) \otimes u \\
&= a_{u,\overline{u},u} \circ [a_{x,y,z} \otimes (\overline{u} \otimes u)] \circ {a_{u,\overline{u},u}}^{-1} \\
&= a_{u,\overline{u},u} \circ (a_{x,y,z} \otimes \mathbf{1}) \circ {a_{u,\overline{u},u}}^{-1}.
\end{aligned}$$

Thus, $\mathrm{id}_{x\otimes y\otimes z} = a_{x,y,z} \otimes \mathbf{1} = r_{x\otimes y\otimes z} \circ a_{x,y,z} \circ {r_{x\otimes y\otimes z}}^{-1}$. Hence, $\mathrm{id}_{x\otimes y\otimes z} = a_{x,y,z}$. $\square$

Remark 4.3. Recall that, in view of Definition 2.11, the unit constraints $\ell_x : x \to \mathbf{1} \otimes x$ and $r_x : x \to x \otimes \mathbf{1}$ are given by natural transformations between suitable homomorphisms of Lie groupoids. In particular, there is smooth dependence on the variable x.

Lemma 4.4. *In any base connected, semistrict Lie 2-group $\{G_1 \rightrightarrows G_0, \otimes, \mathbf{1}\}$, one has $g \otimes \mathbf{1} = g = \mathbf{1} \otimes g$ for each arrow $g \in G_1$.*

Proof. First we prove the identity in a special case, namely when $g \in \mathrm{Aut}(\mathbf{1})$ belongs to the isotropy group at the unit object $\mathbf{1}$.

So, let $\mathbf{s}(g) = \mathbf{t}(g) = \mathbf{1}$. Then, by the naturality of ℓ and the equality of objects $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$, one gets the identity $\mathbf{1} \otimes g = \ell_{\mathbf{1}} \circ g \circ {\ell_{\mathbf{1}}}^{-1}$ in the group $\mathrm{Aut}(\mathbf{1})$. Since the latter group is abelian by Lemma 4.1, the claim follows.

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Next, put $l = \ell_{\mathbf{1}} \in \mathrm{Aut}(\mathbf{1})$ $(\ell_{\mathbf{1}} : \mathbf{1} \to \mathbf{1} \otimes \mathbf{1} = \mathbf{1})$. For each object $x \in G_0$, we take the composition

$$x \xrightarrow{\ell_x} \mathbf{1} \otimes x \xrightarrow{l^{-1} \otimes x} \mathbf{1} \otimes x = x$$

and denote it by $\tilde{\ell}_x \in \mathrm{Aut}(x)$. Since, by Remark 4.3, the map $x \mapsto \ell_x$ is continuous $G_0 \to G_1$, so must be $x \mapsto \tilde{\ell}_x$. Moreover, since $l \in \mathrm{Aut}(\mathbf{1})$,

$$\tilde{\ell}_{\mathbf{1}} = (l^{-1} \otimes \mathbf{1}) \circ \ell_{\mathbf{1}} = l^{-1} \circ l = \mathrm{id}_{\mathbf{1}},$$

by the already established special case. Now, since $\{G_1 \rightrightarrows G_0\}$ is assumed to be an étale Lie groupoid, and since G_0 a connected manifold, the map $x \mapsto \tilde{\ell}_x$ must stay in the identity component $G_0 \subset G_1$ for all x, and thus $\tilde{\ell}_x = \mathrm{id}_x$ for all $x \in G_0$.

Now let $g : x \to x'$ be an arbitrary arrow in G_1. By the naturality of ℓ, the functoriality of $\otimes$, and what we have just observed, the rectangle

$$\begin{array}{ccccccc} x & \xrightarrow{\ell_x} & \mathbf{1} \otimes x & \xrightarrow{l^{-1} \otimes x} & \mathbf{1} \otimes x & = & x \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathbf{1} \otimes g} & & \downarrow{\scriptstyle \mathbf{1} \otimes g} & & \downarrow{\scriptstyle \mathbf{1} \otimes g} \\ x' & \xrightarrow{\ell_{x'}} & \mathbf{1} \otimes x' & \xrightarrow{l^{-1} \otimes x'} & \mathbf{1} \otimes x' & = & x' \end{array}$$

commutes, and its long edges are identity arrows. The claim follows. □

Definition 4.5. A *strict Lie 2-group* is a group object in the 1-category of (étale) Lie groupoids and Lie groupoid homomorphisms.

Proposition 4.6. *Let $\mathbb{G} = \{G_1 \rightrightarrows G_0, \otimes, \mathbf{1}, a_{x,y,z}, \ell_x, r_x\}$ be a base connected, semistrict Lie 2-group. Then the strictification $\mathsf{Strict}(\mathbb{G}) := \{G_1 \rightrightarrows G_0, \otimes, \mathbf{1}\}$, obtained by simply discarding the monoidal constraints $a_{x,y,z}, \ell_x, r_x$, is a strict Lie 2-group, equivalent to $\mathbb{G}$ as a coherent Lie 2-group.*

Proof. In view of Lemmas 4.2 and 4.4, in order to show that $\mathsf{Strict}(\mathbb{G})$ is a strict Lie 2-group, the only thing left to be checked is the existence, for every arrow $g \in G_1$, of an arrow $\overline{g}$ with $g \otimes \overline{g} = \mathrm{id}_{\mathbf{1}} = \overline{g} \otimes g$.

Let $g : x \to y$. Define ${}^t g : \overline{y} \to \overline{x}$, the *transpose* of g, as

$$\overline{y} = \overline{y} \otimes \mathbf{1} = \overline{y} \otimes x \otimes \overline{x} \xrightarrow{\overline{y} \otimes g \otimes \overline{x}} \overline{y} \otimes y \otimes \overline{x} = \mathbf{1} \otimes \overline{x} = \overline{x}. \tag{18}$$

Then put $\overline{g} := {}^t(g^{-1}) : \overline{x} \to \overline{y}$ (the *contragredient* of g). Let us check that $\overline{g}$ defines an inverse for g in the associative monoid $(G_1, \otimes, \mathrm{id}_{\mathbf{1}})$. We have

$$\begin{aligned} g \otimes (\overline{x} \otimes g^{-1}) &= [g \circ \mathrm{id}_x] \otimes [\mathrm{id}_{\mathbf{1}} \circ (\overline{x} \otimes g^{-1})] \\ &= [g \otimes \mathbf{1}] \circ [x \otimes (\overline{x} \otimes g^{-1})] \quad \text{(exchange law)} \\ &= g \circ g^{-1} \quad \text{(Lemmas 4.4, 4.2)} \\ &= \mathrm{id}_y. \end{aligned}$$

