BULL. AUSTRAL. MATH. SOC. MOS 45J05, (45D05)
VOL. 3 (1970), 9-22.

Integro-differential equations of Volterra type

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The aim of this paper is concerned with studying the stability properties of an integro-differential system by reducing it into a scalar integro-differential equation. A theorem is stated about the existence of a maximal solution of such systems and a basic result on integro-differential inequalities. Utilizing these results we obtain sufficient conditions for uniform asymptotic stability of the trivial solution of the integro-differential system of the form

$$x'(t) = F(t, x(t), Ax) , \quad (' \equiv \frac{d}{dt})$$

where $F \in C[R_+ \times S_H \times C(J)]$, $A \in C[C_H, C(J)]$ with $C_H = \lfloor x \in C(J) : \|X\| < H \rfloor$, $J = 0 \le t \le a < \infty$,

$S_H = [x \in R^n : \|x(t)\| < H , H > 0 \text{ for } t \in J]$, $C(J)$ denotes the space of continuous functions, A a continuous operator such that A maps $C(J)$ into $C(J)$. The fruitfulness of the results of the paper are illustrated with two applications.

1.

Corduneanu [1], Levin [3] and Nohel [6] among others have studied the stability properties of solutions of integro-differential equations of Volterra type and many interesting results have been accumulated. Quite recently Lakshmikantham and Rama Mohana Rao [2] investigated such a

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Received 4 April 1970. The authors thank Professor V. Lakshmikantham for his valuable suggestions during the progress of this research.

https://doi.org/10.1017/S0004972700045603 Published online by Cambridge University Press

problem by choosing an appropriate minimal class of functions so that it would be possible to estimate the derivative of the Lyapunov function in terms of a scalar function.

Our aim in this paper is to study the stability properties of the integro-differential system by reducing it into a scalar integro-differential equation. In Section 2, we introduce the notation and a lemma which is used in the study. In Section 3, we shall develop a theorem about the existence of a maximal solution of integro-differential systems and a basic result on integro-differential inequalities. Applying this theory of integro-differential inequalities, we obtain sufficient conditions for uniform asymptotic stability of the trivial solution of integro-differential systems in Section 4. This study includes the results of Corduneanu [1] as a special case. Two applications are given in Section 5 to illustrate the fruitfulness of our results.

2.

We shall use the following notations:

R^n = space of n-vectors,

$\|x\|$ = any convenient norm of $x \in R^n$,

$J = 0 \le t \le a < \infty$,

$S_H = [x \in R^n : \|x(t)\| < H , \ H > 0 , \text{ for } t \in J]$,

R_+ = non-negative real line,

$C[E, F]$ = the class of functions defined and continuous on E taking values in F , where E and F are any convenient spaces,

K = the class of continuous functions $b(r)$, defined and continuous on $0 \le r < H$, $b(0) = 0$ and monotone increasing in r ,

C^+ = the class of non-negative functions defined and continuous on R_+ .

In the presentation of this paper, whenever we employ a vector

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inequality, it is to be understood that the inequality is satisfied componentwise.

Let E_1 and F_1 be two partially ordered sets with the partial ordering "$\leq$" for both sets. We shall assume that the following conditions hold:

(2.1) $$x, y, z \in E_1 , \quad x \leq y , \quad y \leq z \text{ implies } x \leq z ;$$

(2.2) $$x, y \in E_1 , \quad x \leq y , \quad y \leq x \text{ implies } x = y ;$$

(2.3) $$\bar{x}, \bar{y}, \bar{z} \in F_1 , \quad \bar{x} \leq \bar{y} , \quad \bar{y} \leq \bar{z} \text{ implies } \bar{x} \leq \bar{z} ;$$

(2.4) $$\bar{x} \in F_1 , \text{ then } \bar{x} \leq \bar{x} .$$

Furthermore, let P_1 be an operator defined on E_1 taking its values in F_1, the function Q_1 be defined on $E_1 \times E_1$ taking its values in F_1 and m to denote the maximal solution [5] of equation

(2.5) $$P_1(x) = Q_1(x, x) .$$

For $x, y_1, y_2 \in E_1$, we shall assume that

(2.6) $$y_1 \leq y_2 \text{ implies } Q_1(x, y_1) \leq Q_1(x, y_2)$$

and we define a set U_1 as follows:

(2.7) $$U_1 = [x \in E_1 : P_1(x) \leq Q_1(x, x)] .$$

We now state a lemma about the existence of the maximal solution of equation (2.5), the proof of which follows a similar argument as in [5].

