Canad. Math. Bull. Vol. **53** (3), 2010 pp. 550–563
doi:10.4153/CMB-2010-060-8

Representing a Product System Representation as a Contractive Semigroup and Applications to Regular Isometric Dilations

Orr Moshe Shalit

Abstract. In this paper we propose a new technical tool for analyzing representations of Hilbert C^*-product systems. Using this tool, we give a new proof that every doubly commuting representation over $\mathbb{N}^k$ has a regular isometric dilation, and we also prove sufficient conditions for the existence of a regular isometric dilation of representations over more general subsemigroups of $\mathbb{R}_+^k$.

1 Introduction, Preliminaries, and Notation

1.1 Background, Correspondences, Product Systems, and Representations

In the following paragraphs we review the definitions of our main objects of study. The reader familiar with C^*-correspondences, product systems of correspondences, and representations of product systems, may skip ahead to Subsection 1.2.

Definition 1.1 Let A be a C^* algebra. A *Hilbert C^*-correspondence over A* is a (right) Hilbert A-module E that carries an adjointable left action of A.

The following notion of representation of a C^*-correspondence was studied extensively in [4] and turned out to be a very useful tool.

Definition 1.2 Let E be a C^*-correspondence over A, and let H be a Hilbert space. A pair (σ, T) is called a *completely contractive covariant representation* of E on H (or, for brevity, a *c.c. representation*) if the following hold:

(i) $T\colon E \to B(H)$ is a completely contractive linear map;
(ii) $\sigma\colon A \to B(H)$ is a nondegenerate $*$-homomorphism, and
(iii) $T(xa) = T(x)\sigma(a)$ and $T(a \cdot x) = \sigma(a)T(x)$ for all $x \in E$ and all $a \in A$.

Given a C^*-correspondence E and a c.c. representation (σ, T) of E on H, one can form the Hilbert space $E \otimes_\sigma H$, which is defined as the Hausdorff completion of the algebraic tensor product with respect to the inner product

$$\langle x \otimes h, y \otimes g\rangle = \langle h, \sigma(\langle x, y\rangle)g\rangle.$$

Received by the editors September 20, 2007.
Published electronically June 11, 2010.
The author was supported by the Jacobs School of Graduate Studies and the Department of Mathematics at the Technion–I.I.T, and by the Gutwirth Fellowship.
AMS subject classification: **47A20**, 46L08.

One then defines $\tilde{T}: E \otimes_\sigma H \to H$ by $\tilde{T}(x \otimes h) = T(x)h$.

Definition 1.3 A c.c. representation (T, σ) is called *isometric* if for all $x, y \in E$,

$$T(x)^* T(y) = \sigma(\langle x, y \rangle).$$

(This is the case if and only if $\tilde{T}$ is an isometry.) It is called *fully coisometric* if $\tilde{T}$ is a coisometry.

Given two Hilbert C^*-correspondences E and F over A, the *balanced* (or *inner*) tensor product $E \otimes_A F$ is a Hilbert C^*-correspondence over A defined to be the Hausdorff completion of the algebraic tensor product with respect to the inner product

$$\langle x \otimes y, w \otimes z \rangle = \langle y, \langle x, w \rangle \cdot z \rangle, \quad x, w \in E, y, z \in F.$$

The left and right actions are defined as $a \cdot (x \otimes y) = (a \cdot x) \otimes y$ and $(x \otimes y)a = x \otimes (ya)$, respectively, for all $a \in A, x \in E, y \in F$. We will usually omit the subscript A, writing just $E \otimes F$.

Suppose $\mathcal{S}$ is an abelian cancellative semigroup with identity 0, and $p: X \to \mathcal{S}$ is a family of C^*-correspondences over A. Write $X(s)$ for the correspondence $p^{-1}(s)$ for $s \in \mathcal{S}$. We say that X is a (discrete) *product system*[1] over $\mathcal{S}$ if X is a semigroup, p is a semigroup homomorphism and, for each $s, t \in \mathcal{S} \setminus \{0\}$, the map $X(s) \times X(t) \ni (x, y) \mapsto xy \in X(s + t)$ extends to an isomorphism $U_{s,t}$ of correspondences from $X(s) \otimes_A X(t)$ onto $X(s + t)$. The associativity of the multiplication means that, for every $s, t, r \in \mathcal{S}$,

$$U_{s+t,r}(U_{s,t} \otimes I_{X(r)}) = U_{s,t+r}(I_{X(s)} \otimes U_{t,r}).$$

We also require that $X(0) = A$ and that the multiplications $X(0) \times X(s) \to X(s)$ and $X(s) \times X(0) \to X(s)$ are given by the left and right actions of A and $X(s)$.

Definition 1.4 Let H be a Hilbert space, A a C^*-algebra and X a product system of Hilbert A-correspondences over the semigroup $\mathcal{S}$. Assume that $T: X \to B(H)$ and write T_s for the restriction of T to $X(s)$, $s \in \mathcal{S}$, and σ for T_0. T (or (σ, T)) is said to be a *completely contractive covariant representation* of X if

(i) For each $s \in \mathcal{S}$, (σ, T_s) is a c.c. representation of $X(s)$, and
(ii) $T(xy) = T(x)T(y)$ for all $x, y \in X$.

T is said to be an isometric (fully coisometric) representation if it is an isometric (fully coisometric) representation on every fiber $X(s)$.

Since we will not be concerned with any other kind of representation, we will call a completely contractive covariant representation of a product system simply a *representation*.

[1]Product systems of Hilbert spaces over $\mathbb{R}_+$ were introduced by Arveson, and the best reference for such product systems is probably [1]. For product systems of Hilbert C^*-correspondences over $\mathbb{R}_+$, see the survey by Skeide [8]. Product systems over other semigroups were first studied by Fowler [3].

1.2 What This Paper is About

In many ways, representations of product systems are analogous to semigroups of contractions on Hilbert spaces. For example, when $A = \mathbb{C}$ and E is the trivial product system $\mathbb{C} \times [0, \infty)$, then $\{T_t(1)\}_{t \geq 0}$ is a contractive semigroup. Many proofs of results concerning representations are based on the ideas of the proofs of the analogous results concerning contractions on a Hilbert space, with the appropriate, sometimes highly nontrivial, modifications made. For example, the proof given in [5] that every representation has an isometric dilation uses some methods from the classical proof that every contraction on a Hilbert space has an isometric dilation.