Thus, by Lemma 4.2,

$$\begin{aligned} g \otimes \overline{g} &= g \otimes (\overline{x} \otimes\ g^{-1}) \otimes \overline{y} = \mathrm{id}_y \otimes \overline{y} = \mathrm{id}_{\mathbf{1}}, \\ \overline{g} \otimes g &= \overline{x} \otimes g^{-1} \otimes (\overline{y} \otimes (g^{-1})^{-1}) = \overline{x} \otimes \mathrm{id}_x = \mathrm{id}_{\mathbf{1}}. \end{aligned}$$

Next, we define an equivalence Φ of coherent Lie 2-groups between $\mathbb{G}$ and $\mathbb{G}' = \mathsf{Strict}(\mathbb{G})$. As the Lie groupoid homomorphism underlying Φ, we simply take the identity endofunctor of the

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underlying Lie groupoid $\{G_1 \rightrightarrows G_0\}$. Thus, $\Phi(x) = x$ and $\Phi(g) = g$ for all $x \in G_0$ and $g \in G_1$. As the tensor functor constraints associated to Φ, namely as

$$t_{x,y} : \Phi(x) \otimes' \Phi(y) \to \Phi(x \otimes y) \quad \text{and} \quad u : \mathbf{1}' \to \Phi(\mathbf{1}),$$

we take

$$\mathrm{id}_{x\otimes y} : x \otimes y \to x \otimes y \quad \text{and} \quad \ell_{\mathbf{1}} = r_{\mathbf{1}} : \mathbf{1} = \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}, \tag{19}$$

respectively. As shown in the proof of Lemma 4.4, $\ell_x = (\ell_{\mathbf{1}} \otimes x)$ for all $x \in G_0$. Similarly, $r_x = (x \otimes r_{\mathbf{1}}) = (x \otimes \ell_{\mathbf{1}})$ for all x. It follows immediately that Φ is a tensor functor. Since Φ is also a categorical equivalence (in fact, an isomorphism), the proof is finished. □

Put $\Gamma = \mathbf{t}^{-1}(\mathbf{1})$. This is a closed submanifold of G_1, and if $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1 \otimes \gamma_2 : \mathbf{s}(\gamma_1) \otimes \mathbf{s}(\gamma_2) \to \mathbf{1} \otimes \mathbf{1} = \mathbf{1}$ still belongs to Γ. Hence, by Lemma 4.4, $(\Gamma, \otimes)$ is a monoid with unit $1 = \mathrm{id}_{\mathbf{1}}$. The preceding proposition immediately implies the following corollary.

COROLLARY 4.7. *$(\Gamma, \otimes, 1)$ is a (discrete Lie) group.*

Proof. Since $\overline{\mathbf{1}} = \mathbf{1}$, $\gamma \in \Gamma$ implies $\overline{\gamma} \in \Gamma$. □

DEFINITION 4.8. Let $\partial : \Gamma \to G_0$ denote the restriction of the source map $\mathbf{s} : G_1 \to G_0$ to Γ. Define, for all $\gamma \in \Gamma$ and all $x \in G_0$,

$$\gamma \cdot x := \partial(\gamma) \otimes x, \tag{20}$$

$$x * \gamma := x \otimes \gamma \otimes \overline{x}. \tag{21}$$

Clearly, ∂ is a group homomorphism of $\Gamma = (\Gamma, \otimes, 1)$ into $G_0 = (G_0, \otimes, \mathbf{1})$. Moreover, (20) is a smooth action of Γ on the manifold G_0, and (21) is a smooth action of the connected Lie group G_0 on the discrete manifold Γ. Hence in fact the latter action must be trivial. (These assertions follow from Lemmas 4.2 and 4.4.)

Let us recall the notion of crossed module. A *crossed module* consists of a pair of Lie groups (Γ, G_0), together with a homomorphism $\partial : \Gamma \to G_0$ and a left action $(x, \gamma) \mapsto x * \gamma$ of G_0 on Γ by automorphisms of Γ, such that the following two conditions are satisfied:

(Eq) $\partial(x * \gamma) = x\partial(\gamma)x^{-1}$ (equivariance);
(Pf) $\partial(\gamma) * \gamma' = \gamma\gamma'\gamma^{-1}$ (Pfeiffer identity).

To any crossed module one can associate a strict Lie 2-group, as follows. The induced left action $(\gamma, x) \mapsto \gamma \cdot x = \partial(\gamma)x$ of Γ on G_0 defines a translation groupoid $\Gamma \ltimes G_0 = \{\Gamma \times G_0 \rightrightarrows G_0\}$ with source and target given by $\mathbf{s}(\gamma, x) = x$ and $\mathbf{t}(\gamma, x) = \gamma \cdot x$ respectively and with composition law given by $(\gamma', x') \circ (\gamma, x) = (\gamma'\gamma, x)$ (whenever $x' = \gamma \cdot x$). At the same time, G_0 acts on Γ, so that the Cartesian product $\Gamma \times G_0$ carries a natural group structure

$$(\gamma_1, x_1) \otimes (\gamma_2, x_2) := (\gamma_1(x_1 * \gamma_2), x_1x_2). \tag{22}$$

Let us denote the resulting Lie group by $\Gamma \rtimes G_0$ (wreath product).

We contend that the structure $(\Gamma, G_0, \partial, *)$ (Definition 4.8) is a crossed module. To begin with, we note that the two maps

$$\Psi : \Gamma \times G_0 \to G_1, \quad (\gamma, x) \to \gamma \otimes x, \tag{23}$$

$$\Phi : G_1 \to \Gamma \times G_0, \quad g \mapsto (g \otimes \overline{\mathbf{t}(g)}, \mathbf{t}(g)) \tag{24}$$

are inverse bijections, by Lemmas 4.2 and 4.4. Furthermore, Ψ is a homomorphism of Lie groupoids from $\Gamma \ltimes G_0$ into $\{G_1 \rightrightarrows G_0\}$, inducing the identity on the base G_0. The same map is an isomorphism of groups between $(G_1, \otimes, \mathrm{id}_{\mathbf{1}})$ and the wreath product $\Gamma \rtimes G_0$, as the inverse map Φ is easily seen to be a group homomorphism. Thus, $(\Gamma \ltimes G_0, \Gamma \rtimes G_0)$ is a strict Lie 2-group, because so is $\mathsf{Strict}(\mathbb{G})$. In particular, it follows that Definition 4.8 defines a crossed module of Lie groups, the Pfeiffer identity being equivalent to the statement that the composition law of the groupoid $\Gamma \ltimes G_0$ is a group homomorphism with respect to the group structure $\Gamma \rtimes G_0$.