LEMMA 2.1. *Let* P_1 *and* Q_1 *be as defined above. Assume that* Q_1 *possesses the property* (2.6) *and that there exists a function* z_1 *defined on* E_1 *such that* $z_1(E_1) \subset E_1$ *satisfying the conditions*

(2.8) $$P_1\big(z_1(y)\big) = Q_1\big(z_1(y), y\big)$$

and

(2.9) $$P_1(x) \leq Q_1(x, y)$$

together imply that

$$x \leq z_1(y) .$$

Let the set U_1 *defined in* (2.7) *be non-empty. Then,*

(2.10) $$z_1(E_1) \subset U_1 .$$

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Moreover, the existence of $\sup U_1$ *implies the existence of* $\sup z_1(U_1)$ *and vice versa. Also,* $\sup U_1 = \sup z_1(U_1)$. *Their common value is then the maximal solution of* (2.5).

3.

Let $C(J)$ denote the space of continuous functions $u \in C[J, R^n]$ and A be a continuous operator such that A maps $C(J)$ into $C(J)$. For any two continuous functions $u, v \in C[J, R^n]$ the operator A is assumed to satisfy the following property:

$$u(t) \le v(t) \ , \quad 0 \le t \le t_1 \ , \quad t_1 \in (0, \infty)$$

implies

$$Au \le Av \quad \text{for} \quad t = t_1 \ . \tag{3.1}$$

Let $f \in C[J \times R^n \times C(J), R^n]$ and M a constant vector greater than zero such that for $(t, u, v) \in J \times R^n \times C(J)$

$$|f(t, u, v)| \le M \ ,$$

with the function f possessing the quasi-monotone property, that is, for each i , $f_i(t, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n)$ is monotonic increasing in $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$. Now consider the integro-differential system

$$u'(t) = f(t, u(t), Au) \ , \quad u(0) = u_0 \ . \tag{3.2}$$

THEOREM 3.1. *Let* $f(t, u, v)$ *satisfy the above conditions along with* (3.1). *Then there exists a maximal solution* $m(t)$ *of* (3.2) *on* J . *If* $\phi(t)$ *is a continuous vector function satisfying the inequalities*

$$\phi(0) \le m(0) \tag{3.3}$$

and

$$D^+\phi(t) \le f(t, \phi(t), A\phi) \ , \tag{3.4}$$

then,

$$\phi(t) \le m(t) \quad \text{on} \quad J \ .$$

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Proof. Let E_1 be the set of continuous functions $\psi \in C[J, R^n]$, such that

$$\psi(t) \leq m(0) + Mt , \quad t \in J .$$

For ψ, $\rho \in E_1$, $\psi \leq \rho$ we have

$$\psi(t) \leq \rho(t) , \quad t \in J .$$

Similarly, F_1 is a set of vector-valued functions on J with the components of its elements belonging to $[-\infty, \infty]$ and having the same order relationships as the set E_1, that is, for ξ, $\eta \in F_1$, $\xi \leq \eta$ we have

$$\xi(t) \leq \eta(t) .$$

From this it is clear that conditions (2.1) to (2.4) hold. Define the operator P_1 and the function Q_1 as follows:

$$P_1(\psi) = \left\{D^+\psi_i(t)\right\} , \quad i = 1, 2, \ldots, n$$

and

$$Q_1(\psi, \rho) = f_i\big(t, \rho_1(t), \ldots, \rho_{i-1}(t) , \psi_i(t) , \rho_{i+1}(t), \ldots, \rho_n(t) , A\rho_1, A\rho_2, \ldots, A\rho_n\big) .$$

It follows from the behavior of f that Q_1 satisfies (2.6). For $\rho \in E_1$, we define

$$\bar{f}_i(t, u) = f_i\big(t, \rho_1(t), \ldots, \rho_{i-1}(t), u, \rho_{i+1}(t), \ldots, \rho_n(t), A\rho_1, \ldots, A\rho_n\big) .$$