The point of view we adopt in this paper is that one may try to exploit the *results* rather than the *methods* of the theory of contractive semigroups on a Hilbert space when attacking problems concerning representations of product systems. In other words, we wish to find a systematic way to *reduce* (problems concerning) a representation of a product system to (analagous problems concerning) a *semigroup of contractions on a Hilbert space*. This paper contains a first step in this direction. In Section 2, given a product system X over a semigroup $\mathcal{S}$ and representation (σ, T) of X on a Hilbert space H, we construct a Hilbert space $\mathcal{H}$ and a contractive semigroup $\hat{T} = \{\hat{T}_s\}_{s \in \mathcal{S}}$ on $\mathcal{H}$ such that $\hat{T}$ contains all the information regarding the representation. In Section 3, we show that if $\hat{T}$ has a regular isometric dilation, then so does T.

In Section 4, we prove that doubly commuting representations of product systems of Hilbert correspondences over certain subsemigroups of $\mathbb{R}_+^k$ have doubly commuting, regular isometric dilations. This was proved in [9] for the case $\mathcal{S} = \mathbb{N}^k$. Our proof is based on the construction made in Section 2.

This is a good point at which to remark that our approach has some limitations. For example, the construction introduced in Section 2 does not seem to be canonical in any nice way, and we cannot obtain all of the results in [9]. We will illustrate these limitations in Section 5 after proving another sufficient condition for the existence of a regular isometric dilation. One might wonder, indeed, how far can one get by trying to reduce representations of product systems to semigroups of operators on a Hilbert space, as the former are certainly "much more complicated". In this context, let us just mention that in another paper ([7]), we have shown how we can obtain by these methods another result that has not yet been proved by other means, namely the existence of an isometric dilation to a *fully-coisometric* representation of product systems over (a subsemigroup of) $\mathbb{R}_+^k$.

1.3 Notation

A *commensurable semigroup* is a semigroup Σ such that for every N elements $s_1, \dots, s_N \in \Sigma$, there exist $s_0 \in \Sigma$ and $a_1, \dots, a_N \in \mathbb{N}$ such that $s_i = a_i s_0$ for all $i = 1, \dots N$. For example, $\mathbb{N}$ is a commensurable semigroup. If $r \in \mathbb{R}_+$, then $r \cdot \mathbb{Q}_+$ is commensurable, and any commensurable subsemigroup of $\mathbb{R}_+$ is contained in such a semigroup.

Throughout this paper, Ω will denote some fixed set, and $\mathcal{S}$ will denote the semigroup $\mathcal{S} = \sum_{i \in \Omega} \mathcal{S}_i$, where $\mathcal{S}_i$ is a commensurable and unital (*i.e.,* contains 0) subsemigroup of $\mathbb{R}_+$. To be more precise, $\mathcal{S}$ is the subsemigroup of $\mathbb{R}_+^\Omega$ consisting of finitely supported functions s such that $s(j) \in \mathcal{S}_j$ for all $j \in \Omega$. Still another way to describe $\mathcal{S}$ is the following:

$$\mathcal{S} = \Big\{ \sum_{j\in\Omega} \mathbf{e}_j(s_j) : s_j \in \mathcal{S}_j, \text{ all but finitely many } s_j\text{'s are } 0 \Big\},$$

where $\mathbf{e}_i$ is the inclusion of $\mathcal{S}_i$ into $\prod_{j\in\Omega}\mathcal{S}_j$. Here is a good example to keep in mind: if $|\Omega| = k \in \mathbb{N}$, and if $\mathcal{S}_i = \mathbb{N}$ for all $i \in \Omega$, then $\mathcal{S} = \mathbb{N}^k$. We denote by $\mathcal{S} - \mathcal{S}$ the subgroup of $\mathbb{R}^\Omega$ generated by $\mathcal{S}$ (with addition and subtraction defined in the obvious way). For $s \in \mathcal{S} - \mathcal{S}$ we shall denote by s_+ the element in $\mathcal{S}$ that sends $j \in \Omega$ to $\max\{0, s(j)\}$, and $s_- = s_+ - s$. It is worth noting that if $s \in \mathcal{S} - \mathcal{S}$, then s_+ and s_- are both in $\mathcal{S}$.

$\mathcal{S}$ becomes a partially ordered set if one introduces the relation

$$s \leq t \iff s(j) \leq t(j), \quad j \in \Omega.$$

The symbols $<$, $\ngeq$, *etc.*, are to be interpreted in the obvious way.

If $u = \{u_1, \dots, u_N\} \subseteq \Omega$, we let $|u|$ denote the number of elements in u (this notation will only be used for finite sets). We shall denote by $\mathbf{e}[u]$ the element of $\mathbb{R}^\Omega$ having 1 in the ith place for every $i \in u$, and having 0's elsewhere, and we denote $s[u] := \mathbf{e}[u] \cdot s$, where multiplication is pointwise.

2 Representing Representations as Contractive Semigroups on a Hilbert Space

In this section, we can replace $\mathcal{S}$ by any abelian cancellative semigroup with identity 0 and an appropriate partial ordering (for example, $\mathcal{S}$ can be taken to be $\mathbb{R}_+^k$).

Let A be a C^*-algebra, and let X be a discrete product system of C^*-correspondences over $\mathcal{S}$. Let (σ, T) be a completely contractive covariant representation of X on the Hilbert space H. Our assumptions do not imply that $X(0) \otimes H \cong H$. This unfortunate fact will not cause any real trouble, but it will make our exposition a little clumsy.

Define $\mathcal{H}_0$ to be the space of all finitely supported functions f on $\mathcal{S}$ such that for all $0 \neq s \in \mathcal{S}$, $f(s) \in X(s) \otimes_\sigma H$ and such that $f(0) \in H$. We equip $\mathcal{H}_0$ with the inner product $\langle \delta_s \cdot \xi, \delta_t \cdot \eta \rangle = \delta_{s,t} \langle \xi, \eta \rangle$, for all $s, t \in \mathcal{S} - \{0\}, \xi \in X(s) \otimes H, \eta \in X(t) \otimes H$ (where the δ's on the left-hand side are Dirac deltas, the δ on the right-hand side is Kronecker's delta). If s or t is 0, then the inner product is defined similarly. Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$ with respect to this inner product. Note that

$$\mathcal{H} \cong H \oplus \Big(\bigoplus_{0 \neq s \in \mathcal{S}} X(s) \otimes H \Big).$$

We define a family $\hat{T} = \{\hat{T}_s\}_{s\in\mathcal{S}}$ of operators on $\mathcal{H}_0$ as follows. First, we define $\hat{T}_0$ to be the identity. Now assume that $s > 0$. If $t \in \mathcal{S}$ and $t \ngeq s$, then we define $\hat{T}_s(\delta_t \cdot \xi) = 0$ for all $\xi \in X(t) \otimes_\sigma H$ (or all $\xi \in H$, if $t = 0$). If $\xi \in X(s) \otimes_\sigma H$, we define $\hat{T}_s(\delta_s \cdot \xi) = \delta_0 \cdot \tilde{T}_s \xi$. Finally, if $t > s > 0$, we define

$$\hat{T}_s(\delta_t \cdot (x_{t-s} \otimes x_s \otimes h)) = \delta_{t-s} \cdot (x_{t-s} \otimes \tilde{T}_s(x_s \otimes h)). \tag{2.1}$$

https://doi.org/10.4153/CMB-2010-060-8 Published online by Cambridge University Press

Since $\tilde{T}_s$ is a contraction, $\hat{T}_s$ extends uniquely to a contraction in $B(\mathcal{H})$.

