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FRACTIONAL KINETIC EQUATIONS DRIVEN BY GAUSSIAN OR INFINITELY DIVISIBLE NOISE

J. M. ANGULO,* ** *University of Granada*
V. V. ANH *** **** AND
R. MCVINISH,*** ***** *Queensland University of Technology*
M. D. RUIZ-MEDINA,* ****** *University of Granada*

Abstract

In this paper, we consider a certain type of space- and time-fractional kinetic equation with Gaussian or infinitely divisible noise input. The solutions to the equation are provided in the cases of both bounded and unbounded domains, in conjunction with bounds for the variances of the increments. The role of each of the parameters in the equation is investigated with respect to second- and higher-order properties. In particular, it is shown that long-range dependence may arise in the temporal solution under certain conditions on the spatial operators.

Keywords: Fractional diffusion; fractional heat equation; fractional kinetic equation; long-range dependence

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1. Introduction

The Riesz–Bessel fractional diffusion equation subject to Gaussian white noise on rectangular and unbounded domains was introduced in [2]. The equation has the form

$$\frac{\partial}{\partial t}c(t,x)+(I-\Delta)^{\alpha/2}(-\Delta)^{\gamma/2}c(t,x)=\varepsilon(t,x),\qquad c(0,x)=c_0(x),\tag{1}$$

where $t\in\mathbb{R}_+$, $x\in D\subset\mathbb{R}^d$, $\alpha\geq 0$, $\gamma>0$, I is the identity operator, Δ is the Laplacian operator, $\varepsilon(t,x)$ is Gaussian space–time white noise, and $c_0(x)$ is a spatial random field independent of $\varepsilon(t,x)$. Here, $(I-\Delta)^{-\alpha/2}$ is the Bessel potential and $(-\Delta)^{-\gamma/2}$ is the Riesz potential. Apart from in the classical context of heat conduction, an equation of the form (1) with $\gamma=2$ and $\alpha=0$ also arises in neurophysiology [50], [51], for example. Diffusion operators of the form $(-\Delta)^{1+\gamma}$, $\gamma>0$, have been used to define hyperviscosity and to study its effect on the inertial-range scaling of fully developed turbulence [28], [33]. The presence of the Bessel operator is essential for a study of stationary solutions of (1); see [9] for related models.

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* Postal address: Department of Statistics and Operations Research, University of Granada, Campus Fuente Nueva s/n, E-18071 Granada, Spain.
** Email address: jmangulo@ugr.es
*** Postal address: School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, QLD 4001, Australia.
**** Email address: v.anh@qut.edu.au
***** Email address: r.mcvinish@qut.edu.au
****** Email address: mruiz@ugr.es

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Angulo *et al.* [2] studied the existence, sample path regularity, and second-order properties, particularly spatial long memory in the case of unbounded spatial domain, of the solution of (1) as $t \to \infty$. A similar problem was investigated by Bonaccorsi and Tubaro [13] in a Hilbert space setting. However, the issue of temporal memory of the solution of (1) has not been investigated in sufficient detail.

This issue is addressed in the present paper. A fractional-in-time differential operator will be introduced into (1) explicitly, and the effects of the fractional operators in depicting spatial and temporal long memory in the space–time context will be studied (a background on the issue of long memory is provided in the introduction of [2]). In fact, we will consider an extended form of (1):

$$\left(A_n \frac{\partial^{\beta_n}}{\partial t^{\beta_n}} + \cdots + A_1 \frac{\partial^{\beta_1}}{\partial t^{\beta_1}} + A_0 \frac{\partial^{\beta_0}}{\partial t^{\beta_0}}\right) c(t,x) + (I-\Delta)^{\alpha/2}(-\Delta)^{\gamma/2} c(t,x) = \varepsilon(t,x), \tag{2}$$
$$\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 \geq 0, \qquad A_i > 0, \quad i = 0, \ldots, n.$$

Here, the fractional-in-time derivative is the regularized fractional derivative or fractional derivative in the Caputo–Djrbashian sense, i.e.

$$\frac{\partial^\beta u}{\partial t^\beta} = \begin{cases} \dfrac{\partial^m u}{\partial t^m}(t,x) & \text{if } \beta = m \in \mathbb{N}, \\ \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t (t-\tau)^{m-\beta-1} \frac{\partial^m u(\tau,x)}{\partial \tau^m}\, \mathrm{d}\tau & \text{if } m-1 < \beta < m \end{cases}$$

(see [15], [19], and [42]), where $\Gamma(\cdot)$ is the gamma function. The focus in this paper is on the case of $\beta_n \leq 1$, although (2) without the random forcing term covers many important diffusion-type equations as special cases. For example, the equation for generalized Cattaneo diffusion is given by

$$\left(\tau^\gamma \frac{\partial^2}{\partial t^2} + \frac{\partial^{1-\gamma}}{\partial t^{1-\gamma}}\right) c(t,x) = K \frac{\partial^2 c}{\partial x^2}, \qquad 0 < \gamma < 1,$$

τ and K being constants [17]; the time-fractional telegraph equation takes the form

$$\left(\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} + 2\lambda \frac{\partial^\alpha}{\partial t^\alpha}\right) c(t,x) = K \frac{\partial^2 c}{\partial x^2}, \qquad 0 < \alpha \leq 1,$$

λ being a constant [41]; while the fractional Riesz–Bessel diffusion is governed by

$$\frac{\partial^\beta}{\partial t^\beta} c(t,x) = -\mu (I-\Delta)^{\alpha/2}(-\Delta)^{\gamma/2} c(t,x), \qquad \mu > 0. \tag{3}$$

Equation (3) is a fractional diffusion equation when $0 < \beta \leq 1$ and is a fractional wave equation when $1 < \beta \leq 2$; hence, it is also referred to as a fractional diffusion–wave equation. Special cases of (3) have been treated by many authors including Anh and Leonenko [3]–[5], Barkai *et al.* [11], Benson *et al.* [12], Chaves [16], Gorenflo *et al.* [22]–[25], Klafter *et al.* [30], Kochubei [31], Mainardi [35], Meerschaert *et al.* [37]–[39], Metzler and Klafter [40], Saichev and Zaslawsky [45], Schneider [46], Schneider and Wyss [47], Shlesinger *et al.* [48], and Uchaikin and Zolotarev [49].

Equation (2) is capable of encoding various short- and long-range correlation structures in space and time. We will pay attention to the role of the parameters of (2) in these correlation structures of the solution. We consider in detail the behaviour of the solution observed at a single spatial location. The reason for this is that, in many applied areas, for example in turbulence and air pollution, time series data are collected in this fashion (from hot-wire anemometers and air quality monitoring stations). Therefore, in this paper we will study the exact behaviour of the temporal evolution at such spatial locations. A significant finding is that we observe *temporal long-range dependence* even in the case of the infinite-dimensional Ornstein–Uhlenbeck process (1).

In the next section, we provide a connection with the theory of continuous-time random walks (CTRWs) and, hence, a motivation for considering equations of the form (2). In fact, we will show the existence of stochastic processes which are the limits, in the weak sense, of sequences of CTRWs whose probability density functions $p(t, x)$ are governed by general equations of the form

$$A_n \frac{\partial^{\beta_n} p(t, x)}{\partial t^{\beta_n}} + \cdots + A_0 \frac{\partial^{\beta_0} p(t, x)}{\partial t^{\beta_0}} = \mathcal{A} p(t, x), \tag{4}$$

where $\mathcal{A}$ is the infinitesimal generator of a Lévy process. The Riesz–Bessel operator $(I - \Delta)^{\alpha/2}(-\Delta)^{\gamma/2}$ is a special case of $\mathcal{A}$.

For notational simplicity, we concentrate on the fractional kinetic equation

$$\frac{\partial^{\beta}}{\partial t^{\beta}} c(t, x) + (I - \Delta)^{\alpha/2}(-\Delta)^{\gamma/2} c(t, x) = \varepsilon(t, x), \qquad 0 < \beta \leq 1. \tag{5}$$

This will be sufficient for our purposes. Extension to the more general equation (2) is discussed in Subsection 3.3.