Let $\sigma(t)$ be the maximal solution for each i of

$$u' = \bar{f}_i(t, u)$$

such that

$$\sigma(0) = m(0) .$$

The existence of $\sigma(t)$ on J is assured because of the assumptions on f. Now, we define a function $z_1(\rho)$ such that $z_1\big(\rho(t)\big) = \sigma(t)$. From the basic theorem on differential inequalities it follows that the function $z_1(\rho)$ satisfies (2.9) and obviously (2.8) holds. Moreover, the set U_1 is not empty since $m(0) - Mt \in U_1$ and

$$|\sigma'(t)| \le M .$$

Hence, the family of functions $\{\sigma_i(t)\}$ are equi-continuous and uniformly bounded. This proves that $\sup z_1(U_1) = \sup \sigma(t)$ is a continuous vector function on J. The assertion of the theorem now follows from Lemma 2.1.

4.

With respect to the objectives of this paper, consider the following integro-differential system

$$x'(t) = F\big(t, x(t), Ax\big) \tag{4.1}$$

where $F \in C\left[R_+ \times S_H \times C(J), R^n\right]$, $A \in C\left[C_H, C(J)\right]$ with $C_H = \left[x \in C(J) : \|x\| < H\right]$. We shall assume that $F(t, 0, 0) \equiv 0$. Let $x(t) = x\left(t, t_0, x_0\right)$ be any solution of (4.1). In order to avoid repetition we shall only concentrate on uniform asymptotic stability of the trivial solution of the integro-differential system (4.1). For convenience, we shall next define uniform asymptotic stability.

DEFINITION 4.1. The trivial solution of the integro-differential system (4.1) is said to be uniformly asymptotically stable if the following two conditions hold:

(i) for every $\varepsilon > 0$, $t_0 \in J$, there exists a function $\delta = \delta(\varepsilon) > 0$ such that the inequality $\|x_0\| \le \delta$ implies

$$\left\|x\left(t, t_0, x_0\right)\right\| < \varepsilon , \quad t \ge t_0 ;$$

(ii) for every $\varepsilon > 0$, $t_0 \in J$, there exist positive numbers δ_0 and $T = T(\varepsilon)$ such that whenever $\|x_0\| \le \delta_0$,

$$\left\|x\left(t, t_0, x_0\right)\right\| < \varepsilon , \quad t \ge t_0 + T$$

holds.

THEOREM 4.1. *Assume that*

(i) $g \in C\left[R_+ \times R_+ \times R_+, R\right]$, $g(t, 0, 0) \equiv 0$ *and* $g(t, u, v)$ *is*

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non-decreasing in v *for each* (t, u) ;

(ii) $V \in C[R_+ \times S_H, R_+]$, $V(t, 0) \equiv 0$, $V(t, x)$ *is Lipschitzian in* x *for a constant* $L = L(H) > 0$ *and for* $t \in J$, $x(t) \in C(J)$

$$D^+V(t, x(t)) = \limsup_{h\to 0^+} \frac{1}{h}\Big[V\big(t+h, x(t)+hF(t, x(t), Ax)\big) - V(t, x(t))\Big]$$
$$\leq g\big(t, V(t, x(t)), BV\big) ,$$

where $B \in C[C^+, R_+]$;

(iii) there exists a function $a \in K$ *such that for* $(t, x) \in J \times S_H$,

$$a(\|x\|) \leq V(t, x) .$$

Then the uniform asymptotic stability of the trivial solution of the scalar equation

$$(4.2) \qquad r' = g(t, r, Br) , \quad r(t_0) = r_0$$

implies the uniform asymptotic stability of the trivial solution of the integro-differential system (4.1).

Proof. Suppose that the trivial solution of (4.2) is uniformly asymptotically stable. This implies that it is uniformly stable. Let $0 < \varepsilon < H$, $t_0 \in J$ be given. Then, given $a(\varepsilon) > 0$, there exists a $\delta = \delta(\varepsilon) > 0$, such that, whenever $r_0 \leq \delta$ we have

$$(4.3) \qquad r(t, t_0, r_0) < a(\varepsilon) , \quad t \geq t_0 ,$$

where $r(t) = r(t, t_0, r_0)$ is any solution of (4.2). Let us choose $r_0 = L\|x_0\|$ so that by assumption *(ii)* we have $V(t_0, x_0) \leq r_0$. Now we choose $\delta_1(\varepsilon) = \frac{\delta(\varepsilon)}{L}$. Furthermore, we claim that if $\|x_0\| \leq \delta_1(\varepsilon)$, we have