Let us stop to explain what we mean by equation (2.1). There are isomorphisms of correspondences $U_{t-s,s}\colon X(t-s)\otimes X(s)\to X(t)$. Denote their inverses by $U_{t-s,s}^{-1}$. When we write $x_{t-s}\otimes x_s$ for an element of $X(t)$, we actually mean the image of this element by $U_{t-s,s}$, and equation (2.1) should be read as

$$\hat{T}_s(\delta_t\cdot(U_{t-s,s}(x_{t-s}\otimes x_s)\otimes h))=\delta_{t-s}\cdot(x_{t-s}\otimes\tilde{T}_s(x_s\otimes h)),$$

or

$$\hat{T}_s(\delta_t\cdot(\xi\otimes h))=\delta_{t-s}\cdot((I\otimes\tilde{T}_s)(U_{t-s,s}^{-1}\xi\otimes h)).$$

This shows that $\hat{T}$ is well defined.

We now show that $\hat{T}$ is a semigroup. Let $s,t,u\in\mathcal{S}$. If either $s=0$ or $t=0$, then it is clear that the semigroup property $\hat{T}_s\hat{T}_t=\hat{T}_{s+t}$ holds. Assume that $s,t>0$. If $u\not\geq s+t$, then both $\hat{T}_s\hat{T}_t$ and $\hat{T}_{s+t}$ annihilate $\delta_u\cdot\xi$, for all $\xi\in X(u)\otimes H$. Otherwise,[2]

$$\begin{aligned}\hat{T}_s\hat{T}_t(\delta_u(x_{u-s-t}\otimes x_s\otimes x_t\otimes h))&=\hat{T}_s(\delta_{u-t}(x_{u-s-t}\otimes x_s\otimes\tilde{T}_t(x_t\otimes h)))\\&=\delta_{u-s-t}(x_{u-s-t}\otimes\tilde{T}_s(x_s\otimes\tilde{T}_t(x_t\otimes h)))\\&=\delta_{u-s-t}(x_{u-s-t}\otimes\tilde{T}_s(I\otimes\tilde{T}_t)(x_s\otimes x_t\otimes h))\\&=\delta_{u-s-t}(x_{u-s-t}\otimes\tilde{T}_{s+t}(x_s\otimes x_t\otimes h))\\&=\hat{T}_{s+t}(\delta_u(x_{u-s-t}\otimes(x_s\otimes x_t)\otimes h)).\end{aligned}$$

We summarize the construction in the following proposition.

Proposition 2.1 *Let A, X, and $\mathcal{S}$ and (σ,T) be as above, and let*

$$\mathcal{H}=H\oplus\Big(\bigoplus_{0\neq s\in\mathcal{S}}X(s)\otimes_\sigma H\Big).$$

There exists a contractive semigroup $\hat{T}=\{\hat{T}_s\}_{s\in\mathcal{S}}$ on $\mathcal{H}$ such that for all $0\neq s\in\mathcal{S}$, $x\in X(s)$ and $h\in H$, $\hat{T}_s(\delta_s\cdot x\otimes h)=T_s(x)h$. If (σ,S) is another representation of X, and if $\hat{S}$ is the corresponding contractive semigroup, then $\hat{T}=\hat{S}\Rightarrow T=S$.

One immediately sees a limitation in this construction. We cannot say that $\hat{T}$ is unique, or, equivalently, that $\hat{T}=\hat{S}\Leftrightarrow T=S$.

3 Regular Isometric Dilations of Product System Representations

Let H be a Hilbert space, and let $T=\{T_s\}_{s\in\mathcal{S}}$ be a semigroup of contractions over $\mathcal{S}$. A semigroup $V=\{V_s\}_{s\in\mathcal{S}}$ on a Hilbert space $K\supseteq H$ is said to be a *regular dilation of T* if for all $s\in\mathcal{S}-\mathcal{S}$,

$$P_HV_{s_-}^*V_{s_+}\big|_H=T_{s_-}^*T_{s_+}.$$

[2]Strictly speaking, this only takes care of the case $u>s+t$, but the case $u=s+t$ is handled in a similar manner. This annoying issue will come up again and again throughout the paper. Assuming that σ is unital, $X(0)\otimes H\cong H$, and one does not have to separate the reasoning for the $X(s)\otimes H$ blocks and the H blocks.

https://doi.org/10.4153/CMB-2010-060-8 Published online by Cambridge University Press

Here and henceforth, P_H will denote the orthogonal projection from K onto H. V is said to be an *isometric* dilation if it consists of isometries. An isometric dilation V is said to be a *minimal* isometric dilation if

$$K = \bigvee_{s\in\mathcal{S}} V_s H.$$

In [6], we collected various results concerning isometric dilations of semigroups, all of them direct consequences of [10, Sections I.7 and I.9].

The notion of regular isometric dilations can be naturally extended to representations of product systems.

Definition 3.1 Let X be a product system over $\mathcal{S}$, and let (σ, T) be a representation of X on a Hilbert space H. An isometric representation (ρ, V) on a Hilbert space $K \supseteq H$ is said to be a *regular isometric dilation* if for all $a \in A = X(0)$, H reduces $\rho(a)$ and $\rho(a)\mid_H = \sigma(a)$, and for all $s \in \mathcal{S} - \mathcal{S}$,

$$P_{X(s_-)\otimes H}\tilde{V}_{s_-}^* \tilde{V}_{s_+} \mid_{X(s_+)\otimes H} = \tilde{T}_{s_-}^* \tilde{T}_{s_+}.$$

Here, $P_{X(s_-)\otimes H}$ denotes the orthogonal projection of $X(s_-) \otimes_\rho K$ onto $X(s_-) \otimes_\rho H$. (ρ, V) is said to be a *minimal* dilation if

$$K = \bigvee \{V(x)h : x \in X, h \in H\}.$$

In [9], Solel studied regular isometric dilation of product system representations over $\mathbb{N}^k$ and proved some necessary and sufficient conditions for the existence of a regular isometric dilation. One of our aims in this paper is to show how the construction of Proposition 2.1 can be used to generalize some of the results in [9]. The following proposition is the main tool.