We first consider (5) for the case of bounded spatial domains in Subsection 3.1. We formulate a solution to the Dirichlet problem via the eigenfunction expansion of the associated Green function. Sharp bounds are obtained for the variance of the increments in space and time. In the case of unbounded spatial domains (Subsection 3.2), we obtain a solution in terms of the Fourier transform of the associated Green function. At each time $t \in \mathbb{R}_+$, the solution is a homogeneous random field. We calculate its spatial spectral density and then obtain a similar bound for the variance of the increments. In both cases studied, the solutions are asymptotically stationary in time. We derive the corresponding spectral densities and bounds on the variances of the increments, as well as the joint spatio-temporal spectral density in the second case. These results allow us to describe the memory nature (both spatial and temporal) of the solutions. In Section 4, we provide some results in the case when the Gaussian noise in (5) is replaced by infinitely divisible noise. In particular, sufficient conditions for the existence of higher-order moments of the solution, and corresponding higher-order spectral densities, will be derived.

2. Stochastic processes associated with fractional kinetic equations

This section provides a CTRW pathway to the general fractional kinetic equation (4). We first define and recall some salient features of the CTRW model. In this model, a random walker starts at $r = 0$ at time $t = 0$ and proceeds by successive random jumps. It is generally assumed that the waiting times τ_i, $i = 1, 2, \ldots$, between consecutive jumps are independent, identically distributed random variables with probability density $\psi(\tau)$, and that the jumps ξ_i, $i = 1, 2, \ldots$, are independent, identically distributed random vectors in $\mathbb{R}^d$ with probability density $\lambda(\xi)$. It is further assumed that τ_i and ξ_i are independent. Let $T_0 = 0$ and let $T_n = \sum_{i=1}^{n} \tau_i$ be the time of the nth jump. For $t \geq 0$, we define $N_t = \max\{n \geq 0 \colon T_n \leq t\}$. The position of the random walker at time t is then given by $X_t = \sum_{i=1}^{N_t} \xi_i$. The stochastic process $\{X_t\}_{t\geq 0}$ is called a *continuous-time random walk.*

The probability density $p(t, r)$ of finding the random walker at position r at time t is governed by the equation

$$p(t,r) = \delta(r)\Psi(t) + \int_0^t \psi(t-s)\bigg[\int_{\mathbb{R}^d} \lambda(r-r')p(s,r')\,\mathrm{d}r'\bigg]\,\mathrm{d}s, \qquad t \in \mathbb{R}_+, r \in \mathbb{R}^d, \quad (6)$$

where $\delta(r)$ is the Dirac delta function, used here to reflect the initial condition $p(0, r) = \delta(r)$, and $\Psi(t)$ is the survival probability, which is related to the waiting time density through $\Psi(t) = 1 - \int_0^t \psi(s)\,\mathrm{d}s$ [26, pp. 118–119], [36]. The first term on the right-hand side of (6) expresses the persistence of the initial position $r = 0$, while the second term describes the contribution to $p(t, r)$ from the walker via the jump $r - r'$ at instant t after a waiting time $t - s$. The evolution equation (6) is known as the *master equation* of the continuous-time random walk X_t.

From now on, we consider the one-dimensional case, $d = 1$, which is sufficient for our purpose. Mainardi *et al.* [36] noted that if ψ is the Mittag-Leffler density, that is, if

$$\psi(t) = t^{\beta-1}E_{\beta,\beta}(-t^{\beta}), \qquad 0 < \beta \leq 1,$$

where $E_{\alpha,\beta}(x)$ is the two-parameter Mittag-Leffler function (see [19, pp. 1–6], for example), which can be defined by the series expansion

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C}, \alpha > 0, \beta > 0,$$

then (6) may be written as

$$\frac{\partial^{\beta} p(t,x)}{\partial t^{\beta}} = -p(t,x) + \int_{-\infty}^{\infty} \lambda(x-y)p(t,y)\,\mathrm{d}y.$$

To consider the more general fractional-in-time operator given in (4), with $\beta_n \leq 1$, we require the class of extended Mittag-Leffler distributions defined in [6]. Distributions in this class have densities described by the Laplace transform

$$\tilde{\psi}(s) = (1 + A_1 s^{\beta_1} + \cdots + A_n s^{\beta_n})^{-1}, \qquad s > 0,$$

where $1 \geq \beta_n > \cdots > \beta_1 > 0$ and $A_i > 0$, $i = 1, \ldots, n$. Following [36], we work in the Fourier–Laplace domain so that (6) becomes

$$\frac{1-\tilde{\psi}(s)}{s\tilde{\psi}(s)}[s\hat{\tilde{p}}(\kappa,s) - 1] = [\hat{\lambda}(\kappa) - 1]\hat{\tilde{p}}(\kappa,s),$$

with '$\tilde{\ }$' and '$\hat{\ }$' denoting the Laplace and Fourier transforms, respectively. Hence, with the extended Mittag-Leffler distribution as the waiting time distribution of the CTRW, we have

$$(A_n s^{\beta_n - 1} + \cdots + A_1 s^{\beta_1 - 1})[s\hat{\tilde{p}}(\kappa,s) - 1] = [\hat{\lambda}(\kappa) - 1]\hat{\tilde{p}}(\kappa,s).$$

By inverting the Laplace and Fourier transforms, we arrive at the extended form

$$A_n \frac{\partial^{\beta_n} p(t,x)}{\partial t^{\beta_n}} + \cdots + A_1 \frac{\partial^{\beta_1} p(t,x)}{\partial t^{\beta_1}} = -p(t,x) + \int_{-\infty}^{\infty} \lambda(x-y)p(t,y)\,\mathrm{d}y.$$

We will now restrict our attention to the case of $n = 1$, although the results will apply to the general case.

Let λ be an infinitely divisible distribution with characteristic function

$$\hat{\lambda}(\omega) = \exp\biggl(-\tfrac{1}{2}\sigma^2 + \mathrm{i}\gamma\omega + \int_{\mathbb{R}\setminus\{0\}} [\mathrm{e}^{\mathrm{i}\omega x} - 1 - \mathrm{i}\omega x\, \mathbf{1}_{\{|x|\le 1\}}]\nu(\mathrm{d}x)\biggr),$$

where $\sigma, \gamma \in \mathbb{R}$, $\nu(\mathrm{d}x)$ is a Lévy measure [34], and $\mathbf{1}_{\{\cdot\}}$ is an indicator function. Let c be a positive constant and λ^{*c} be the distribution with characteristic function $(\hat{\lambda}(\omega))^c$. When c is a positive integer, λ^{*c} is the distribution of the sum of c independent random variables with distribution λ. For $c < 1$, the distribution λ^{*c} represents a convolution root of λ. Our interest is in the behaviour of solutions to the master equation

$$c\frac{\partial^\beta p(t,x)}{\partial t^\beta} = -p(t,x) + \int_{-\infty}^{\infty} \lambda^{*c}(x-y)p(t,y)\,\mathrm{d}y, \tag{7}$$

as $c \to 0$. The CTRW corresponding to (7) with $c > 0$ has waiting time density

$$c^{-1}t^{\beta-1}E_{\beta,\beta}\biggl(\frac{-t^\beta}{c}\biggr)$$

and jump distribution λ^{*c}. We will show that the master equation converges to the fractional space–time equation

$$\frac{\partial^\beta p(t,x)}{\partial t^\beta} = \mathcal{A}p(t,x) \tag{8}$$

(in a sense to be specified below), where $\mathcal{A}$ is the infinitesimal generator of the Lévy motion associated with the infinitely divisible distribution λ, and the sequence of random walks X_t^c, which (7) describes, converges weakly. Interesting examples of $\mathcal{A}$ include the inverse of the Riesz potential $(-\Delta)^\alpha$, $\alpha \in (0,1]$, which generates 2α-stable motion, and the inverse of the composition of the Riesz and Bessel potentials $(-\Delta)^\alpha(I-\Delta)^\gamma$, $\alpha \in (0,1]$, $\alpha+\gamma \in [0,1]$, which generates Riesz–Bessel Lévy motion [7]. The following theorem has been proved by Meerschaert and Scheffler [37] in a more general context. An alternative proof is provided here to indicate that there is further scope for extension to general operators that are fractional in time and space.