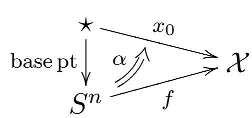
Summarizing, we have proved the following theorem.

Theorem 4.9. *Every connected stacky Lie group can be presented by a crossed module (Γ, G_0), with Γ discrete, and with G_0 connected and simply connected.*

5. Relation to the fundamental group

Our purpose, in this last section, is to show that there is an isomorphism (of groups) between $\pi_1(\mathcal{G})$ (the fundamental group of $\mathcal{G}$) and the group $\Gamma = (\Gamma, \otimes, 1)$ constructed in the last section (Corollary 4.7). The isomorphism in question is of course noncanonical, as the construction of Γ itself was noncanonical.

Definition 5.1. Let $x_0 : \star \to \mathcal{X}$ be a point of a differentiable stack. The *(smooth) nth homotopy set* of $\mathcal{X}$ at x_0, denoted by $\pi_n(\mathcal{X}; x_0)$, is the set of equivalence classes of maps of differentiable stacks $f : S^n \to \mathcal{X}$ for which there exists a 2-isomorphism α like that in the following diagram



(25)

modulo the homotopy equivalence relation

$$f_0 \sim f_1 \text{ if and only if there is } F : S^n \times \mathbb{R} \to \mathcal{X} \text{ such that}$$
$$F(\star, \cdot) \Leftrightarrow i_0, \quad F(\cdot, 0) \Leftrightarrow f_0, \quad F(\cdot, 1) \Leftrightarrow f_1,$$

where $i_0 : \mathbb{R} \to \mathcal{X}$ denotes the constant map $\mathbb{R} \to \star \to \mathcal{X}$.

Notation 5.2. For $\mathcal{X} = \mathcal{G}$ a stacky group, and $x_0 = \mathbf{1} : \star \to \mathcal{G}$ the unit of the stacky group, we shall use the abbreviation $\pi_n(\mathcal{G}) := \pi_n(\mathcal{G}; \mathbf{1})$.

The usual group structure on $\pi_n(\mathcal{X}; x_0)$, $n \geqq 1$ (given by concatenation of loops for $n = 1$) makes still sense in view of the following lemma.

Lemma 5.3. *Any element $[g] \in \pi_n(\mathcal{X}; x_0)$ has a representative $g : S^n \to \mathcal{X}$ which is constant near the base point x_0; namely, there exists an open subset $x_0 \in U \subset S^n$ such that the restriction of g to U factors through $x_0 : \star \to \mathcal{X}$ (up to 2-isomorphism).*

Obviously, any equivalence of stacky Lie groups $\mathcal{G} \xrightarrow{\simeq} \mathcal{G}'$ canonically induces isomorphisms of groups $\pi_n(\mathcal{G}) \xrightarrow{\sim} \pi_n(\mathcal{G}')$ for all $n \geqq 1$. Hence, for our purposes, there will be no loss of generality to assume that $\mathcal{G}$ actually *is* a crossed module of the type considered in the previous section. Then we have the following immediate consequence (see also the proof of Lemma 5.7).

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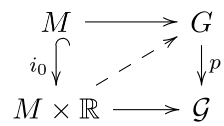
LEMMA 5.4. *The stack map $p : G \to \mathcal{G}$ is a representable surjective submersion, with the following lifting property: given a 2-commutative square*

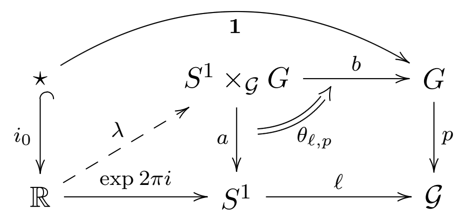


(26)

with M a manifold, there exists a unique map of differentiable stacks, as indicated in the diagram, for which the upper and lower triangle 2-commute (the upper triangle, of course, will be then strictly commutative).

We say that $p : G \to \mathcal{G}$ is a *smooth fibration.*

LEMMA 5.5. *For any loop ℓ representing a class $[\ell] \in \pi_1(\mathcal{G})$, there exists some smooth lift λ fitting as indicated in the following diagram.*



(27)

In other words, λ is a lift of $t \mapsto \exp(2\pi i t)$ to $S^1 \times_{\mathcal{G}} G$ with the property that $(b \circ \lambda)(0) = \mathbf{1} \in G$, or, equivalently, that $b \circ \lambda$ is a lift of $\ell \circ [\exp 2\pi i]$ through $\mathbf{1} \in G$.

Proof. We have a 2-isomorphism between the two stack morphisms

$$\ell \circ [\exp 2\pi i] \circ i_0 \quad \text{and} \quad p \circ \mathbf{1},$$

by (25), because of the assumption $\ell \in \pi_1(\mathcal{G})$. Hence, by Lemma 5.4, there exists a unique lift $f : \mathbb{R} \to G$ with $f \circ i_0 = \mathbf{1}$ (i.e., $f(0) = \mathbf{1}$) and with $p \circ f$ 2-isomorphic to $\ell \circ [\exp 2\pi i]$. By the pullback property of $S^1 \times_{\mathcal{G}} G$, we find $\lambda : \mathbb{R} \to S^1 \times_{\mathcal{G}} G$ such that $b \circ \lambda = f$ and $a \circ \lambda = \exp 2\pi i$. This is precisely what we wanted. $\square$

Remark 5.6. Of course, the preceding lemma holds for any choice of an HS-bibundle E representing ℓ, not just for the canonical pullback of stacks $E = S^1 \times_{\mathcal{G}} G$. A similar remark applies to the next result.

To correctly understand the next lemma, recall that there is a canonical HS-bibundle structure on

$$S^1 \xleftarrow{a} E := S^1 \times_{\mathcal{G}} G \xrightarrow{b} G,$$

and, therefore, a canonical principal right action of the Lie groupoid $\mathcal{G} = \{G_1 \rightrightarrows G\}$ on E along the map b. Hence, for any pair of elements $e_0, e_1 \in E$ with $a(e_0) = a(e_1)$, we have a unique arrow $g : b(e_1) \to b(e_0) \in G_1$, denoted by ${e_0}^{-1} e_1$, such that $e_1 = e_0 \cdot g$.