Proposition 3.2 *Let A be a C^*-algebra, let $X = \{X(s)\}_{s\in\mathcal{S}}$ be a product system of A-correspondences over $\mathcal{S}$, and let (σ, T) be a representation of X on a Hilbert space H. Let $\hat{T}$ and $\mathcal{H}$ be as in Proposition* 2.1. *Assume that $\hat{T}$ has a regular isometric dilation. Then there exists a Hilbert space $K \supseteq H$ and an isometric representation V of X on K, such that*

(i) *P_H commutes with $V_0(A)$, and $V_0(a)P_H = \sigma(a)P_H$, for all $a \in A$;*
(ii) *$P_{X(s_-)\otimes H}\tilde{V}_{s_-}^* \tilde{V}_{s_+}\big|_{X(s_+)\otimes H} = \tilde{T}_{s_-}^* \tilde{T}_{s_+}$ for all $s \in \mathcal{S} - \mathcal{S}$;*
(iii) *$K = \bigvee \{V(x)h : x \in X, h \in H\}$;*
(iv) *$P_H V_s(x)\big|_{K\ominus H} = 0$ for all $s \in \mathcal{S}$, $x \in X(s)$.*

That is, if $\hat{T}$ has a regular isometric dilation, then so does T. If σ is nondegenerate and X is essential (that is, $AX(s)$ is dense in $X(s)$ for all $s \in \mathcal{S}$), then V_0 is also nondegenerate.

Remark 3.3 The results also hold in the W^* setting, that is, if A is a W^*-algebra, X is a product system of W^*-correspondences and σ is normal, then V_0 is also normal. For a proof, see [7, Theorem 5.2].

https://doi.org/10.4153/CMB-2010-060-8 Published online by Cambridge University Press

Proof Construct $\mathcal{H}$ and $\hat{T}$ as in the previous section.

Let $\hat{V} = \{\hat{V}_s\}_{s\in\mathcal{S}}$ be a minimal, regular, isometric dilation of $\hat{T}$ on some Hilbert space $\mathcal{K}$. Minimality means that

$$\mathcal{K} = \bigvee\{\hat{V}_t(\delta_s \cdot (x \otimes h)) : s, t \in \mathcal{S}, x \in X(s), h \in H\}.$$

Introduce the Hilbert space K,

$$K = \bigvee\{\hat{V}_s(\delta_s \cdot (x \otimes h)) : s \in \mathcal{S}, x \in X(s), h \in H\}.$$

We consider H as embedded in K (or in $\mathcal{H}$ or in $\mathcal{K}$) by the identification $h \leftrightarrow \delta_0 \cdot h$. Next, we define a left action of A on $\mathcal{H}$ by $a \cdot (\delta_s \cdot x \otimes h) = \delta_s \cdot ax \otimes h$, for all $a \in A, s \in \mathcal{S} - \{0\}, x \in X(s)$ and $h \in H$, and

$$a \cdot (\delta_0 \cdot h) = \delta_0 \cdot \sigma(a)h, \quad a \in A, h \in H. \tag{3.1}$$

By [2, Lemma 4.2], this extends to a bounded linear operator on $\mathcal{H}$. Indeed, this follows from the following inequality:

$$\begin{aligned}
\Big\| \sum_{i=1}^{n} ax_i \otimes h_i \Big\|^2 &= \sum_{i,j=1}^{n} \langle h_i, \sigma(\langle ax_i, ax_j \rangle) h_j \rangle \\
&= \Big\langle \big(\sigma(\langle ax_i, ax_j\rangle)\big) (h_1, \dots, h_n)^T, (h_1, \dots, h_n)^T \Big\rangle_{H^{(n)}} \\
(*) \qquad &\leq \|a\|^2 \Big\langle \big(\sigma(\langle x_i, x_j\rangle)\big) (h_1, \dots, h_n)^T, (h_1, \dots, h_n)^T \Big\rangle_{H^{(n)}} \\
&= \|a\|^2 \Big\| \sum_{i=1}^{n} x_i \otimes h_i \Big\|^2.
\end{aligned}$$

The inequality $(*)$ follows from the complete positivity of σ and from $(\langle ax_i, ax_j \rangle) \leq \|a\|^2 (\langle x_i, x_j \rangle)$, which is the content of the cited lemma.

In fact, this is a $*$-representation (and it is faithful if σ is). Indeed, it is clear that this is a homomorphism of algebras. To see that it is a $*$-representation, it is enough to take $s \in \mathcal{S}, x, y \in X(s)$ and $h, k \in H$ and to compute

$$\langle ax \otimes h, y \otimes k \rangle = \langle h, \sigma(\langle ax, y \rangle)k \rangle = \langle h, \sigma(\langle x, a^* y \rangle)k \rangle = \langle x \otimes h, a^* y \otimes k \rangle$$

(recall that the left action of A on X(s) is adjointable). Note that this left action commutes with $\hat{T}$:

$$a\hat{T}_s(\delta_t x_{t-s} \otimes x_s \otimes h) = \delta_{t-s} a x_{t-s} \otimes T_s(x_s)h = \hat{T}_s(\delta_t a x_{t-s} \otimes x_s \otimes h),$$

or

$$a\hat{T}_s(\delta_s x_s \otimes h) = \delta_0 \sigma(a) T_s(x_s)h = \delta_0 T_s(a x_s)h = \hat{T}_s(\delta_s a x_s \otimes h).$$

We shall now define a representation V of X on K. We wish to define V_0 by the following rules:

$$V_0(a)\hat{V}_s(\delta_s \cdot x_s \otimes h) = \hat{V}_s(\delta_s \cdot ax_s \otimes h),$$
$$V_0(a)(\delta_0 \cdot h) = \delta_0 \cdot \sigma(a)h.$$

To see that this extends to a bounded, linear operator on K, let $\sum_t \hat{V}_t(\delta_t \cdot x_t \otimes h_t) \in K$ (a finite sum), and compute

$$\begin{aligned} \|\sum_t \hat{V}_t(\delta_t \cdot ax_t \otimes h_t)\|^2 &= \sum_{s,t} \langle \hat{V}_s(\delta_s \cdot ax_s \otimes h_s), \hat{V}_t(\delta_t \cdot ax_t \otimes h_t)\rangle \\ &= \sum_{s,t} \langle \hat{V}^*_{(s-t)_-} \hat{V}_{(s-t)_+}(\delta_s \cdot ax_s \otimes h_s), \delta_t \cdot ax_t \otimes h_t\rangle \\ (*) \qquad &= \sum_{s,t} \langle \hat{T}^*_{(s-t)_-} \hat{T}_{(s-t)_+}(\delta_s \cdot ax_s \otimes h_s), \delta_t \cdot ax_t \otimes h_t\rangle \\ &= \sum_{s,t} \langle \hat{T}^*_{(s-t)_-} \hat{T}_{(s-t)_+}(\delta_s \cdot a^*ax_s \otimes h_s), \delta_t \cdot x_t \otimes h_t\rangle \\ &= \sum_{s,t} \langle \hat{V}_s(\delta_s \cdot a^*ax_s \otimes h_s), \hat{V}_t(\delta_t \cdot x_t \otimes h_t)\rangle. \end{aligned}$$