Theorem 1. *Let $p_c(t,x)$ be the solution to (7) subject to the initial condition $p_c(0,x) = f(x) \in C_0$, the Banach space of continuous functions with decay at infinity. Then*

(i) *the sequence of solutions $p_c(t,x)$ converges pointwise in $\mathbb{R}$ to a mild solution of (8) under the same initial condition, and*

(ii) *the sequence of continuous-time random walks described by (7) converges in the weak sense.*

Proof. (i) The solution to (7) is seen to exist and be unique for $f \in C_0$ by using the representation (6) and applying the Banach fixed point theorem. The solution may be written as the expectation $\mathrm{E}(f(x+X_t^c))$, with X_t^c being the corresponding continuous-time random walk. As f is bounded, $f(x+X_t^c)$ is uniformly integrable; hence, we need only show that $X_t^c \overset{\mathrm{D}}{=} X_t$ for fixed t. If $L(t)$ is the Lévy motion with infinitesimal generator $\mathcal{A}$, then we may

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write $X_t^c \xrightarrow{\text{D}} L(cN_t^c)$, where N_t^c is the counting process defined in the following manner. Set $T^c(0) = 0$ and

$$T^c(n) = \sum_{i=1}^{n} c^{1/\beta} J_i,$$

where n is an integer and J_i are independent Mittag-Leffler random variables. Note that the probability density function of the random variable $c^{1/\beta} J_i$ has Laplace transform $(1 + cs^\beta)^{-1}$. The process $T^c(n)$ gives the nth jump time of the CTRW and the counting process N_t^c is given by

$$N_t^c = \max\{n \geq 0 \colon T^c(n) \leq t\}.$$

We show convergence of the Laplace transform of the distribution of cN_t^c, i.e.

$$\mathrm{E}(\mathrm{e}^{-ucN_t^c}) = \sum_{n=0}^{\infty} \mathrm{e}^{-ucn}\, \mathrm{P}(N_t^c = n).$$

Taking the Laplace transform with respect to time, we find that

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-st}\, \mathrm{E}(\mathrm{e}^{-ucN_t^c})\, \mathrm{d}t &= \sum_{n=0}^{\infty} \mathrm{e}^{-ucn} \int_0^\infty \mathrm{e}^{-st}\, \mathrm{P}(N_t^c = n)\, \mathrm{d}t \\ &= \frac{cs^{\beta-1}}{1 + cs^\beta} \sum_{n=0}^{\infty} \mathrm{e}^{-ucn} (1 + cs^\beta)^{-n} \\ &= \frac{cs^{\beta-1}}{1 + cs^\beta} \frac{1 + cs^\beta}{1 + cs^\beta - \mathrm{e}^{-uc}} \\ &\to \frac{s^{\beta-1}}{u + s^\beta}. \end{aligned} \tag{9}$$

Convergence of Laplace transforms implies convergence of the original function; hence, $\mathrm{E}(\mathrm{e}^{-ucN_t^c})$ converges. Inverting the Laplace transform of (9) with respect to s gives $E_{\beta,1}(-ut^\beta)$ and $E_{\beta,1}(0) = 1$ (see, for example, [36, Equation (3.4)]). Hence, cN_t^c converges in distribution as $c \to 0$ for fixed t. As a result, the sequence of solutions $p_c(t, x)$ converges to some function $q(t, x)$. Standard conditioning arguments can now be used to yield

$$q(t, x) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega x} E_{\beta,1}(-\varphi(\omega) t^\beta) \hat{f}(\omega)\, \mathrm{d}\omega \tag{10}$$

(provided that the integral exists), where $\hat{f}$ is the Fourier transform of f and $\varphi(\omega) = -\ln \hat{\lambda}(\omega)$. For an f in the space of Schwartz distributions, we may use a result of [20] (see also [3]) to show that (10) is the unique solution to (8).

(ii) To show that the continuous-time random walk converges in the weak sense to a well-defined stochastic process, we need only show convergence of the process cN_t^c since the random walk is given by $L(cN_t^c)$. Furthermore, we need only show convergence of cN_t^c in the sense of finite-dimensional distributions as it is increasing and, hence, tight. Convergence in finite-dimensional distributions of a tight sequence implies convergence in the weak sense.

We will prove convergence in finite-dimensional distributions using an argument of [37]. Take $T^c(n)$ and N_t^c as defined in the proof of part (i), and note that

$$\{T^c(n) \leq t\} = \{N_t^c \geq n\}.$$

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Fix $0 < t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geq 0$; then

$$\begin{aligned} \mathrm{P}\{cN_{t_i}^c < x_i \text{ for all } i\} &= \mathrm{P}\{N_{t_i}^c < c^{-1}x_i \text{ for all } i\} \\ &= \mathrm{P}\{T^c([c^{-1}x_i]) > t_i \text{ for all } i\}. \end{aligned}$$

(Note that here $[x]$ denotes the integer part of x.) By a theorem of Skorokhod (see Theorem 16.14 of [29]), to show that $T^c([c^{-1}x])$ converges weakly to a Lévy process L_t, we need only show that $T^c([c^{-1}x]) \stackrel{\mathrm{D}}{=} L_1$ when $T^c(n)$ is a random walk on the integers. Let $c = 1/n$; the density of the distribution of $T^{1/n}(n)$ has Laplace transform $(1 + s^\beta/n)^n$. Letting $n \to \infty$, we see that $(1 + s^\beta/n)^n \to \exp(-s^\beta)$ and, therefore, that $T^{1/n}(n)$ converges to a β-stable random variable. It follows that $T^{1/n}([nx])$ converges in the sense of finite-dimensional distributions to a β-stable subordinator and, so, cN_t^c converges in finite-dimensional distributions and in the weak sense.

Remark 1. The theorem can be adapted to show that a sequence of solutions to

$$A_n \frac{\partial^{\beta_n} p(t,x)}{\partial t^{\beta_n}} + \cdots + A_1 \frac{\partial^{\beta_1} p(t,x)}{\partial t^{\beta_1}} = c^{-1}\biggl(-p(t,x) + \int_{-\infty}^{\infty} \lambda^{*c}(x-y)p(t,y)\,\mathrm{d}y\biggr)$$

will converge to a mild solution of (4) as $c \to 0$. This is achieved by replacing the waiting time distribution by an extended Mittag-Leffler distribution with Laplace transform

$$\tilde{\psi}(s) = (1 + cA_1 s^{\beta_1} + \cdots + cA_n s^{\beta_n})^{-1}, \qquad s > 0.$$

The sequence of continuous-time random walks that this equation describes will also converge in the weak sense.

Remark 2. A more general equation is obtained by allowing $\mathcal{A}$ in (8) to be the infinitesimal generator of a Markov process. The restriction that β satisfy $0 < \beta \leq 1$ is still required in this remark. A special case of this is the fractional Fokker–Planck equation studied by Metzler and Klafter [40], among others. We are able to propose an alternative interpretation of this equation and provide an explicit construction of a stochastic process whose marginal distribution satisfies the fractional Fokker–Planck equation. This is achieved by taking the limit of a sequence of solutions to a fractional-in-time linear Boltzmann equation.

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Following [32], any linear operator $\mathcal{P}\colon L^1 \to L^1$ satisfying

$$\mathcal{P}f \geq 0, \qquad \|\mathcal{P}f\| = \|f\|,$$

for $f \geq 0$ and $f \in L^1$, is called a Markov operator. In this paper, for $p \in [1, \infty)$ we let $L^p := L^p(I)$, $I \in \mathbb{R}^m$, be the space of all measurable functions $f\colon I \to \mathbb{R}$ with finite norm $\|f\|_p := (\int_I |f(x)|^p\,\mathrm{d}x)^{1/p}$. We shall write $\|\cdot\|$ for the norm if p is clear from the context. A measurable function $K\colon \Omega \times \Omega \to \mathbb{R}$ satisfying

$$K(x,y) \geq 0, \qquad \int_\Omega K(x,y)\,\mathrm{d}x = 1$$

is called a stochastic kernel. Clearly, the operator defined by

$$\mathcal{P}f(x) = \int_\Omega K(x,y)f(y)\,\mathrm{d}y, \qquad f \in L^1,$$

is a Markov operator. An example of a Markov operator is obtained by letting $K(x, y) = \lambda(x - y)$. This is the convolution operator, which was used in the master equation. Using a general Markov operator instead of a convolution operator allows the jump distribution to depend on the current position of the walk.