LEMMA 5.7. *The difference $\lambda(0)^{-1}\lambda(1)$ is the same for all the liftings $\lambda : \mathbb{R} \to S^1 \times_{\mathcal{G}} G$ (associated with a given representative loop ℓ, fixed once and for all) that were considered in the previous lemma, i.e., those liftings fitting in the diagram (27).*

The proof will make use of the following simple observation.

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Lemma 5.8. *The stabilizer subgroup* $\mathrm{Aut}(\mathbf{1}) = \mathbf{s}^{-1}(\mathbf{1}) \cap \mathbf{t}^{-1}(\mathbf{1})$ *is contained in the center of the group* $\Gamma = (\Gamma, \otimes, 1)$.

Proof. Let $\gamma \in \Gamma$, and $\alpha \in \mathrm{Aut}(\mathbf{1})$. Then

$$\begin{aligned}
\gamma \otimes \alpha &= (\mathrm{id}_{\mathbf{1}} \circ \gamma) \otimes (\alpha \circ \mathrm{id}_{\mathbf{1}}) \\
&= (\mathrm{id}_{\mathbf{1}} \otimes \alpha) \circ (\gamma \otimes \mathrm{id}_{\mathbf{1}}) \quad \text{(exchange law)} \\
&= \alpha \circ \gamma \quad \text{(Lemma 4.4)} \\
&= (\alpha \otimes \mathrm{id}_{\mathbf{1}}) \circ (\mathrm{id}_{\mathbf{1}} \otimes \gamma) \quad \text{(Lemma 4.4)} \\
&= (\alpha \circ \mathrm{id}_{\mathbf{1}}) \otimes (\mathrm{id}_{\mathbf{1}} \circ \gamma) \quad \text{(exchange law)} \\
&= \alpha \otimes \gamma.
\end{aligned}$$
□

Proof of Lemma 5.7. Since $\mathcal{G}$ is presented by the action groupoid $\Gamma \ltimes G \rightrightarrows G$, an HS-bibundle of the kind considered above is actually the same thing as a principal right Γ-bundle $a : E \to S^1$ given with a Γ-equivariant map $b : E \to G$, where Γ acts on the right on G by $x \cdot \gamma := \gamma^{-1} \cdot x$. Hence, if λ, μ are two liftings of the kind considered in (27), and they differ at zero by an element γ_0, namely $\mu(0) = \lambda(0) \cdot \gamma_0$, they will differ by the same γ_0 for all t, because of the uniqueness of lifting for a given initial condition (the map $a : E \to S^1$ is étale, because of the discreteness of Γ). Thus, there exists $\alpha_0 \in \Gamma$ such that

$$\mu(0) = \lambda(0) \cdot \alpha_0 \quad \text{and} \quad \mu(1) = \lambda(1) \cdot \alpha_0.$$

One necessarily has $\alpha_0 \in \mathrm{Aut}(\mathbf{1})$, because $b(\lambda(0)) = \mathbf{1} = b(\mu(0))$.

Now, from the assumption

$$\lambda(1) = \lambda(0) \cdot \gamma \quad \text{and} \quad \mu(1) = \mu(0) \cdot \delta,$$

it follows, by the principality of the action of Γ on E, that

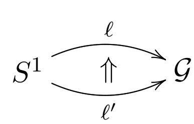
$$\delta = {\alpha_0}^{-1} \gamma \alpha_0 \quad \text{in } \Gamma.$$

By Lemma 5.8, we conclude that $\gamma = \delta$, as contended. □

As observed in the course of the last proof, the assumption $b(\lambda(0)) = \mathbf{1}$ implies that the difference $\lambda(0)^{-1}\lambda(1)$ is an element of $\mathbf{t}^{-1}(\mathbf{1}) = \Gamma$. Thus, we obtain a well defined map into Γ from the set of representative loops; to each representative loop ℓ, one associates the boundary difference $\partial_1(\ell) := \lambda(0)^{-1}\lambda(1)$, for an arbitrary lifting λ as in Lemma 5.5.

Lemma 5.9. *If two loops* $\ell, \ell' : S^1 \to \mathcal{G}$ *represent the same homotopy class* $[\ell] = [\ell'] \in \pi_1(\mathcal{G})$*, then their boundary differences are equal:* $\partial_1(\ell) = \partial_1(\ell') \in \Gamma$.

Proof. Suppose, as a first step, that there is a 2-isomorphism α relating ℓ and ℓ'.



Let $S^1 \xleftarrow{a} E := S^1 \times_{\mathcal{G}} G \xrightarrow{b} G$ be the pullback of p along ℓ, and let E', a', b' be the analogous pullback along ℓ'. Recall that both a, a' are principal right Γ-bundles (canonically), and that both b, b' are equivariant maps.

By the stacky pullback universal property, there exists a canonical smooth map $\tilde{\alpha} : E' \to E$, which is Γ-equivariant, and which commutes with the HS-bibundle maps: $a \circ \tilde{\alpha} = a'$, and $b \circ \tilde{\alpha} = b'$.

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Now, choose any lifting $\lambda' : \mathbb{R} \to E'$, with $a' \circ \lambda' = \exp(2\pi i\,\text{-})$, and with $(b' \circ \lambda')(0) = \mathbf{1}$. The composition $\lambda := \tilde{\alpha} \circ \lambda'$ then satisfies $a \circ \lambda = \exp(2\pi i\,\text{-})$, $(b \circ \lambda)(0) = \mathbf{1}$, and is therefore itself a lifting of the type considered in (27). By the Γ-equivariance of $\tilde{\alpha}$, the boundary differences for λ' and λ must be the same. This proves the lemma in the special case there exists $\alpha : \ell' \Rightarrow \ell$.