(The computation would have worked for finite sums including summands from H also). Step $(*)$ is justified because $\hat{V}$ is a regular dilation of $\hat{T}$. This will be used repeatedly. We conclude that if $a \in A$ is unitary, then

$$\Big\|\sum_t \hat{V}_t(\delta_t \cdot ax_t \otimes h_t)\Big\| = \Big\|\sum_t \hat{V}_t(\delta_t \cdot x_t \otimes h_t)\Big\|.$$

For general $a \in A$, we may write $a = \sum_{i=1}^4 \lambda_i u_i$, where u_i is unitary and $|\lambda_i| \leq 2\|a\|$. Thus,

$$\Big\|\sum_t \hat{V}_t(\delta_t \cdot ax_t \otimes h_t)\Big\| = \Big\|\sum_{i=1}^4 \lambda_i \sum_t \hat{V}_t(\delta_t u_i \cdot x_t \otimes h_t)\Big\| \leq 8\|a\| \Big\|\sum_t \hat{V}_t(\delta_t \cdot x_t \otimes h_t)\Big\|.$$

In fact, we will soon see that V_0 is a representation, so this quite a lousy estimate. But we make it only to show that $V_0(a)$ can be extended to a well-defined operator on K.

It is immediate that V_0 is linear and multiplicative. To see that it is $*$-preserving, let $s, t \in \mathcal{S}$, $x \in X(s)$, $x' \in X(t)$ and $h, h' \in H$.

$$\begin{aligned} \langle V_0(a)^*\hat{V}_s(\delta_s \cdot x \otimes h), \hat{V}_t(\delta_t \cdot x' \otimes h')\rangle &= \langle \hat{V}_s(\delta_s \cdot x \otimes h), V_0(a)\hat{V}_t(\delta_t \cdot x' \otimes h')\rangle \\ &= \langle \hat{V}_s(\delta_s \cdot x \otimes h), \hat{V}_t(\delta_t \cdot ax' \otimes h')\rangle \\ &= \langle \hat{V}^*_{(s-t)_-} \hat{V}_{(s-t)_+}(\delta_s \cdot x \otimes h), \delta_t \cdot ax' \otimes h'\rangle \\ &= \langle \hat{T}^*_{(s-t)_-} \hat{T}_{(s-t)_+}(\delta_s \cdot x \otimes h), \delta_t \cdot ax' \otimes h'\rangle \\ &= \langle \hat{T}^*_{(s-t)_-} \hat{T}_{(s-t)_+}(\delta_s \cdot a^*x \otimes h), \delta_t \cdot x' \otimes h'\rangle \\ &= \langle \hat{V}_s(\delta_s \cdot a^*x \otimes h), \hat{V}_t(\delta_t \cdot x' \otimes h')\rangle \\ &= \langle V_0(a^*)\hat{V}_s(\delta_s \cdot x \otimes h), \hat{V}_t(\delta_t \cdot x' \otimes h')\rangle. \end{aligned}$$

Thus, $V_0(a)^* = V_0(a^*)$.

By (3.1), H reduces $V_0(A)$, and $V_0(a)\big|_H = \sigma(a)\big|_H$ (under the appropriate identifications). The assertion about the nondegeneracy of V_0 is clear from the definitions.

To define V_s for $s > 0$, we will show that the rule

$$V_s(x_s)\hat{V}_t(\delta_t \cdot x_t \otimes h) = \hat{V}_{s+t}(\delta_{s+t} \cdot x_s \otimes x_t \otimes h) \tag{3.2}$$

can be extended to a well-defined operator on K. Let $\sum \hat{V}_{t_i}(\delta_{t_i} \cdot x_i \otimes h_i)$ be a finite sum in K, and let $s \in \mathcal{S}$, $x_s \in X(s)$. To estimate

$$\begin{aligned}
\|\textstyle\sum \hat{V}_{t_i+s}(\delta_{t_i+s}\cdot x_s \otimes x_i \otimes h_i)\|^2 \\
&= \textstyle\sum \langle \hat{V}_{t_i+s}(\delta_{t_i+s} \cdot x_s \otimes x_i \otimes h_i), \hat{V}_{t_j+s}(\delta_{t_j+s} \cdot x_s \otimes x_j \otimes h_j)\rangle \\
&= \textstyle\sum \langle \hat{V}_s\hat{V}_{t_i}(\delta_{t_i+s} \cdot x_s \otimes x_i \otimes h_i), \hat{V}_s\hat{V}_{t_j}(\delta_{t_j+s} \cdot x_s \otimes x_j \otimes h_j)\rangle \\
&= \textstyle\sum \langle \hat{V}_{t_i}(\delta_{t_i+s} \cdot x_s \otimes x_i \otimes h_i), \hat{V}_{t_j}(\delta_{t_j+s} \cdot x_s \otimes x_j \otimes h_j)\rangle,
\end{aligned}$$

we look at each summand of the last equation. Denoting $\xi_i = x_i \otimes h_i$, we have

$$\begin{aligned}
\big\langle \hat{V}_{t_i}(\delta_{t_i+s} \cdot x_s \otimes \xi_i), \hat{V}_{t_j}(\delta_{t_j+s} \cdot x_s \otimes \xi_j)\big\rangle \\
&= \big\langle \hat{V}^*_{(t_i-t_j)_-}\hat{V}_{(t_i-t_j)_+}(\delta_{t_i+s} \cdot x_s \otimes \xi_i), \delta_{t_j+s} \cdot x_s \otimes \xi_j\big\rangle \\
&= \big\langle \hat{T}^*_{(t_i-t_j)_-}\hat{T}_{(t_i-t_j)_+}(\delta_{t_i+s} \cdot x_s \otimes \xi_i), \delta_{t_j+s} \cdot x_s \otimes \xi_j\big\rangle \\
&= \big\langle \delta_{t_j+s} \cdot x_s \otimes \big(I \otimes \tilde{T}^*_{(t_i-t_j)_-}\big)\big(I \otimes \tilde{T}_{(t_i-t_j)_+}\big)\xi_i, \delta_{t_j+s} \cdot x_s \otimes \xi_j\big\rangle \\
&= \big\langle \delta_{t_j} \cdot \big(I \otimes \tilde{T}^*_{(t_i-t_j)_-}\big)\big(I \otimes \tilde{T}_{(t_i-t_j)_+}\big)\xi_i, \delta_{t_j} \cdot |x_s|^2\xi_j\big\rangle \\
&= \big\langle \hat{T}^*_{(t_i-t_j)_-}\hat{T}_{(t_i-t_j)_+}(\delta_{t_i} \cdot \xi_i), \delta_{t_j} \cdot |x_s|^2\xi_j\big\rangle \\
&= \big\langle \hat{V}_{t_i}(\delta_{t_i} \cdot |x_s|\xi_i), \hat{V}_{t_j}(\delta_{t_j} \cdot |x_s|\xi_j)\big\rangle \\
&= \big\langle V_0(|x_s|)\hat{V}_{t_i}(\delta_{t_i} \cdot \xi_i), V_0(|x_s|)\hat{V}_{t_j}(\delta_{t_j} \cdot \xi_j)\big\rangle,
\end{aligned}$$