Let $\mathcal{P}$ be a Markov operator with stochastic kernel and let N_t be the counting process with Mittag-Leffler jump times. Define the function $u(t, x)$ by

$$u(t, x) = \sum_{k=0}^{\infty} \mathrm{P}(N_t = k)\mathcal{P}^k f(x).$$

Making Laplace transforms with respect to the time variable gives [27]

$$\tilde{u}(s, x) = \sum_{k=0}^{\infty} \frac{s^{\beta-1}}{(1 + s^{\beta})^{k+1}} \mathcal{P}^k f(x).$$

Applying the Caputo fractional differentiation with respect to time for Laplace transforms gives

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{s^{2\beta-1}}{(1 + s^{\beta})^{k+1}} \mathcal{P}^k f(x) - s^{\beta-1} f(x) \\
&\quad = \sum_{k=0}^{\infty} \frac{s^{\beta-1}}{(1 + s^{\beta})^{k+1}} \mathcal{P}^{k+1} f(x) - \sum_{k=0}^{\infty} \frac{s^{\beta-1}}{(1 + s^{\beta})^{k+1}} \mathcal{P}^k f(x) \\
&\quad = \mathcal{P}\tilde{u}(s, x) - \tilde{u}(s, x).
\end{aligned}$$

Inverting the Laplace transform then yields the equation

$$\frac{\partial^{\beta} u(t, x)}{\partial t^{\beta}} = \mathcal{P}u(t, x) - u(t, x), \tag{11}$$

subject to the initial condition

$$u(0, x) = f(x).$$

Again, if $\mathcal{P}$ is the convolution operator, we obtain the master equation (7). For $\beta = 1$, (11) is the linear Boltzman equation and so we refer to this equation as the fractional-in-time linear Boltzmann equation.

We now introduce a scale parameter c into the Mittag-Leffler jump distribution, as in the previous section, and let the Markov operators $\mathcal{P}_c$ form a continuous-time semigroup at time $t = c$. Equation (11) then becomes

$$\frac{\partial^{\beta} u_c(t, x)}{\partial t^{\beta}} = c^{-1}(\mathcal{P}_c - I)u_c(t, x), \tag{12}$$

and taking the limit $c \to 0$, as in the proof of Theorem 1, gives the desired equation (8) with $\mathcal{A}$ being the infinitesimal generator of $\mathcal{P}_t$. The solution to (12) is given by the expectation

$$u_c(t, x) = \mathrm{E}\, f(Y(cN_t^c) + x),$$

where $Y(t)$ is the Markov process with semigroup $\mathcal{P}_t$. Note that the conditional expectation

$$\mathrm{E}(f(Y(cN_t^c) + x) \mid cN_t^c = \zeta) = v(\zeta, x),$$

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where $v(t, x)$ is the solution to (8) with $\beta = 1$, and subject to the initial condition $v(0, x) = f(x)$. Hence, we can write the solution to (12) as

$$u_c(t, x) = \int_0^\infty v(\zeta, x)\sigma_{t,c}(\mathrm{d}\zeta),$$

where $\sigma_{t,c}$ is the distribution of cN_t^c. Assuming that $f(\cdot)$ is a bounded and continuous function, it follows that $v(t, \cdot)$ is bounded for all $t \geq 0$. The dominated convergence theorem can then be applied to conclude that the limiting solution has the representation

$$u_0(t, x) = \int_0^\infty v(\lambda, x)t^{-\beta} M(t^{-\beta}\lambda; \beta)\,\mathrm{d}\lambda, \tag{13}$$

where $M(\lambda; \beta)$ is a special case of the Wright function given by

$$M(\lambda; \beta) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-\lambda)^{n-1}}{\Gamma(n)}\Gamma(\beta n)\sin(\beta n\pi).$$

We note that representation (13) is equivalent to [10, Equation (25)] when $\mathcal{A}$ is taken to be the generator of a Feller semigroup associated with an infinitely divisible law. The representation (13) suggests that numerical solution of the equation is possible by first solving the equation with $\beta = 1$ and then performing the integration over time numerically.

3. Gaussian random fields

In studying the behaviour of the sample paths of the solution to (5), it will be useful to recall some results on the geometry of random fields [1]. Let $\{X(t), t \in \mathbb{R}^d\}$ be a zero-mean Gaussian random field with stationary increments and continuous covariance function. Define $\sigma^2(t) = \mathrm{E}(X(t+s) - X(t))^2$. If there exists a $\beta \in (0, 1)$ such that

$$\beta = \sup\{\beta : \sigma(t) = o(|t|^\beta), |t| \downarrow 0\} = \inf\{\beta : |t|^\beta = o(\sigma(t))\},$$

then $\{X(t), t \in \mathbb{R}^d\}$ is called an index-β Gaussian field. The following results hold with probability one.

- $\dim_{\mathrm{H}}(\mathrm{graph}(X)) = d + 1 - \beta$, where $\dim_{\mathrm{H}}$ is the Hausdorff dimension and $\mathrm{graph}(X) := \{(t, X(t)),\ t \in \mathbb{R}^d\}$.
- X satisfies a stochastic Hölder continuity condition of order α for any $\alpha < \beta$. Furthermore, for any $\alpha > \beta$, X fails to satisfy any uniform Hölder condition of order α.

The following subsections examine the conditions under which a solution to (5) exists, and establish some properties of that solution. Propositions 2–4 and 6 give the conditions on the operators of (5) for the solution, or a restriction of it, to be an index-β Gaussian random field. Hence, these propositions will allow the dimension and the Hölder continuity of the solution to be determined.

3.1. The fractional kinetic equation on a bounded spatial domain

For $t > 0$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d$, we denote

$$W_{tx} = W((0, t] \times (0, x_1] \times \cdots \times (0, x_d]),$$

where W is a Gaussian additive set function on the Borel sets of $\mathbb{R}^{d+1}$ such that there is a version of W_{tx} that is a continuous, real-valued zero-mean Gaussian random field with covariance function

$$\mathrm{E}(W_{sx}W_{ty}) = (s \wedge t)(x_1 \wedge y_1) \cdots (x_d \wedge y_d).$$

For a measurable square-integrable function $f(t, x)$, the Itô stochastic integral $\iint f(t, x)\,\mathrm{d}W_{tx}$ is defined in [14] (see also the Appendix of [50]). Here, and where not specified in the following, the range of integration is $\mathbb{R} \times \mathbb{R}^d$. The following properties hold:

$$\mathrm{E}\biggl(\iint f(t, x)\,\mathrm{d}W_{tx}\biggr)^2 = \iint f^2(t, x)\,\mathrm{d}x\,\mathrm{d}t,$$

$$\mathrm{E}\biggl(\iint f(t, x)\,\mathrm{d}W_{tx}\biggr)\biggl(\iint g(t, x)\,\mathrm{d}W_{tx}\biggr) = \iint f(t, x)g(t, x)\,\mathrm{d}x\,\mathrm{d}t.$$

Clearly, if f and g have disjoint support then the two stochastic integrals are uncorrelated and, hence, independent. Define a generalized random function ε by the stochastic integral representation $\varepsilon = (f, \varepsilon) = \iint f(t, x)\,\mathrm{d}W_{tx}$ for $f \in C_0^\infty(\mathbb{R}^{d+1})$, the space of infinitely differentiable functions with compact support in $\mathbb{R}^{d+1}$. This is a mean-square continuous linear function with respect to the L^2-norm over $C_0^\infty(\mathbb{R}^{d+1})$ and, following [44, Sections I.1.3 and I.5.1], we may treat ε as a random Schwartz distribution. Formally, we identify $\varepsilon(t, x)$ with $\partial^{d+1}W_{tx}/\partial x_1 \cdots \partial x_d \partial t$ and call it space–time white noise.

We will consider the problem (5) for $x \in (0, L_1) \times \cdots \times (0, L_d) = D_L$, $L = (L_1, \dots, L_d)$, with initial condition $c(0, x) = 0$ and Dirichlet boundary conditions. As will be proved in the next result, a solution to this problem can be formulated in terms of the following function:

$$G(t, x; s, y) = \begin{cases} \displaystyle\sum_{k \in \mathbb{N}_*^d} \phi_k(x)\phi_k(y)(t - s)^{\beta - 1}E_{\beta,\beta}(-\lambda_k(t - s)^\beta), & t \geq s, \\ 0, & t < s, \end{cases} \tag{14}$$

for all $x, y \in D_L$. Here, $\mathbb{N}_* = \{1, 2, 3, \dots\}$; $k = (k_1 \dots, k_d)$ with $k_i \in \mathbb{N}_*$;

$$\phi_k(x) = \biggl(\frac{2^d}{V_L}\biggr)^{1/2} \prod_{i=1}^d \sin\biggl(\frac{k_i \pi x_i}{L_i}\biggr)$$

are the eigenfunctions of the Laplacian in the Dirichlet problem; $V_L = \prod_{i=1}^d L_i$ is the volume of the parallelepiped D_L; and $\lambda_k = (1 + w_k)^{\alpha/2}w_k^{\gamma/2}$ are the eigenvalues of $(I - \Delta)^{\alpha/2}(-\Delta)^{\gamma/2}$, with $w_k = \sum_{i=1}^d k_i^2\pi^2/L_i^2 = \pi^2\sum_{i=1}^d (k_i/L_i)^2 = \pi^2|k|_L^2$ being the eigenvalues of $-\Delta$. The function G defined in (14) can be interpreted as the Green function of the corresponding deterministic problem in the weak (or generalized) sense.