Next, let $L : S^1 \times \mathbb{R} \to \mathcal{G}$ be a homotopy between the loops $\ell_0 := L(\text{-}, 0)$ and $\ell_1 := L(\text{-}, 1)$. We want to show that $\partial_1(\ell_0) = \partial_1(\ell_1)$. By the same argument used in the proof of Lemma 5.5, we can find a lifting $\Lambda : \mathbb{R} \times \mathbb{R} \to (S^1 \times \mathbb{R}) \times_{\mathcal{G}} G$ such that $\mathrm{pr}_1 \circ \Lambda = \exp(2\pi i\,\text{-}) \times \mathrm{id}_{\mathbb{R}}$ and $\mathrm{pr}_2(\Lambda(0, s)) = \mathbf{1} \in G$ for all $s \in \mathbb{R}$, where $\mathrm{pr}_1, \mathrm{pr}_2$ denote the two projections

$$S^1 \times \mathbb{R} \longleftarrow (S^1 \times \mathbb{R}) \times_{\mathcal{G}} G \longrightarrow G.$$

Put $\ell_s := L(\text{-}, s)$, for each $s \in \mathbb{R}$. One has a canonical identification between the fiber ${\mathrm{pr}_1}^{-1}(S^1 \times \{s\})$ and the pullback $S^1 \times_{\mathcal{G}} G$ along the loop ℓ_s. For each $s \in \mathbb{R}$, $\lambda_s := \Lambda(\text{-}, s)$ then gets identified to a lifting of the type considered in (27) relative to ℓ_s. Then, the map $s \mapsto \lambda_s(0)^{-1}\lambda_s(1)$ yields a smooth path in Γ connecting $\partial_1(\ell_0)$ and $\partial_1(\ell_1)$. □

The last lemma shows that there is a well defined *boundary map*

$$\partial_1 : \pi_1(\mathcal{G}) \longrightarrow \Gamma. \tag{28}$$

This map is the precise analogue of the usual boundary map in the long exact sequence of homotopy groups associated with the 'stack fibration' $\Gamma \hookrightarrow G \to \mathcal{G}$.

LEMMA 5.10. *The boundary map (28) is a surjection.*

Proof. Let $\gamma_0 \in \Gamma$ be given. We will construct a loop $\ell_0 : S^1 \to \mathcal{G}$ with $\partial_1(\ell_0) = \gamma_0$. The construction will of course make use of the connectedness of the base G.

Suppose we have constructed a smooth curve $f : \mathbb{R} \to G$ with the properties $f(0) = \mathbf{1}$ and $f(t) = \gamma_0 \cdot f(t + 1)$ for all $t \in \mathbb{R}$. Then ℓ_0 may be obtained as follows. Put

$$E := (\mathbb{R} \times \Gamma)/\sim \quad \text{where } (t, \gamma) \sim (t + k, {\gamma_0}^{-k}\gamma)\ \forall k \in \mathbb{Z}. \tag{29}$$

This is evidently a smooth manifold. Define two projections

$$S^1 \overset{a}{\longleftarrow} E \overset{b}{\longrightarrow} G$$

by setting

$$a([t, \gamma]) := \exp(2\pi i t),$$
$$b([t, \gamma]) := \gamma^{-1} \cdot f(t),$$

where $[t, \gamma]$ denotes the equivalence class of the pair (t, γ) with respect to the equivalence relation (29). Finally, let Γ act on E from the right by

$$[t, \gamma] \cdot \gamma' := [t, \gamma\gamma'].$$

One obtains in this way an HS-bibundle representing a loop ℓ_0. By choosing the lifting $\lambda = \{t \mapsto [t, 1]\} : \mathbb{R} \to E$, one immediately sees that $\partial_1(\ell_0) = \gamma_0$.

It only remains to show how to construct a curve f with the desired properties. Choose first any smooth curve $\alpha : (-\frac{1}{4}, \frac{1}{4}) \to G$ with $\alpha(0) = \mathbf{1}$, and translate it by ${\gamma_0}^{-1}$, namely, consider $\beta : (\frac{3}{4}, \frac{5}{4}) \to G$ given by $\beta(t) = {\gamma_0}^{-1} \cdot \alpha(t - 1)$. By connectedness of G, one can then find a smooth path $f : (-\frac{1}{8}, \frac{9}{8}) \to G$ such that f restricts to α on $(-\frac{1}{8}, \frac{1}{8})$, and to β on $(\frac{7}{8}, \frac{9}{8})$. Finally, one extends f to all of $\mathbb{R}$ simply by imposing the required γ_0-periodicity $f(t) = \gamma_0 \cdot f(t + 1)$. □

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Lemma 5.11. *The boundary map (28) is an injection.*

Proof. Let $\ell, \ell' : S^1 \to G$ be two loops such that $\partial_1(\ell) = \partial_1(\ell') =: \gamma_0$, and let

$$S^1 \xleftarrow{a} E \xrightarrow{b} G, \quad S^1 \xleftarrow{a'} E' \xrightarrow{b'} G$$

be the corresponding HS-bibundles. Choose respective liftings $\lambda : \mathbb{R} \to E$, $\lambda' : \mathbb{R} \to E'$ of the type considered in Lemma 5.5, and put $f := b \circ \lambda$, $f' := b' \circ \lambda'$.

Note that one has the periodicity relations $f(t) = \gamma_0 \cdot f(t+1)$, $f'(t) = \gamma_0 \cdot f'(t+1)$, by the assumption $\partial_1(\ell) = \partial_1(\ell') = \gamma_0$. Then, by using the same technique as in the previous proof, one can construct a homotopy $F : \mathbb{R} \times \mathbb{R} \to G$ between $f = F(\text{-}, 0)$ and $f' = F(\text{-}, 1)$ with the periodicity property $F(t, s) = \gamma_0 \cdot F(t+1, s)$ for all $s, t \in \mathbb{R}$. (This uses the simply connectedness of G.)

One obtains again a bibundle

$$E := (\mathbb{R} \times \mathbb{R} \times \Gamma)/\sim \quad \text{with } (t, s, \gamma) \sim (t+k, s, {\gamma_0}^{-k}\gamma) \ \forall k \in \mathbb{Z},$$
$$E \ni [t, s, \gamma] \mapsto (\exp(2\pi i t), s) \in S^1 \times \mathbb{R},$$
$$E \ni [t, s, \gamma] \mapsto \gamma^{-1} \cdot F(t, s) \in G,$$
$$[t, s, \gamma] \cdot \gamma' := [t, s, \gamma\gamma'].$$

The reader can check that this gives a homotopy between ℓ and ℓ'. □

Lemma 5.12. *The boundary map (28) is a homomorphism of groups.*

Proof. Let ℓ_0, ℓ_1 be any two loops in $\mathcal{G}$. We must show that $\partial_1(\ell_0 \odot \ell_1) = \partial_1(\ell_0)\partial_1(\ell_1)$, where $\ell_0 \odot \ell_1$ denotes the concatenation of the two loops.