$$\|x(t, t_0, x_0)\| < \varepsilon , \quad t \geq t_0 .$$

Suppose that this claim is not true, then for $t_1 > t_0$ we have

$$\|x(t_1, t_0, x_0)\| = \varepsilon$$

and

$$\|x(t, t_0, x_0)\| \le \varepsilon \ , \quad t_0 \le t \le t_1 \ ,$$

which implies that

$$a(\varepsilon) \le V(t_1, x(t_1)) \tag{4.4}$$

and

$$\|x(t)\| < H \ , \quad t \in [t_0, t_1] \ .$$

Utilizing the assumption *(ii)* and standard computation yields

$$D^+m(t) \le g(t, m(t), Bm) \ , \quad t \in [t_0, t_1]$$

where

$$m(t) = V(t, x(t)) \ .$$

Now, taking r_0 as defined above and applying Theorem 3.1 we obtain

$$V(t, x(t)) \le r(t) \ , \quad t \in [t_0, t\,] \ . \tag{4.5}$$

Inequalities (4.3), (4.4) and (4.5) lead to the following contradiction

$$a(\varepsilon) \le V(t_1, x(t_1)) \le r(t_1) < a(\varepsilon) \ .$$

Hence, the first condition of Definition 4.1 holds. Now, for $\varepsilon = H$ and $\hat{\delta}_0 = \delta_1(H)$ it follows that

$$V(t, x(t)) \le r(t) \ , \quad t \ge t_0 \ . \tag{4.6}$$

Also from the uniform asymptotic stability of the trivial solution of (4.2), we have, given $a(\varepsilon) > 0$, $t_0 \in J$, there exists a positive number δ_0 and $T(\varepsilon)$ such that for $r_0 \le \delta_0$ it implies that

$$r(t, t_0, r_0) < a(\varepsilon) \ , \quad t \ge t_0 + T \ . \tag{4.7}$$

Let $\delta_0^* = \min\left[\hat{\delta}_0, \frac{\delta_0}{L}\right]$. Suppose that there exists a sequence $\{t_k\}$,

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$t_k \geq t_0 + T$ with $t_k \to \infty$ as $k \to \infty$, such that

$$\|x(t_k, t_0, x_0)\| \geq \varepsilon ,$$

where $x(t, t_0, x_0)$ is any solution of (4.1) starting in $\|x_0\| \leq \delta_0^*$.
Thus in view of assumption *(iii)* of the theorem and inequalities (4.6) and (4.7) we have the following contradiction

$$a(\varepsilon) \leq V\left[t_k, x(t_k)\right] \leq r(t_k) < a(\varepsilon) .$$

Hence, the second condition of Definition 4.1 holds and the proof is complete.

REMARK. In particular, if

$$F(t, x, Ax) = G(t, x) + \int_{t_0}^{t} K(t, s, x(s))ds ,$$

then equation (4.1) reduces to a perturbed differential system with integral perturbations, that is,

$$x'(t) = G(t, x(t)) + \int_{t_0}^{t} K(t, s, x(s))ds .$$

5. Applications

Consider the system of ordinary differential equations,

(5.1) $$x' = G(t, x)$$

and integro-differential equations

(5.2) $$x'(t) = G(t, x(t)) + \int_{t_0}^{t} K(t, s, x(s))ds$$

where $G \in C\left[R_+ \times S_H, R^n\right]$, $G(t, x)$ satisfies a Lipschitz condition in x for a constant $\lambda(H) > 0$ and $G(t, 0) \equiv 0$. Let $K(t, s, x)$ be defined and continuous on $0 \leq s \leq t < \infty$, $\|x\| < H \leq \infty$, $K(t, s, 0) \equiv 0$.