Proposition 1. *For $\beta \leq 1$ and $(\alpha + \gamma)(2\beta - 1) > \beta d$, the stochastic fractional kinetic equation (5), defined on D_L, with the initial condition $c(0, x) = 0$, for all x, and homogeneous boundary conditions admits the following solution in the mean-square sense (and, therefore, in the distribution sense):*

$$c(t, x) = \int_0^t \int_{D_L} G(t, x; s, y)\,\mathrm{d}W_{sy}, \qquad t > 0, x \in D_L, \tag{15}$$

where the integral is interpreted in the mean-square sense and $G(t, x; s, y)$ is defined by (14).

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Proof. We recall, from Chapter 5 of [42], that the solution to the linear fractional differential equation

$$\frac{\mathrm{d}^{\beta} f(t)}{\mathrm{d}t^{\beta}} + \lambda f(t) = \delta(t) \tag{16}$$

is given by the function

$$G_{\lambda}(t) = t^{\beta-1} E_{\beta,\beta}(-\lambda t^{\beta}). \tag{17}$$

From the asymptotic properties of the two-parameter Mittag-Leffler function (see [19, Theorem 1.3-3]), it can be seen that (17) is square integrable on $\mathbb{R}_+$ for $\beta > \frac{1}{2}$. The proof now follows along the same lines as that of Proposition 2.1 of [2]. We require, for a fixed $(t, x) \in \mathbb{R}_+ \times D_L$, the convergence in the mean-square sense of the following random sequence of partial sums:

$$S_n^{t,x} = \int_{\mathbb{R}_+} \int_{D_L} \sum_{l \le n} G_{\lambda_l}(t-s) H(t-s) \phi_l(x) \phi_l(y) \, \mathrm{d}W_{s,y}, \qquad n \in \mathbb{N}_*.$$

This is equivalent to the convergence in $L^2(\mathbb{R}_+ \times D_L)$ of the sequence of functions

$$\tilde{S}_n^{t,x}(s, y) = \sum_{l \le n} G_{\lambda_l}(t-s) H(t-s) \phi_l(x) \phi_l(y), \qquad n \in \mathbb{N}_*.$$

Here, $H(t-s) = \mathbf{1}_{[0,t]}(s)$ represents the Heaviside function. Now, for $m, n \in \mathbb{N}_*$ with $m < n$,

$$\begin{aligned}
\|\tilde{S}_n^{t,x} - \tilde{S}_m^{t,x}\| &= \int_{\mathbb{R}_+} \int_{D_L} \biggl(\sum_{m<l\le n} G_{\lambda_l}(t-s) H(t-s) \phi_l(s) \phi_l(y) \biggr)^2 \mathrm{d}y \, \mathrm{d}s \\
&= \int_0^t \int_{D_L} \sum_{m<l\le n} G_{\lambda_l}^2(t-s) \phi_l^2(x) \phi_l^2(y) \, \mathrm{d}y \, \mathrm{d}s \\
&= \int_0^t \sum_{m<l\le n} G_{\lambda_l}^2(t-s) \phi_l^2(x) \, \mathrm{d}s \\
&= \sum_{m<l\le n} \phi_l^2(x) \int_0^t G_{\lambda_l}^2(t-s) \, \mathrm{d}s \\
&\le \int_0^{\infty} G_1^2(s) \, \mathrm{d}s \sum_{m<l\le n} \phi_l^2(x) \lambda_l^{1/\beta-2} \\
&\le \frac{2^d}{V_L} \int_0^{\infty} G_1^2(s) \, \mathrm{d}s \sum_{m<l\le n} \lambda_l^{1/\beta-2}.
\end{aligned}$$

As $\lambda_l \sim l^{(\alpha+\gamma)/d}$, the sum $\sum_{l\in\mathbb{N}_*} \lambda^{1/\beta-2}$ is finite if $(\alpha+\gamma)(2\beta-1) > \beta d$ and, hence, we have convergence in the mean-square sense. From (16) and (17), we may take the fractional-in-time derivative of (15) to show that it solves (5) in the mean-square sense.

Remark 3. The solution to (5) with random initial condition in the rectangular domain case, and $\beta \le 1$, is given by

$$c(t, x) = \int_{D_L} \sum_{k\in\mathbb{N}_*^d} \phi_k(x) \phi_k(y) E_{\beta,1}(-\lambda_k t^{\beta}) c_0(y) \, \mathrm{d}y + \int_0^t \int_{D_L} G(t, x; s, y) \, \mathrm{d}W_{sy}, \tag{18}$$

with conditions on the parameters as given in Proposition 1 and assuming that $c_0(x)$ has finite variance. Convergence, in the mean-square sense, of the first term on the right-hand side of (18) is guaranteed if

$$\int_{D_L}\int_{D_L} \sum_{k\in\mathbb{N}_*^d} \phi_k(x)\phi_k(y_1)E_{\beta,1}(-\lambda_k t^\beta) \sum_{k\in\mathbb{N}_*^d} \phi_k(x)\phi_k(y_2)E_{\beta,1}(-\lambda_k t^\beta) \times \operatorname{cov}[c_0(y_1), c_0(y_2)]\,\mathrm{d}y_1\,\mathrm{d}y_2 < \infty,$$

which generally requires that

$$\sum_{k\in\mathbb{N}_*^d} \phi_k(x)\phi_k(y)E_{\beta,1}(-\lambda_k t^\beta) \in L_1(D_L).$$

From the asymptotics of the Mittag-Leffler function [19], this means that $\alpha+\gamma > d$, which is satisfied if the condition of Proposition 1 is satisfied, i.e. $\beta \le 1$. To see that (18) solves (5), we recall a result of [20] (also see [3]), namely that the unique solution in $L^p([0,T])$ to the Cauchy problem of the fractional differential equation

$$\frac{\mathrm{d}^\beta u(t)}{\mathrm{d}t^\beta} + au(t) = 0, \qquad u(0) = 1, \qquad a > 0,$$

is given by $u(t) = E_{\beta,1}(-at^\beta)$. Clearly, $\lim_{t\to 0} c(t,x) = c_0(x)$ and, taking fractional derivatives with respect to time, we see that (18) solves (5) in the mean-square sense.

Proposition 2. *The solution (15) of the fractional kinetic equation (5) satisfies the following inequalities:*

$$\mathrm{E}(c(t,x)-c(s,x))^2 \le K|t-s|^{2\beta-1-\beta d/(\alpha+\gamma)}, \quad \textit{with} \quad \beta d < (\alpha+\gamma)(2\beta-1), \tag{19}$$

$$\mathrm{E}[c(t,x)-c(t,y)]^2 \le M_L|x-y|^{(\alpha+\gamma)(2-1/\beta)-d}, \quad \textit{with} \quad \beta d < (\alpha+\gamma)(2\beta-1) < \beta(d+2). \tag{20}$$

Here, $t, s \in \mathbb{R}_+$, $x, y \in D_L$, and K and M_L are constants independent of t and s, and x and y, respectively. Also, the following inequality holds:

$$\mathrm{E}(c(t,x)-c(t,y))^2 \le \begin{cases} \tilde{M}_L|x-y|^2 + \tilde{M}'_L|x-y|^2(-\log|x-y|), & (\alpha+\gamma)(2\beta-1) = \beta(d+2), \\ \tilde{\tilde{M}}_L|x-y|^2, & (\alpha+\gamma)(2\beta-1) > \beta(d+2), \end{cases} \tag{21}$$

where $\tilde{M}_L$, $\tilde{M}'_L$, and $\tilde{\tilde{M}}_L$ are constants independent of x and y.

Remark 4. Note that the constant in the inequality (19) does not depend on time, unlike in Proposition 2.3 and Remark 2.2 of [2]. The slightly sharper result given above is due to a different proof.