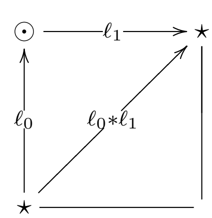
The idea behind the proof is very simple. One considers the following map of differentiable stacks

$$S^1 \times S^1 \xrightarrow{\ell_0 \times \ell_1} \mathcal{G} \times \mathcal{G} \xrightarrow{\mu} \mathcal{G}$$

and its composition $\ell_0 * \ell_1$ with the diagonal embedding $S^1 \hookrightarrow S^1 \times S^1$. By considering HS-bibundle presentations for ℓ_0, ℓ_1 and then for the composition $\mu \circ (\ell_0 \times \ell_1)$, and by playing a bit with liftings, one can explicitly check that

$$\partial_1(\ell_0 * \ell_1) = \partial_1(\ell_0)\partial_1(\ell_1).$$

Moreover, by composing the above map of differentiable stacks with the exponential covering $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, one sees that the loops $\ell_0 \odot \ell_1$ and $\ell_0 * \ell_1$ correspond to the boundary of the triangle above the diagonal in the square $[0, 1] \times [0, 1] \subset \mathbb{R} \times \mathbb{R}$.



Modulo some obvious technicalities, this shows that $\ell_0 \odot \ell_1$ is homotopic to $\ell_0 * \ell_1$. The proof is now complete. □

We may summarize our conclusions as follows.

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Theorem 5.13. *Every connected stacky Lie group $\mathcal{G}$ can be presented as a crossed module of the form $(\pi_1(\mathcal{G}), G)$, with $\pi_1(\mathcal{G})$ the fundamental group of $\mathcal{G}$ (viewed as a discrete Lie group), and G a connected and simply connected Lie group.*

6. Examples

In this final section we illustrate our main theorem on a couple of concrete examples coming from the theory of Hopfish algebras and from Poisson geometry, and speculate on the possible applications of our result to the linearization theory of Poisson manifolds.

6.1 A first example: quantum tori

The notion of Hopfish algebra [BTW08, TWZ07] is a generalization of that of Hopf algebra. The original motivating examples of this notion are *quantum tori*. A quantum torus fails to be a Hopf algebra even though, being the quantization of a group, it should be a quantum group in a certain sense. The key idea behind Hopfish algebras is that instead of describing the coproduct and counit as *morphisms* (of algebras), one should describe them more generally as (algebra) *bimodules*. In fact, the notion of Hopfish algebra is dual to that of 'group object in the category of differentiable stacks'. The stacky group corresponding to a quantum torus under this duality is described in [BTW08, § 2]. We proceed to recall its construction.

The underlying differentiable stack is the étale stack $[S^1/\mathbb{Z}]$ presented by the translation groupoid $\mathbb{Z} \ltimes S^1 \rightrightarrows S^1$ associated to any action of $\mathbb{Z}$ on S^1 of the form: write S^1 additively, i.e., parameterize it by $\theta \in \mathbb{R}$ via the exponential map $\theta \mapsto \exp(2\pi i\theta)$

$$k \cdot \theta = \theta + \lambda k \tag{30a}$$

with λ an irrational real number. The structure maps are given by

$$\left.\begin{array}{l} \mathbf{t}(\theta, k) := \theta + \lambda k, \quad \mathbf{s}(\theta, k) := \theta, \\ (\theta_2, k_2) \cdot (\theta_1, k_1) := (\theta_1, k_2 + k_1) \quad (\text{whenever } \theta_1 = \theta_2 + \lambda k), \\ \mathrm{id}(\theta) = (\theta, 0) \quad \text{and} \quad i(\theta, k) = (\theta - \lambda k, -k). \end{array}\right\} \tag{30b}$$

The multiplication bibundle

$$\begin{array}{ccccc} (S^1 \times \mathbb{Z}) \times (S^1 \times \mathbb{Z}) & \circlearrowright & S^1 \times S^1 \times \mathbb{Z} & \circlearrowleft & S^1 \times \mathbb{Z} \\ \Downarrow & \overset{J_l}{\swarrow} & & \overset{J_r}{\searrow} & \Downarrow \\ S^1 \times S^1 & & & & S^1 \end{array} \tag{31}$$

has left and right moment maps given by $J_l(\theta_1, \theta_2, k) := (\theta_1, \theta_2)$ and $J_r(\theta_1, \theta_2, k) := \lambda k + \theta_1 + \theta_2$ respectively, and left and right groupoid actions given by

$$(\theta_1, l_1; \theta_2, l_2) \cdot (\theta_1, \theta_2, k) := (\theta_1 + \lambda l_1, \theta_2 + \lambda l_2, k - l_1 - l_2)$$

and

$$(\theta_1, \theta_2, k) \cdot (\theta_1 + \theta_2 + \lambda(k - l), l) := (\theta_1, \theta_2, k - l)$$

respectively; in these equations, because of the definition of the moment maps and of the source and target maps, only the groupoid elements of the indicated form can act on a given bibundle element (θ_1, θ_2, k). The identity bibundle is given by

$$\begin{array}{ccccc} \mathrm{pt} & & \mathbb{Z} & \circlearrowleft & S^1 \times \mathbb{Z} \\ \downdownarrows & \swarrow & & \searrow^{J_r} & \downdownarrows \\ \mathrm{pt} & & & & S^1 \end{array}$$

with $J_r(k) := \lambda k$ and with right action $k \cdot (\lambda(k-l), l) := k - l$. The inverse bibundle

$$\begin{array}{ccccc} S^1 \times \mathbb{Z} & \circlearrowleft & S^1 \times \mathbb{Z} & \circlearrowright & S^1 \times \mathbb{Z} \\ \downdownarrows & \swarrow^{J_l} & & \searrow^{J_r} & \downdownarrows \\ S^1 & & & & S^1 \end{array}$$

has moment maps $J_l(\theta, k) := \theta + \lambda k$ and $J_r(\theta) := -\theta$, and left and right actions

$$(\theta + \lambda k, l_1) \cdot (\theta, k) := (\theta, k + l_1) \quad \text{and} \quad (\theta, k) \cdot (-\theta - \lambda l_2, l_2) := (\theta + \lambda l_2, k - l_2).$$

According to our main theorem (Theorem 4.9), the étale stacky Lie group thus defined ought to be equivalent to a crossed module of the form

$$\mathbb{Z} \times \mathbb{Z} \xrightarrow{\tilde{\lambda} k + n} \mathbb{R} \text{ with trivial action of } \mathbb{R} \text{ on } \mathbb{Z} \times \mathbb{Z}. \tag{32}$$