COROLLARY 5.1. *Assume that*

(i) the trivial solution of the unperturbed system (5.1) *is*

uniformly asymptotically stable;

(ii) $H \in C[R_+ \times R_+ \times R_+, R_+]$, $H(t, s, 0) \equiv 0$, $H(t, s, r)$ *is nondecreasing in* r *and*

$\|K(t, s, x)\| \le H(t, s, \|x\|)$, $t, s \in J$, $x \in S_H$;

(iii) for every $d > 0$ *, there exists a* $\tau_d \ge 0$ *and a function* $h_d(t, s)$ *continuous on* $\tau_d \le s \le t < \infty$ *such that*

$$H\left(t, s, a^{-1}(r)\right) \le h_d(t, s) , \quad a \in K$$

for all $d \le r$ *and* $0 \le s \le t$ *, with*

$$G_d(t) = \int_t^{t+1} \left\{\int_{t_0}^{\xi} h_d(\xi, s)ds\right\}d\xi \to 0 \quad as \quad t \to \infty .$$

Then the trivial solution of the perturbed system (5.2) *is uniformly asymptotically stable.*

Proof. Since the trivial solution of (5.1) is uniformly asymptotically stable, applying Massera's theorem [4], there exists a Lyapunov function $V(t, x)$ satisfying

$$a(\|x\|) \le V(t, x) , \quad a \in K , \tag{5.3}$$

$$|V(t, x)-V(t, y)| \le L\|x-y\| \tag{5.4}$$

and

$$D^+V_{(5.1)}(t, x) \le -C\left(V(t, x)\right) , \quad C \in K . \tag{5.5}$$

From condition *(ii)* of Theorem 4.1, inequalities (5.3), (5.4), (5.5), assumption *(ii)* of the corollary and the monotonic property in H , we obtain

$$D^+m(t) \le -C\left(m(t)\right) + L\int_{t_0}^{t} H\left[t, s, a^{-1}\left(m(s)\right)\right]ds$$

where $m(t) = V\left(t, x(t)\right)$. Therefore

$$g(t, r, Br) = -C(r) + L\int_{t_0}^{t} H\left[t, s, a^{-1}\left(r(s)\right)\right]ds .$$

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Now, it remains to verify the uniform asymptotic stability of the trivial solution of

$$r' = g(t,\ r,\ Br)\ . \tag{5.6}$$

This can be done by applying condition *(iii)* of the corollary and a similar argument as presented in [7]. The solution of equation (5.6) can be written as

$$r(t,\ t_0,\ r_0) = r_0 - \int_{t_0}^{t} C(r(s))ds + L\int_{t_0}^{t}\int_{t_0}^{\xi} H\left[\xi,\ s,\ a^{-1}(r(s))\right]dsd\xi$$

and for $0 < d \le r(s)$ between t_0 and t, we have

$$r(t,\ t_0,\ r_0) \le r_0 - \int_{t_0}^{t} C(r(s))ds + L\int_{t_0}^{t}\int_{t_0}^{\xi} h_d(\xi,\ s)dsd\xi\ .$$

For $t \ge t_0 \ge 1$ and applying Lemma 3.4 in [7] we obtain

$$r(t) = r(t,\ t_0,\ r_0) \le r_0 - \int_{t_0}^{t} C(r(s))ds + L\int_{t_0-1}^{t} G_d(\xi)d\xi\ . \tag{5.7}$$

Define $Q_d(t) = \sup\left[G_d(\xi)\ :\ t-1 \le \xi < \infty\right]$. Then $Q_d(t) \to 0$ as $t \to \infty$ and

$$r(t) \le r_0 - C(d)\left[t-t_0\right] + LQ_d(t_0)\left[t-t_0+1\right]\ . \tag{5.8}$$

Let $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ so that $0 < 2\delta < \varepsilon$. Also choose $T_1 = T_1(\varepsilon) \ge \tau_d + 1$ so that

$$2LQ_\delta(T_1) < \min\left[C(\delta),\ \varepsilon\right]\ . \tag{5.9}$$

Now, for $r_0 \le \delta$ and $t_0 \ge T_1$, we claim that

$$r(t) < \varepsilon \quad \text{for} \quad t_0 \le t < \infty\ . \tag{5.10}$$

Suppose this is not true. Let T_3 be the first point such that $r(T_3) = \varepsilon$ and let $T_2 < T_3$ be the last point such that $r(T_2) = \delta$. Then $\delta \le r(t) \le \varepsilon$ on $[T_2,\ T_3]$, hence by (5.8),