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Proof of Proposition 2. We have

$$\begin{aligned}
\mathrm{E}(c(t,x)-c(t,y))^2 &= \int_0^t \int_{D_L} \sum_{k\in\mathbb{N}_*^d} [\phi_k(x)-\phi_k(y)]^2 \phi_k^2(z) G_{\lambda_k}^2(t-r)\,\mathrm{d}z\,\mathrm{d}r \\
&= \sum_{k\in\mathbb{N}_*^d} [\phi_k(x)-\phi_k(y)]^2 \lambda_k^{1/\beta-2} \int_0^{t\lambda_k} G_1^2(t-r)\,\mathrm{d}r \\
&\le \int_0^\infty G_1^2(s)\,\mathrm{d}s \sum_{k\in\mathbb{N}_*^d} [\phi_k(x)-\phi_k(y)]^2 \lambda_k^{1/\beta-2} \\
&\le \int_0^\infty G_1^2(s)\,\mathrm{d}s \sum_{k\in\mathbb{N}_*^d} [\phi_k(x)-\phi_k(y)]^2 \omega_k^{(\alpha+\gamma)(1/2\beta-1)}.
\end{aligned}$$

Following the proof of Proposition 2.3 of [2], we then have

$$\begin{aligned}
&\mathrm{E}(c(t,x)-c(t,y))^2 \\
&\quad\le \int_0^\infty G_1^2(s)\,\mathrm{d}s \bigg(\frac{2^d}{V_L}\bigg) \sum_{k\in\mathbb{N}_*^d} [2\wedge(\pi|x-y||k|_L)]^2 \omega_k^{(\alpha+\gamma)(1/2\beta-1)} \\
&\quad\le \int_0^\infty G_1^2(s)\,\mathrm{d}s \bigg(\frac{2^d}{V_L\pi^{(\alpha+\gamma)/\beta}}\bigg) \sum_{k\in\mathbb{N}_*^d} \bigg(\frac{4}{|k|_L^{(\alpha+\gamma)(2-1/\beta)}} \wedge \frac{\pi^2|x-y|^2}{|k|_L^{(\alpha+\gamma)(2-1/\beta)-2}}\bigg).
\end{aligned}$$

The remainder of the proof of (19) and (21) follows the proof of Proposition 2.3 of [2] with $\alpha+\gamma$ replaced by $(\alpha+\gamma)(2-1/\beta)$. For the proof of (19), note that the solution $c(t,x)$ can be written as

$$c(t,x) = \sum_{k\in\mathbb{N}_*^d} \phi_k(x) \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-\lambda_k(t-s)^\beta)\,\mathrm{d}B_k(s),$$

where $B_k(t)$ are independent Brownian motions. This representation follows from the orthogonality of the eigenfunctions of the negative Laplacian. We define the temporal limiting process

$$\begin{aligned}
\tilde{c}(t,x) &= \sum_{k\in\mathbb{N}_*^d} \phi_k(x) \int_{-\infty}^t (t-s)^{\beta-1} E_{\beta,\beta}(-\lambda_k(t-s)^\beta)\,\mathrm{d}B_k(s) \\
&= c(t,x) + \sum_{k\in\mathbb{N}_*^d} \phi_k(x) \int_{-\infty}^0 (t-s)^{\beta-1} E_{\beta,\beta}(-\lambda_k(t-s)^\beta)\,\mathrm{d}B_k(s). \qquad (22)
\end{aligned}$$

Clearly, $c(t,x)$ and $\tilde{c}(t,x)-c(t,x)$ are independent and, hence,

$$\mathrm{E}((c(t+\tau,x)-c(t,x))^2) \le \mathrm{E}((\tilde{c}(t+\tau,x)-\tilde{c}(t,x))^2).$$

The upper bound follows from the proof of Proposition 3, below.

Remark 5. From Proposition 2, the moduli of continuity of the random field can be determined as in [2]:

$$\omega_t(\delta) = \sup_{|t-s|<\delta} |c(t,x) - c(s,x)| = X\delta^{\beta-1/2-\beta d/2(\alpha+\gamma)} + K\delta^{\beta-1/2-\beta d/2(\alpha+\gamma)}(-\log\delta)^{1/2},$$

$$\omega_x(\delta) = \sup_{\|x-y\|<\delta} |c(t,x) - c(t,y)| = Y\delta^{(\alpha+\gamma)(1-1/2\beta)-d/2} + L\delta^{(\alpha+\gamma)(1-1/2\beta)-d/2}(-\log\delta)^{1/2},$$

$$\begin{aligned}\omega_{t,x}(\delta) = \sup_{\|(t,x)-(s,y)\|<\delta} |c(t,x) - c(s,x)| &= Z\delta^{\beta-1/2-\beta d/2(\alpha+\gamma)} \\ &\quad + M\delta^{\beta-1/2-\beta d/2(\alpha+\gamma)}(-\log\delta)^{1/2},\end{aligned}$$

where X, Y, and Z are almost surely finite, positive random variables and K, L, and M are positive constants.

Proposition 3. *Let $c(t,x)$ be the solution of (5) given by (15). For a fixed $x \in D_L$, its asymptotic temporal spectral density is*

$$f_x(\omega) = \frac{1}{2\pi} \sum_{k\in\mathbb{N}_*^d} \phi_k^2(x)(|\omega|^{2\beta} + 2\cos(\tfrac{1}{2}\pi\beta)|\omega|^\beta\lambda_k + \lambda_k^2)^{-1}. \tag{23}$$

Also, for $(2\beta-1)(\alpha+\gamma) > \beta d$, the asymptotic variance of the increments, $\sigma_x^2(\tau)$, satisfies

$$c_1|\tau|^{2\beta-1-d\beta/(\alpha+\gamma)} \le \sigma_x^2(\tau) \le c_2|\tau|^{2\beta-1-d\beta/(\alpha+\gamma)} \tag{24}$$

for some constants c_1 and c_2. That is, at a fixed $x \in D_L$, the temporal limiting process is an index-$\frac{1}{2}(2\beta - 1 - d\beta/(\alpha+\gamma))$ process.

Proof. Note that $c(t,x)$ converges, in the mean-square sense, to the random field $\tilde{c}(t,x)$ defined in (22). The spectral density of the stochastic integral in (22) is given by [8, Equation (3.15)]. The temporal spectral density (23) follows from the fact that the $B_k(t)$ are independent.

The upper bound in (24) is established by arguments similar to those in the proof of Proposition 2.6 of [2], which we will briefly outline. By definition,

$$\begin{aligned}
2\pi\sigma_x^2(\tau) &= \int_{\mathbb{R}} |e^{i\omega\tau} - 1|^2 f_x(\omega)\,d\omega \\
&= \sum_{k\in\mathbb{N}_*^d} \phi_k^2(x) \int_{\mathbb{R}} |e^{i\omega\tau} - 1|^2 (|\omega|^{2\beta} + 2\cos(\tfrac{1}{2}\pi\beta)|\omega|^\beta\lambda_k + \lambda_k^2)^{-1}\,d\omega \\
&\le \sum_{k\in\mathbb{N}_*^d} \phi_k^2(x) \int_{\mathbb{R}} |e^{i\omega\tau} - 1|^2 (|\omega|^{2\beta} + \lambda_k^2)^{-1}\,d\omega \\
&\le \frac{2^d}{V_L} \int_{\mathbb{R}} |e^{i\eta} - 1|^2 \Bigg[\sum_{k\in\mathbb{N}_+^d} \frac{\tau^{2\beta-1}}{\tau^{2\beta}(\pi^2|k|_L^2)^{\alpha+\gamma} + \eta^{2\beta}}\Bigg]\,d\eta \\
&\le \frac{2^d}{V_L} \int_{\mathbb{R}} |e^{i\eta} - 1|^2 \int_{[0,\infty)^d} \frac{\tau^{2\beta-1}}{\tau^{2\beta}(\pi^2|z|_L^2)^{\alpha+\gamma} + \eta^{2\beta}}\,dz\,d\eta \\
&\le \tau^{2\beta-1-d\beta/(\alpha+\gamma)} \frac{2^d}{V_L} \int_{\mathbb{R}} \int_{[0,\infty)^d} |e^{i\eta} - 1|^2 \frac{dz\,d\eta}{(\pi^2|z|_L^2)^{\alpha+\gamma} + \eta^{2\beta}},
\end{aligned}$$