We now check directly that this is indeed the case. For simplicity, we may and will pass to the following equivalent (as a coherent Lie 2-group) crossed module

$$\mathbb{Z} \xrightarrow{\lambda k} S^1 \text{ with trivial action of } S^1 \text{ on } \mathbb{Z} \tag{33}$$

where $\lambda = \mathrm{pr}(\tilde{\lambda})$ and where $\mathrm{pr} : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ denotes the quotient projection corresponding to the action of the second copy of $\mathbb{Z}$. Referring back to § 4, in order to see that the stacky Lie group structure introduced above corresponds to (33), we will first describe the new coherent Lie 2-group structure on the Lie groupoid corresponding to (33). The latter groupoid turns out to be the same underlying translation groupoid $\mathbb{Z} \ltimes S^1$ we considered in (30). The new multiplication m^c is given by the (strict) homomorphism of Lie groupoids

$$\begin{cases} m_1^c : (k_1, \theta_1; k_2, \theta_2) \mapsto (k_1 + k_2, \theta_1 + \theta_2) & \text{on the level of arrows,} \\ m_0^c : (\theta_1, \theta_2) \mapsto \theta_1 + \theta_2 & \text{on the level of objects.} \end{cases}$$

The associated bibundle is $(S^1 \times S^1) \times_{m_0^c, S^1, \mathbf{t}} (S^1 \times \mathbb{Z})$ with J_l^c the projection to $S^1 \times S^1$ and with J_r^c induced by the source map of $\mathbb{Z} \ltimes S^1$. We have an isomorphism of bibundles

$$\varphi : (S^1 \times S^1) \times_{m_0^c, S^1, \mathbf{t}} (S^1 \times \mathbb{Z}) \xrightarrow{\sim} S^1 \times S^1 \times \mathbb{Z},$$

given by

$$(\theta_1, \theta_2, \theta_1 + \theta_2 - \lambda k, k) \mapsto (\theta_1, \theta_2, -k);$$

indeed, φ preserves the moment maps:

$$J_r \circ \varphi(\theta_1, \theta_2, \theta_1 + \theta_2 - \lambda k, k) = \theta_1 + \theta_2 - \lambda k = J_r^c(\theta_1, \theta_2, \theta_1 + \theta_2 - \lambda k, k),$$

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and also the left and right actions:

$$\begin{aligned}
\varphi((\theta_1, l_1; \theta_2, l_2) \cdot (\theta_1, \theta_2, \theta_1 + \theta_2 - \lambda k, k)) &= \varphi((\theta_1 + \lambda l_1, \theta_2 + \lambda l_2, k + l_1 + l_2)) \\
&= (\theta_1 + \lambda l_1, \theta_2 + \lambda l_2, -k - l_1 - l_2) \\
&= (\theta_1, l_1; \theta_2, l_2) \cdot \varphi((\theta_1, \theta_2, \theta_1 + \theta_2 - \lambda k, k)), \\
\varphi((\theta_1, \theta_2, \theta_1 + \theta_2 - \lambda k, k) \cdot (\theta_1 + \theta_2 - \lambda k - \lambda l, l)) &= \varphi((\theta_1, \theta_2, \theta_1 + \theta_2 - \lambda k - \lambda l, k + l)) \\
&= (\theta_1, \theta_2, -k - l) \\
&= (\theta_1, \theta_2, -k) \cdot (\theta_1 + \theta_2 - \lambda k - \lambda l, l).
\end{aligned}$$

Thus φ is a 2-morphism between the multiplication bibundle (31) and that of (33). Similarly, one can find other 2-morphisms between the identity bibundles and the inverse bibundles:

$$\mathrm{pt} \times_{0, S^1, \mathbf{t}} S^1 \times \mathbb{Z} \xrightarrow{\sim} S^1 \quad \text{given by } (\mathrm{pt}, -\lambda k, k) \mapsto -k,$$

and

$$S^1 \times_{-\Box, S^1, \mathbf{t}} S^1 \times \mathbb{Z} \xrightarrow{\sim} S^1 \times \mathbb{Z} \quad \text{given by } (\theta + \lambda k, -\theta, -k) \mapsto (\theta, k).$$

It is routine to verify that these 2-morphisms provide an equivalence of stacky Lie groups.

6.2 A second example: Poisson homotopy groups

Recall from [CM09] that a *cotangent path* in a Poisson manifold (M, π) is a path $a : [0, 1] \to T^*M$ of class C^1 that obeys the differential equation

$$\frac{d\gamma_a}{dt}(t) = \pi^{\#}(a(t))$$

where $\pi^{\#} : T^*M \to TM$ denotes the vector bundle map associated to the Poisson bivector π and γ_a denotes the base path in M onto which a projects. A *cotangent homotopy* is a variation of cotangent paths $\{a_\epsilon : 0 \leqq \epsilon \leqq 1\}$ of class C^2 in ϵ with the property that the base paths γ_{a_ϵ} have fixed end points and for which the unique solution $b(\epsilon, t)$ to the equation

$$\partial_t b - \partial_\epsilon a = T_\nabla(a, b), \quad b(\epsilon, 0) = 0$$

(T_∇ being the torsion of any covariant derivative ∇ on T^*M) satisfies $b(\epsilon, 1) = 0$ for all ϵ.

As in [TZ06a], we consider the Banach manifold $\boldsymbol{P}_0$ of all cotangent paths which satisfy the following boundary conditions:

$$a(0) = 0, a(1) = 0 \quad \text{and} \quad \dot{a}(0) = 0, \dot{a}(1) = 0.$$

The path foliation $\boldsymbol{\mathcal{F}}_0$ on $\boldsymbol{P}_0$ consists by definition of the cotangent homotopy classes, and has leaves of finite codimension. Consider the corresponding holonomy (Banach) groupoid $\mathrm{Hol}(\boldsymbol{\mathcal{F}}_0) \rightrightarrows \boldsymbol{P}_0$. Choose a complete transversal $P \subset \boldsymbol{P}_0$ for the foliation $\boldsymbol{\mathcal{F}}_0$. The restriction of the holonomy groupoid to P will be an étale (finite-dimensional) Lie groupoid, which we denote by $\mathcal{H} = \{H \rightrightarrows P\}$. Different choices of P will yield Morita equivalent Lie groupoids $\mathcal{H}$.