(again, this argument works also if some ξ's are in H). This means that

$$\begin{aligned}
\|\textstyle\sum \hat{V}_{t_i+s}(\delta_{t_i+s} \cdot x_s \otimes x_i \otimes h_i)\|^2 &= \|V_0(|x_s|)\textstyle\sum \hat{V}_{t_i}(\delta_{t_i} \cdot x_i \otimes h_i)\|^2 \\
&\leq \|V_0(|x_s|)\|^2\big\|\textstyle\sum \hat{V}_{t_i}(\delta_{t_i} \cdot x_i \otimes h_i)\big\|^2,
\end{aligned}$$

so the mapping V_s defined in (3.2) does extend to a well-defined operator on K. Now it is clear from the definitions that for all $s \in \mathcal{S}$, (V_0, V_s) is a covariant representation of $X(s)$ on K. We now show that it is isometric. Let $s, t, u \in \mathcal{S}$, $x, y \in X(s)$, $x_t \in X(t)$,

https://doi.org/10.4153/CMB-2010-060-8 Published online by Cambridge University Press

$x_u \in X(u)$ and $h, g \in H$. Then

$$\begin{aligned}
&\langle V_s(x)^* V_s(y) \hat{V}_t \delta_t \cdot x_t \otimes h, \hat{V}_u \delta_u \cdot x_u \otimes g \rangle \\
&\qquad = \langle \hat{V}_{t+s} \delta_{t+s} \cdot y \otimes x_t \otimes h, \hat{V}_{u+s} \delta_{u+s} \cdot x \otimes x_u \otimes g \rangle \\
&\qquad = \langle \hat{V}^*_{(t-u)_-} \hat{V}_{(t-u)_+} \delta_{t+s} \cdot y \otimes x_t \otimes h, \delta_{u+s} \cdot x \otimes x_u \otimes g \rangle \\
(*) &\qquad = \langle \hat{V}^*_{(t-u)_-} \hat{V}_{(t-u)_+} \delta_t \cdot x_t \otimes h, \delta_u \cdot \langle y, x \rangle x_u \otimes g \rangle \\
&\qquad = \langle \hat{V}_t \delta_t \cdot x_t \otimes h, \hat{V}_u \delta_u \cdot \langle y, x \rangle x_u \otimes g \rangle \\
&\qquad = \langle V_0(\langle x, y \rangle) \hat{V}_t \delta_t \cdot x_t \otimes h, \hat{V}_u \delta_u \cdot x_u \otimes g \rangle .
\end{aligned}$$

The justification of $(*)$ was essentially carried out in the proof that $V_s(x_s)$ is well defined. Let us, for a change, show that this computation works also for the case $u = 0$:

$$\begin{aligned}
\langle V_s(x)^* V_s(y) \hat{V}_t \delta_t \cdot x_t \otimes h, \delta_0 \cdot g \rangle &= \langle \hat{V}_{t+s} \delta_{t+s} \cdot y \otimes x_t \otimes h, \hat{V}_s \delta_s \cdot x \otimes g \rangle \\
&= \langle \hat{V}_t \delta_{t+s} \cdot y \otimes x_t \otimes h, \delta_s \cdot x \otimes g \rangle \\
&= \langle \hat{T}_t \delta_{t+s} \cdot y \otimes x_t \otimes h, \delta_s \cdot x \otimes g \rangle \\
&= \langle \delta_s \cdot y \otimes T_t(x_t) \otimes h, \delta_s \cdot x \otimes g \rangle \\
&= \langle T_t(x_t) \otimes h, \sigma(\langle y, x \rangle) g \rangle \\
&= \langle \hat{T}_t \delta_t \cdot x_t \otimes h, V_0(\langle y, x \rangle) \delta_0 \cdot g \rangle \\
&= \langle \hat{V}_t \delta_t \cdot x_t \otimes h, V_0(\langle y, x \rangle) \delta_0 \cdot g \rangle \\
&= \langle V_0(\langle x, y \rangle) \hat{V}_t \delta_t \cdot x_t \otimes h, \delta_0 \cdot g \rangle .
\end{aligned}$$

We have constructed a family $V = \{V_s\}_{s \in \mathcal{S}}$ of maps such that (V_0, V_s) is an isometric covariant representation of $X(s)$ on K. To show that V is a product system representation of X, we need to show that the "semigroup property" holds.

Let $h \in H$, $s, t, u \in \mathcal{S}$, and let x_s, x_t, x_u be in $X(s), X(t), X(u)$, respectively. Then

$$\begin{aligned}
V_{s+t}(x_s \otimes x_t) \hat{V}_u (\delta_u \cdot x_u \otimes h) &= \hat{V}_{s+t+u} (\delta_{s+t+u} \cdot x_s \otimes x_t \otimes x_u \otimes h) \\
&= V_s(x_s) \hat{V}_{t+u} (\delta_{t+u} \cdot x_t \otimes x_u \otimes h) \\
&= V_s(x_s) V_t(x_t) \hat{V}_u (\delta_u \cdot x_u \otimes h),
\end{aligned}$$

so the semigroup property holds.

We have yet to show that V is a minimal regular dilation of T. To see that it is a regular dilation, let $s \in \mathcal{S} - \mathcal{S}$, $x_+ \in X(s_+)$, $x_- \in X(s_-)$ and $h = \delta_0 \cdot h, g = \delta_0 \cdot g \in H$.