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which completes the proof of the upper bound. In the construction of the lower bound, we will assume for simplicity that $d = 1$ and $x/L = p/q$, with p and q being coprime. Then,

$$\begin{aligned}2\pi\sigma_x^2(\tau) &= \sum_{k\in\mathbb{N}_*}\phi_k^2(x)\int_{\mathbb{R}}|\mathrm{e}^{\mathrm{i}\omega\tau}-1|^2(|\omega|^{2\beta}+2\cos(\tfrac{1}{2}\pi\beta)|\omega|^{\beta}\lambda_k+\lambda_k^2)^{-1}\,\mathrm{d}\omega\\ &= \frac{2^d}{V_L}\int_{\mathbb{R}}|\mathrm{e}^{\mathrm{i}\omega\tau}-1|^2\\ &\quad\times\bigg[\sum_{l=0}^{\infty}\sum_{m=1}^{q-1}\sin^2\bigg(\frac{2\pi pm}{q}\bigg)(|\omega|^{2\beta}+2\cos(\tfrac{1}{2}\pi\beta)|\omega|^{\beta}\lambda_{lq+m}+\lambda_{lq+m}^2)^{-1}\bigg]\,\mathrm{d}\omega.\end{aligned}$$

Note that the eigenvalues are increasing in k and that $\lambda_k \le c|k|_L^{\alpha+\gamma}$ for a finite constant c. Hence, there are finite constants c_1 and c_2 such that

$$\begin{aligned}2\pi\sigma_x^2(\tau) &\ge \frac{2^d}{V_L}\bigg(\sum_{m=1}^{q-1}\sin^2\bigg(\frac{2\pi pm}{q}\bigg)\bigg)\\ &\quad\times\int_{\mathbb{R}}|\mathrm{e}^{\mathrm{i}\omega\tau}-1|^2\bigg[\sum_{l=1}^{\infty}(|\omega|^{2\beta}+c_1|\omega|^{\beta}|lq|_L^{\alpha+\gamma}+c_1|lq|_L^{2(\alpha+\gamma)})^{-1}\bigg]\,\mathrm{d}\omega\\ &\ge \frac{2^{d-1}}{V_L}\int_{\mathbb{R}}|\mathrm{e}^{\mathrm{i}\omega\tau}-1|^2\int_q^{\infty}(|\omega|^{2\beta}+c_1|\omega|^{\beta}|z|_L^{\alpha+\gamma}+c_2|z|_L^{2(\alpha+\gamma)})^{-1}\,\mathrm{d}z\,\mathrm{d}\omega\\ &\ge \tau^{2\beta-1}\frac{2^{d-1}}{V_L}\int_{\mathbb{R}}|\mathrm{e}^{\mathrm{i}u}-1|^2\int_q^{\infty}(|u|^{2\beta}+c_1|u|^{\beta}\tau^{\beta}|z|_L^{\alpha+\gamma}+c_2\tau^{2\beta}|z|_L^{2(\alpha+\gamma)})^{-1}\,\mathrm{d}z\,\mathrm{d}u\\ &\ge \tau^{2\beta-1-\beta/(\alpha+\gamma)}\frac{2^{d-1}}{V_L}\int_{\mathbb{R}}|\mathrm{e}^{\mathrm{i}u}-1|^2\\ &\quad\times\int_{q\tau^{\beta/(\alpha+\gamma)}}^{\infty}(|u|^{2\beta}+c_1|u|^{\beta}|y|_L^{\alpha+\gamma}+c_2|y|_L^{2(\alpha+\gamma)})^{-1}\,\mathrm{d}x\,\mathrm{d}u.\end{aligned}$$

So, for rational x/L, the proposition is proved. When x/L is irrational, we note that

$$\begin{aligned}&\mathrm{E}((c(t+\tau,x+z)-c(t,x+z)+c(t+\tau,x)-c(t,x))^2)\\ &\quad\le\sum_{k\in\mathbb{N}_*^d}[\phi_k(x+z)-\phi_k(x)]^2\int_{\mathbb{R}}|\mathrm{e}^{\mathrm{i}\omega\tau}-1|^2\big(|\omega|^{2\beta}+2\cos\big(\tfrac{1}{2}\pi\beta\big)|\omega|^{\beta}\lambda_k+\lambda_k^2\big)^{-1}\,\mathrm{d}\omega\\ &\quad\le o_{|z|}(1)|\tau|^{2\beta-1-\beta d/(\alpha+\gamma)},\end{aligned}$$

and the proposition is established for irrational x/L by application of the triangle inequality.

Remark 6. In Proposition 2.6 of [2], the asymptotic temporal covariance function is seen to decay exponentially fast as $|\tau|\to\infty$. It is clear from (23) that the temporal covariance function does not display long-range dependence for $\beta \ne 1$; however, the decay is much slower than exponential. For $\beta < 1$, it can be seen from (22) and Theorem 1.3-5 of [19] that the asymptotic temporal covariance function is given by

$$R_x(\tau)=\sum_{k\in\mathbb{N}_+^d}\phi_k^2(x)\int_0^{\infty}\mathrm{e}^{-|\tau|y}\frac{y^{\beta}\sin(\pi\beta)\,\mathrm{d}y}{(y^{\beta}+\lambda_k)(y^{2\beta}+2\cos(\pi\beta)y^{\beta}\lambda_k+\lambda_k^2)}. \tag{25}$$

Making the change of variable $u = |\tau|y$ in (25) and applying the dominated convergence theorem, we have

$$\lim_{|\tau|\to\infty} |\tau|^{1+\beta} R_x(\tau) = \left(\sum_{k\in\mathbb{N}_+^d} \phi_k^2(x)\lambda_k^{-3}\right) \sin(\pi\beta)\Gamma(1+\beta). \tag{26}$$

Remark 7. In the previous discussion, we have assumed that the domain is a bounded rectangle of $\mathbb{R}^d$. However, this assumption can be weakened considerably. Assume that D is a bounded region in $\mathbb{R}^d$ with smooth boundary ∂D and Dirichlet boundary conditions. The following properties of the negative Laplacian on D are needed (see Chapter 6 of [18]).

Let λ_k denote the eigenvalues of the negative Laplacian on the d-dimensional cube, written in increasing order and repeated according to multiplicity. Then there exists a constant $c > 0$ such that $\lim_{k\to\infty} \lambda_k k^{-2/d} = c$.

Let $\lambda_k(D)$ be the eigenvalues of the negative Laplacian acting on $L^2(D)$. Then $\lambda_k(D)$ is a monotonically decreasing function of the region in the sense that if $D_1 \subset D_2$ then $\lambda_k(D_1) \leq \lambda_k(D_2)$.

Let $\phi_k(x)$ be the eigenfunctions of the negative Laplacian on D with Dirichlet boundary conditions. The sequence of eigenfunctions forms a complete orthonormal set in $L^2(D)$. Under additional conditions on the smoothness of ∂D, the eigenfunctions will lie in the domain $C_0^\infty(\overline{D})$, the space of smooth functions on D whose partial derivatives can be continuously extended to the closure of D and vanish on the boundary ∂D.

From the spectral theory of linear self-adjoint operators, the eigenvalues defining the spectrum of the operator $(I-\Delta)^{\alpha/2}(-\Delta)^{\gamma/2}$ are $\omega_k = (1+\lambda_k)^{\alpha/2}\lambda_k^{\gamma/2}$, where λ_k are the eigenvalues of the negative Laplacian on D. Hence, the eigenvalues of $(I-\Delta)^{\alpha/2}(-\Delta)^{\gamma/2}$ on D, written in increasing order and repeated according to multiplicity, satisfy the bounds

$$c_1 k^{(\alpha+\gamma)/d} \leq \omega_k \leq c_2 k^{(\alpha+\gamma)/d},$$

for some positive constants c_1, c_2, and $k \geq 1$. Thus, with the exception of (20), (21), and the lower bound in (24), the results of this section hold, using the same proofs.

3.2. The fractional kinetic equation on an unbounded spatial domain

In this section we study the mean-square solution to (5) under the 'zero' initial condition $c(0, x) = 0$, with $x \in \mathbb{R}^d$. We formulate a solution to the problem in terms of the Fourier transform of a spatially and temporally homogeneous Green function. For $t \in \mathbb{R}_+$, we denote by $\hat{\varepsilon}_t(\lambda)$, $\lambda \in \mathbb{R}^d$, the complex-valued, generalized random function defined by the following weak-sense mean-square identity in $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^d$:

$$\varepsilon(t, x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\lambda\rangle} \hat{\varepsilon}_t(\mathrm{d}\lambda).$$

In addition, $\mathrm{E}(\hat{\varepsilon}_t(\mathrm{d}\lambda)\overline{\hat{\varepsilon}_s(\mathrm{d}\mu)}) = \delta(t-s)\delta(\lambda-\mu)\,\mathrm{d}\lambda\,\mathrm{d}\mu$, for all $\lambda, \mu \in \mathbb{R}^d$ and $t, s \in \mathbb{R}_+$, where a bar denotes complex conjugate. The process $\hat{\varepsilon}_t(\cdot)$ can be interpreted as the spatial spectral process of $\varepsilon_t(x) := \varepsilon(t, x)$, $x \in \mathbb{R}^d$, in the weak sense in time (see [52, p. 101 and p. 112]).