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Choosing the transversal P conveniently, one can accommodate for the following Lie groupoid homomorphisms, which we call source $\underline{s}$ and target $\underline{t}$, unit $\underline{e}$, and inverse $\underline{i}$.

$$
\begin{array}{ccc}
H \xrightarrow{\underline{s}_1,\underline{t}_1} M & \qquad M \xrightarrow{\underline{e}_1} H & \qquad H \xrightarrow{\underline{i}_1} H \\
\Downarrow \qquad \Downarrow & \qquad \Downarrow \qquad \Downarrow & \qquad \Downarrow \qquad \Downarrow \\
P \xrightarrow{\underline{s}_0,\underline{t}_0} M & \qquad M \xrightarrow{\underline{e}_0} P & \qquad P \xrightarrow{\underline{i}_0} P \\
[a(\epsilon,t)] \mapsto \gamma_a(0,0), \gamma_a(0,1) & \qquad x \mapsto \mathrm{id}_{0_x} & \qquad [a(\epsilon,t)] \mapsto [a(\epsilon,1-t)]
\end{array}
$$

The source and target homomorphisms $\underline{s}$ and $\underline{t}$ are *surjective submersions* of Lie groupoids; compare [TZ06a, proof of Lemma 4.2]. There is no analogous choice of P leading to a description of the operation of path concatenation

$$(a_1, a_0) \mapsto a_1 \odot a_0 = \begin{cases} 2a_0(2t), & 0 \leqq t \leqq \frac{1}{2}, \\ 2a_1(2t-1), & \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

as a strict Lie groupoid homomorphism. One can, however, easily describe path concatenation as an HS-morphism

$$
\begin{array}{ccccc}
H \times_{\underline{s}_1,M,\underline{t}_1} H & \circlearrowright & E & \circlearrowleft & H \\
\Downarrow & \swarrow^{J_l} & & \searrow^{J_r} & \Downarrow \\
P \times_{\underline{s}_0,M,\underline{t}_0} P & & & & P
\end{array}
$$

by taking

$$E := {\mathbf{s}_{\mathrm{Hol}(\boldsymbol{\mathcal{F}}_0)}}^{-1}(P) \cap {\mathbf{t}_{\mathrm{Hol}(\boldsymbol{\mathcal{F}}_0)}}^{-1}(\odot(P \times_{\underline{s}_0,M,\underline{t}_0} P))$$

with moment maps $J_l := \odot^{-1} \circ \mathbf{t}_{\mathrm{Hol}(\boldsymbol{\mathcal{F}}_0)}$ and $J_r := \mathbf{s}_{\mathrm{Hol}(\boldsymbol{\mathcal{F}}_0)}$.

Now consider any symplectic leaf S in M. The restriction of $\mathcal{H}$ to the submanifold $P_S := \underline{s}_0^{-1}(S) \subset P$ defines another étale Lie groupoid $\mathcal{H}_S = \{H_S \rightrightarrows P_S\}$; this makes sense just because $\underline{s}_0^{-1}(S)$ is an $\mathcal{H}$-invariant submanifold of P, by the $\mathcal{H}$-invariance of $\underline{s}_0$ (cf. again the proof of Lemma 4.2 in [TZ06a]). As in general a symplectic leaf S in M is the subset of all $x \in M$ which can be joined to any of its points $x_0 \in S$ by some cotangent path [CM09, p. 8], the combined map $(\underline{s}_0, \underline{t}_0) : P_S \to S \times S$ is a surjective submersion, so that the fiber $G_0 := (\underline{s}_0, \underline{t}_0)^{-1}((x_0, x_0)) \subset P_S$ is an $\mathcal{H}_S$-invariant submanifold of P_S. Hence $\mathcal{G}_{x_0} := \{G_1 \rightrightarrows G_0\}$ is an étale Lie groupoid, where $G_1 := \mathcal{H}_S|_{G_0}$. The structure morphisms discussed in the preceding paragraph restrict to structure morphisms on $\mathcal{G}_{x_0}$ and make the latter into an étale stacky Lie group,[2] which, sticking to the terminology of [CM09] (compare below), we call the *Poisson homotopy group* at $x_0 \in M$.

Unlike the previous Example 6.1, in the case of Poisson homotopy groups there is no obvious way to make the concatenation operation into a strict groupoid homomorphism; our theorem is therefore a nontrivial result in this case.

6.3 Potential applications of Theorem 4.9

In their recent paper [CM09] on the linearization of Poisson structures around symplectic leaves, Crainic and Mărcuţ consider a certain principal bundle over any symplectic leaf $S \subset M$ in a Poisson manifold (M, π). The total space $P = P_{x_0}$ of this principal bundle (which they call the *Poisson homotopy cover* of S) is defined to be the space of cotangent homotopy classes

[2] We omit the discussion of the coherence 2-morphism, for which we refer the reader to [TZ06a].

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of cotangent paths originating at some fixed point x_0 on S. The classes of paths starting and ending at x_0 form what in [CM09] is called the *Poisson homotopy group* $G = G_{x_0}$. This group acts naturally on P from the right and makes the latter into a principal G-bundle over S.

A delicate issue here is that in general P and G need not exist as smooth manifolds [CF04], although they always do as differentiable stacks. This is related to the integrability problem for Lie algebroids [CF03]. Thus one has either to work under a smoothness assumption, as in [CM09], or to allow differentiable stacks into the game. The second option leads to the notion of Poisson homotopy group introduced in the preceding example.

When P and G exist as smooth manifolds, the natural candidate considered in [CM09] for a local linear model of the Poisson manifold (M, π) around the leaf S is provided by

$$P \times_G \mathfrak{g}_{x_0}^*$$

where $\mathfrak{g}_{x_0}$, the *isotropy Lie algebra* of the Poisson manifold at x_0, is the conormal space to S at x_0 with Lie bracket induced by the Poisson bracket, and where the action of G on $\mathfrak{g}_{x_0}^*$ can be identified with the coadjoint action of G on its own Lie algebra. On the other hand, as this local model always exists as a differentiable stack, it is conceivable that it might also exist as a smooth manifold (i.e., be representable) even in cases where P and G do not. One might then ask whether the linearization theorem still holds in such cases. As our theorem is essentially a normal form result for étale stacky Lie groups, we expect it to play a role in the analysis of these problems. Note that on the level of stacks one can always define the leafwise symplectic structure required by the local linearization theorem [TZ06b]. All of this seems to be pointing towards an interesting potential generalization of the result of Crainic and Mărcuţ.

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