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$$\varepsilon = r(T_3) \le [LQ_\delta(T_2)-C(\delta)][T_3-T_2] + LQ_d(T_2) + r_o \le LQ_d(T_1) + r_o < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,$$

a contradiction, proving (5.10). This proves the uniform stability of the trivial solution of (5.6). For the rest of the proof, choose $\delta_o = \delta(\varepsilon)$, $T_o = T_1(\varepsilon)$. Fix $t_o \ge T_o$ and $r_o \le \delta_o$. Then (5.10) implies that

$$r(t, t_o, r_o) < \varepsilon \quad \text{on} \quad [t_o, \infty) .$$

For $\eta > 0$, choose $\delta(\eta)$ and $T_1(\eta)$ as before so that (5.9) holds. Choose

$$T = [C(\delta)T_1(\eta)+2LQ_\delta(1)+2\delta][C(\delta)]^{-1} \ge T_1(\eta) ,$$

which does not depend on t_o or r_o. We now claim that

(5.11) $\quad r(t_1, t_o, r_o) < \delta$ for some t_1 in $[t_o+T_1, t_o+T]$.

Suppose that our claim is false, then

$$r(t_1, t_o, r_o) \ge \delta \quad \text{on} \quad [t_o+T_1, t_o+T] .$$

Let $y_o = r(t_o+T_1, t_o, r_o)$. Then

$$0 < \delta \le r(t_o+T, t_o+T_1, y_o) \le [LQ_\delta(t_o+T_1)-C(\delta)][T-T_1] + LQ_\delta(t_o+T_1) + y_o \le -\frac{C(\delta)}{2}[T-T_1] + LQ_\delta(1) + \delta = 0 ,$$

a contradiction, proving (5.11). Thus by (5.10)

$$r\Big(t, t_1, r(t_1, t_o, r_o)\Big) < \eta , \text{ on } [t_1, \infty] ,$$

since $t_1 \ge t_o+T_1 \ge T_1$ and $r(t_1, t_o, r_o) < \delta$. Hence

$$r(t, t_o, r_o) < \eta \quad \text{for} \quad t \ge t_o+T .$$

Since η is arbitrary $r(t, t_o, r_o) \to 0$ as $t \to \infty$. Also as T depends only on η and δ depends only on ε, the trivial solution of (5.6) is uniformly asymptotically stable. Consequently by Theorem 4.1 the stated result follows.

COROLLARY 5.2. *Assume that*

(i) *the trivial solution of the unperturbed system* (5.1) *is exponentially stable;*

(ii) *the function* $K(t, s, x)$ *is defined and continuous on* $0 \le s \le t < \infty$, $\|x\| < H \le \infty$

$$\|K(t, s, x)\| \le K_1(t, s)w(s, \|x\|)$$

where $K_1(t, s) \ge 0$ *is defined and continuous on* $0 \le s \le t < \infty$, $w(t, r) \in C|R_+ \times R_+, R|$, $w(t, 0) \equiv 0$, *satisfies a Lipschitz condition in* r *and non-decreasing in* r *;*

(iii) *there exists a function* $\tilde{h}(t, s)$ *defined and continuous on* $0 \le s \le t < \infty$ *and satisfying the inequality*

$$\tilde{h}(t, s) \le M_1 \exp[-\lambda_1(t-s)] \text{ , for } 0 \le s \le t ,$$

$\lambda_1, M_1 > 0$, *where*

$$\tilde{h}(t, s) = \int_s^t K_1(\tau, s)\exp[-\lambda_1(t-\tau)]d\tau .$$

Then the trivial solution of the perturbed system (5.2) *is exponentially stable.*

Proof. Since the trivial solution of (5.1) is exponentially stable, there exists a Lyapunov function $V(t, x)$ satisfying the following:

(i) $\|x\| \le V(t, x)$;

(ii) $|V(t, x) - V(t, y)| \le L\|x-y\|$;

(iii) $D^+V_{(5.1)}(t, x) \le - \lambda V(t, x)$, $\lambda > 0$.

It is easy to show that

$$g(t, r, Br) = - \lambda r + L \int_0^t K_1(t, s)w(s, r(s))ds .$$

Now, by using similar arguments as in [1], it is easy to show that the trivial solution of (5.6) is exponentially stable and applying Theorem 4.1 we obtain the desired result.

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