Using the fact that $\hat{V}$ is a regular dilation of $\hat{T}$, we compute

$$\begin{aligned}\langle \tilde{V}_{s_-}^* \tilde{V}_{s_+}(x_+ \otimes \delta_0 \cdot h), (x_- \otimes \delta_0 \cdot g)\rangle &= \langle \hat{V}_{s_+}(\delta_{s_+} x_+ \otimes h), \hat{V}_{s_-}(\delta_{s_-} x_- \otimes g)\rangle \\ &= \langle \hat{V}_{s_-}^* \hat{V}_{s_+}(\delta_{s_+} x_+ \otimes h), \delta_{s_-} x_- \otimes g\rangle \\ &= \langle \hat{T}_{s_-}^* \hat{T}_{s_+}(\delta_{s_+} x_+ \otimes h), \delta_{s_-} x_- \otimes g\rangle \\ &= \langle \tilde{T}_{s_+}(x_+ \otimes h), \tilde{T}_{s_-}(x_- \otimes g)\rangle \\ &= \langle \tilde{T}_{s_-}^* \tilde{T}_{s+}(x_+ \otimes h), x_- \otimes g\rangle.\end{aligned}$$

V is a minimal dilation of T, because

$$\begin{aligned}K &= \bigvee \{\hat{V}_s(\delta_s \cdot (x \otimes h)) : s \in \mathcal{S}, x \in X(s), h \in H\} \\ &= \bigvee \{V_s(x)(\delta_0 \cdot h) : s \in \mathcal{S}, x \in X(s), h \in H\}.\end{aligned}$$

Finally, let us note that item (iv) from the statement of the proposition is true for any minimal isometric dilation (of any c.c. representation of a product system over any semigroup). Indeed, let V be a minimal isometric dilation of T on K. Let $x_s \in X(s), x_t \in X(t)$ and $h \in H$. Then

$$\begin{aligned}P_H V_s(x_s) V_t(x_t) h &= P_H V_{s+t}(x_s \otimes x_t) h \\ &= T_{s+t}(x_s \otimes x_t) h = T_s(x_s) T_t(x_t) h \\ &= P_H V_s(x_s) P_H V_t(x_t) h.\end{aligned}$$

But $K = \bigvee\{V_s(x)h : s \in \mathcal{S}, x \in X(s), h \in H\}$, so $P_H V_s(x_s) P_H = P_H V_s(x_s)$, from which item (iv) follows. ■

It is worth noting that, as commensurable semigroups are countable, if $\mathcal{S} = \sum_{i=1}^{\infty} \mathcal{S}_i$, then, using the notation of the above proposition, separability of H implies that K is separable. It is also worth recording the following result, the proof of which essentially appears in the proof of [9, Proposition 3.7].

Proposition 3.4 *Let X be a product system over $\mathcal{S}$, and let T be a representation of X. A minimal, regular, isometric dilation of T is unique up to unitary equivalence.*

4 Regular Isometric Dilations of Doubly Commuting Representations

It is well known that in order for a k-tuple $(T_1, T_2, \ldots, T_k)$ of contractions to have a commuting isometric dilation, it is not enough to assume that the contractions commute. One of the simplest sufficient conditions that one can impose on $(T_1, T_2, \ldots, T_k)$ is that it *doubly commute*, that is

$$T_j T_k = T_k T_j \quad \text{and} \quad T_j^* T_k = T_k T_j^*$$

https://doi.org/10.4153/CMB-2010-060-8 Published online by Cambridge University Press

for all $j \neq k$. Under this assumption, the k-tuple $(T_1, T_2, \dots, T_k)$ actually has *regular* unitary dilation. In fact, if the k-tuple $(T_1, T_2, \dots, T_k)$ doubly commutes, then it also has a *doubly commuting*, regular, *isometric* dilation (see [6, Proposition 3.5] for the simple explanation). This fruitful notion of double commutation can be generalized to representations as follows.

Definition 4.1 A representation (σ, T) of a product system X over $\mathcal{S}$ is said to *doubly commute* if

$$(I_{\mathbf{e}_k(s_k)} \otimes \tilde{T}_{\mathbf{e}_j(s_j)})(t \otimes I_H)(I_{\mathbf{e}_j(s_j)} \otimes \tilde{T}^*_{\mathbf{e}_k(s_k)}) = \tilde{T}^*_{\mathbf{e}_k(s_k)} \tilde{T}_{\mathbf{e}_j(s_j)}$$

for all $j \neq k$ and all nonzero $s_j \in \mathcal{S}_j, s_k \in \mathcal{S}_k$, where t stands for the isomorphism between $X(\mathbf{e}_j(s_j)) \otimes X(\mathbf{e}_k(s_k))$ and $X(\mathbf{e}_k(s_k)) \otimes X(\mathbf{e}_j(s_j))$, and I_s is shorthand for $I_{X(s)}$.

The following theorem appeared already as [9, Theorem 3.10] (for the case $\mathcal{S} = \mathbb{N}^k$). We give a new proof here.

Theorem 4.2 *Let A be a C^*-algebra, let $X = \{X(s)\}_{s \in \mathcal{S}}$ be a product system of A-correspondences over $\mathcal{S}$, and let (σ, T) be a doubly commuting representation of X on a Hilbert space H. There exists a Hilbert space $K \supseteq H$ and a minimal, doubly commuting, regular isometric representation V of X on K.*

Proof Construct $\mathcal{H}$ and $\hat{T}$ as in Section 2.

We now show that $\hat{T}_{\mathbf{e}_j(s_j)}$ and $\hat{T}_{\mathbf{e}_k(s_k)}$ doubly commute for all $j \neq k$, and all $s_j \in \mathcal{S}_j, s_k \in \mathcal{S}_k$. Let $t \in \mathcal{S}$, $x \in X(t)$, $y \in X(\mathbf{e}_j(s_j))$ and $h \in H$. Using the assumption that T is a doubly commuting representation,

$$\begin{aligned}
&\hat{T}^*_{\mathbf{e}_k(s_k)} \hat{T}_{\mathbf{e}_j(s_j)} (\delta_{t+\mathbf{e}_j(s_j)} \cdot x \otimes y \otimes h) \\
&\quad = \hat{T}^*_{\mathbf{e}_k(s_k)} \big(\delta_t \cdot x \otimes \tilde{T}_{\mathbf{e}_j(s_j)}(y \otimes h) \big) \\
&\quad = \delta_{t+\mathbf{e}_k(s_k)} \cdot x \otimes \tilde{T}^*_{\mathbf{e}_k(s_k)} \tilde{T}_{\mathbf{e}_j(s_j)} (y \otimes h) \\
&\quad = \delta_{t+\mathbf{e}_k(s_k)} \cdot x \otimes \big((I_{\mathbf{e}_k(s_k)} \otimes \tilde{T}_{\mathbf{e}_j(s_j)})(t \otimes I_H)(I_{\mathbf{e}_j(s_j)} \otimes \tilde{T}^*_{\mathbf{e}_k(s_k)})(y \otimes h) \big) \\
&\quad = \hat{T}_{\mathbf{e}_j(s_j)} \hat{T}^*_{\mathbf{e}_k(s_j)} (\delta_{t+\mathbf{e}_j(s_j)} \cdot x \otimes y \otimes h),
\end{aligned}$$

where we have written t for the isomorphism between $X(\mathbf{e}_j(s_j)) \otimes X(\mathbf{e}_k(s_k))$ and $X(\mathbf{e}_k(s_k)) \otimes X(\mathbf{e}_j(s_j))$, and we have not written the isomorphisms between $X(s) \otimes X(t)$ and $X(s+t)$.

By [6, Corollary 3.7], there exists a minimal, regular isometric dilation $\hat{V} = \{\hat{V}_s\}_{s \in \mathcal{S}}$ of $\hat{T}$ on some Hilbert space $\mathcal{K}$, such that $\hat{V}_{\mathbf{e}_j(s_j)}$ and $\hat{V}_{\mathbf{e}_k(s_k)}$ doubly commute for all $j \neq k, s_j \in \mathcal{S}_j, s_k \in \mathcal{S}_k$.

Proposition 3.2 gives a minimal, regular isometric dilation V of T on some Hilbert space K.

To see that V is doubly commuting, one computes what one should using the fact that $\hat{V}$ is a minimal, doubly commuting, regular isometric dilation of $\hat{T}$ (all the five adjectives attached to $\hat{V}$ play a part). This takes about four pages of handwritten computations, so is omitted. Let us indicate how it is done. For any $i \in \Omega$, $s_i \in \mathcal{S}_i$, write $\tilde{V}_i$ for $\tilde{V}_{X(\mathbf{e}_i(s_i))}$, I_i for $I_{X(\mathbf{e}_i(s_i))}$, and so on. Taking $j \neq k$, $s_j \in \mathcal{S}_j$, $s_k \in \mathcal{S}_k$, operate with

$$\tilde{V}_k(I_k \otimes \tilde{V}_j)(t_{j,k} \otimes I_J)(I_j \otimes \tilde{V}_k^*)$$

and with

$$\tilde{V}_k \tilde{V}_k^* \tilde{V}_j$$

on a typical element of $X(\mathbf{e}_j(s_j)) \otimes K$ of the form:

$$x \otimes \hat{V}_s(\delta_s \cdot x_s \otimes h), \tag{4.1}$$

to see that what you get is the same. One has to separate the cases where $\mathbf{e}_k(s_k) \leq s$ and $\mathbf{e}_k(s_k) \not\leq s$ (this is the case where the fact that $\hat{V}$ is a doubly commuting semigroup comes in). Because $\tilde{V}_k$ is an isometry and the elements (4.1) span $X(\mathbf{e}_j(s_j)) \otimes K$, one has

$$\tilde{V}_k^* \tilde{V}_j = (I_k \otimes \tilde{V}_j)(t_{j,k} \otimes I_J)(I_j \otimes \tilde{V}_k^*).$$

That will conclude the proof. ■

5 A Sufficient Condition for the Existence of a Regular Isometric Dilation

Using the above methods, one can, quite easily, arrive at the following result, which is, for the case $\mathcal{S} = \mathbb{N}^k$, one half of Theorem 3.5 of [9].

Theorem 5.1 *Let X be a product system over* $\mathcal{S}$, *and let T be a representation of X. If*

$$\sum_{u \subseteq v} (-1)^{|u|} \Big(I_{s[v]-s[u]} \otimes \tilde{T}_{s[u]}^* \tilde{T}_{s[u]} \Big) \geq 0 \tag{5.1}$$

for all finite subsets $v \subseteq \Omega$ *and all* $s \in \mathcal{S}$, *then T has a regular isometric dilation.*

Proof Here are the main lines of the proof. Construct $\hat{T}$ as in section 2. From (5.1), it follows that $\hat{T}$ satisfies

$$\sum_{u \subseteq v} (-1)^{|u|} \hat{T}_{s[u]}^* \hat{T}_{s[u]} \geq 0,$$

for all finite subsets $v \subseteq \Omega$ and all $s \in \mathcal{S}$, which, by Proposition 3.5 and Theorem 3.6 in [6], is a necessary and sufficient condition for the existence of a regular isometric dilation $\hat{V}$ of $\hat{T}$. The result now follows from Proposition 3.2. ■

Among other reasons, this example has been put forward to illustrate the limitations of our method. By [9, Theorem 3.5], when $\mathcal{S} = \mathbb{N}^k$, equation (5.1) is a *necessary*, as well as a sufficient, condition that T has a regular isometric dilation. But our construction "works only in one direction", so we are able to prove only sufficient conditions (roughly speaking). We believe that, using the methods of [9] combined with commensurability considerations, one would be able to show that (5.1) is indeed a necessary condition for the existence of a regular isometric dilation (over $\mathcal{S}$). Whether or not the constructions of Section 2 can be modified to give the other direction remains to be answered.

Acknowledgments This research is part of the author's Ph.D. thesis, written under the supervision of Baruch Solel, whose guidance and encouragement were invaluable.

https://doi.org/10.4153/CMB-2010-060-8 Published online by Cambridge University Press

References

[1] W. Arveson, *Noncommutative dynamics and E-semigroups.* Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[2] E. C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists.* London Mathematical Society Lecture Note Series, 210, Cambridge University Press, Cambridge, 1995.

[3] N. J. Fowler, *Discrete product systems of Hilbert bimodules.* Pacific J. Math. **204**, no. 2 (2002), 335–375. doi:10.2140/pjm.2002.204.335

[4] P. Muhly and B. Solel, *Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes.* J. Funct. Anal. **158**(1998), no. 2, 389–457. doi:10.1006/jfan.1998.3294

[5] ———, *Quantum Markov processes (Correspondences and Dilations).* Internat. J. Math. **13**(2002), no. 8, 863–906. doi:10.1142/S0129167X02001514

[6] O. M. Shalit, *Dilation theorems for contractive semigroups.*, 2007, http://arxiv.org/abs/1004.0723v1

[7] ———, *E_0-dilation of strongly commuting CP_0-semigroups.*, J. Funct. Anal. **255**(2008), no. 1, 46–89. doi:10.1016/j.jfa.2008.04.003

[8] M. Skeide, *Product Systems; a Survey with commutants in view.* In: Quantum stochastics and information, World Sci. Publ., Hackensack, NJ, 2008.

[9] B. Solel, *Regular dilations of representations of product systems.* Math. Proc. R. Ir. Acad. **108**(2008), no. 1, 89–110. doi:10.3318/PRIA.2008.108.1.89

[10] B. Sekefal'vi-Nad' and C. Fojaş, *Harmonic analysis of operators in Hilbert space.* Izdat "Mir", Moscow, 1970.

Department of Mathematics, Technion, Haifa, Israel
e-mail: orrms@tx.technion.ac.il

https://doi.org/10.4153/CMB-2010-060-8 Published online by Cambridge University Press