Proposition 4. *A real-valued zero-mean solution (in the mean-square sense) of (5) defined on* $\mathbb{R}_+ \times \mathbb{R}^d$, *under zero initial condition and assuming that* $(2\beta-1)(\alpha+\gamma) > \beta d$, *is given by*

$$c(t, x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle x,\lambda\rangle} \int_0^t G_{(1+|\lambda|^2)^{\alpha/2}|\lambda|^\gamma}(t-s)\hat{\varepsilon}_s(\mathrm{d}\lambda)\,\mathrm{d}s, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{27}$$

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where the integrals are interpreted in the mean-square sense. In addition, if $\gamma(2\beta - 1) < \beta d$*, then the process is asymptotically stationary, with spectral density*

$$f(\omega, \lambda) = \frac{1}{2\pi(|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^{\beta}(1 + |\lambda|^2)^{\alpha/2}|\lambda|^{\gamma} + (1 + |\lambda|^2)^{\alpha}|\lambda|^{2\gamma})} \tag{28}$$

for all $\omega \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}^d$*. If the process is stationary then, for a fixed time* $t_0 \in \mathbb{R}_+$*, the random field is a fractional Riesz–Bessel random field with spectral density*

$$f_{t_0}(\lambda) = \frac{1}{2\pi}\biggl(\int_{\mathbb{R}} \frac{\mathrm{d}u}{|u|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|u|^{\beta} + 1}\biggr)(|\lambda|^{\gamma}(1 + |\lambda|^2)^{\alpha/2})^{1/\beta - 2}, \tag{29}$$

and

$$\lim_{|z|\to 0} \frac{\mathrm{E}((c(t_0, x + z) - c(t_0, x))^2)}{|z|^{(\alpha+\gamma)(2-1/\beta)-d}} = \text{const.}$$

Thus, at each fixed time $t_0 \in \mathbb{R}_+$*, the asymptotically stationary process* $c(t_0, x)$ *is an index-*$((\alpha + \gamma)(2\beta - 1) - \beta d)/2\beta$ *random field.*

Proof. We first determine the range of parameters for which the integral in (27) is defined. For this, it is necessary to show that the Green function is in $L^2([0, T] \times \mathbb{R}^d)$. By the Plancherel formula (see, e.g. [21]),

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} G^2(s, x)\,\mathrm{d}x\,\mathrm{d}s &= \int_0^T \int_{\mathbb{R}^d} G^2_{(1+|\lambda|^2)^{\alpha/2}|\lambda|^{\gamma}}(s)\,\mathrm{d}\lambda\,\mathrm{d}s \\ &= \int_{\mathbb{R}^d} \int_0^{T((1+|\lambda|^2)^{\alpha/2}|\lambda|^{\gamma})^{1/\beta-2}} ((1 + |\lambda|^2)^{\alpha/2}|\lambda|^{\gamma})^{1/\beta-2} G_1^2(s)\,\mathrm{d}s\,\mathrm{d}\lambda.\end{aligned}$$

The domain of integration may be split into the ranges $|\lambda| < \delta$ and $|\lambda| > \delta$ to show that the integral is finite for $(2\beta - 1)(\alpha + \gamma) > \beta d$ and finite, positive T. It should also be clear that, for $\gamma(2\beta - 1) < \beta d$ and $(2\beta - 1)(\alpha + \gamma) > \beta d$, the Green function is square integrable on $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ since $G_1(t)$ is square integrable on $\mathbb{R}_+$ for $\beta > \frac{1}{2}$. To show that (27) is a solution, in the mean-square sense, of (5), we simply take the fractional derivative of $c(t, x)$ with respect to time. Noting that $(1 + |\lambda|^2)^{\alpha/2}|\lambda|^{\gamma}$ is the transfer function associated with $(I - \Delta)^{\alpha/2}(-\Delta)^{\gamma/2}$, that (17) is the Green function of fractional differential equation (16), and that the interchange of differentiation and integration is justified since both are defined in the mean-square sense, it follows that (27) solves (5). To determine the spectral density, we first determine the covariance function, which is given by

$$\mathrm{E}(c(t + \tau, x)\overline{c(t, y)}) = \int_{\mathbb{R}^d} \int_0^{\infty} \mathrm{e}^{\mathrm{i}\langle x-y,\lambda\rangle} G_{(1+|\lambda|^2)^{\alpha/2}|\lambda|^{\gamma}}(\tau + s) G_{(1+|\lambda|^2)^{\alpha/2}|\lambda|^{\gamma}}(s)\,\mathrm{d}s\,\mathrm{d}\lambda.$$

Equation (28) now follows from [8, Equation (3.15)], and (29) follows by integrating ω out of (28). The variance of the spatial increments at any fixed time t_0 is given by

$$\mathrm{E}((c(t_0, x + z) - c(t_0, x))^2) = (2\pi)^{d/2} \int_0^{\infty} \biggl(\frac{2^{(2-d)/2}}{\Gamma(\frac{1}{2}d)} - \frac{J_{(d-2)/2}(r|z|)}{(r|z|)^{(d-2)/2}}\biggr) f_{t_0}(r) r^{d-1}\,\mathrm{d}r,$$

where $J_\kappa(s)$ is the Bessel function of order κ. Application of the dominated convergence theorem yields

$$\lim_{|z|\to 0} \frac{\mathrm{E}((c(t_0, x+z) - c(t_0, x))^2)}{|z|^{(\alpha+\gamma)(2-1/\beta)-d}} = (2\pi)^{(d-2)/2}\bigg(\int_0^\infty \bigg(\frac{2^{(d-2)/2}}{\Gamma(\frac{1}{2}d)} - \frac{J_{(d-2)/2}(r)}{r^{(d-2)/2}}\bigg) r^{d-(\alpha+\gamma)(2-1/\beta)-1}\,\mathrm{d}r\bigg) \times \bigg(\int_{\mathbb{R}} \frac{\mathrm{d}u}{|u|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|u|^\beta + 1}\bigg).$$

Remark 8. The solution of (5) under random initial conditions will simply be the sum of (27) and the solution to the Cauchy problem without forcing noise, given in [3].

Remark 9. Other than with a brief comment in their Remark 3.3, Angulo *et al.* [2] did not investigate the decay of the asymptotic temporal covariance function in the case of an unbounded spatial domain. It can be seen that the covariance function decays more slowly than an exponential function and, if $d/\gamma < 2$, that the temporal process will exhibit long-range dependence. From [2, Proposition 3.3], it follows that the asymptotic temporal covariance function is given by

$$R_x(\tau) = \int_{\mathbb{R}^d} \frac{\exp\{-|\tau||\lambda|^\gamma(1+|\lambda|^2)^{\alpha/2}\}}{2|\lambda|^\gamma(1+|\lambda|^2)^{\alpha/2}}\,\mathrm{d}\lambda.$$

A change to spherical coordinates yields

$$R_x(\tau) = S_d \int_0^\infty \frac{\exp\{-|\tau|\rho^\gamma(1+\rho^2)^{\alpha/2}\}}{\rho^\gamma(1+\rho^2)^{\alpha/2}}\rho^{d-1}\,\mathrm{d}\rho,$$

where S_d is a constant resulting from the integration over the angular spherical coordinates. Making the change of variable $u = |\tau|\rho^\gamma$, we have

$$R_x(\tau) = S_d \int_0^\infty \frac{\exp\{-u(1+u^2/|\tau|^2)^{\alpha/2}\}}{\gamma(1+u^2/|\tau|^2)^{\alpha/2}} u^{d/\gamma-2}|\tau|^{1-d/\gamma}\,\mathrm{d}u.$$

From the dominated convergence theorem,

$$\lim_{|\tau|\to\infty} |\tau|^{d/\gamma-1} R_x(\tau) = \frac{S_d}{\gamma}\Gamma\bigg(\frac{d}{\gamma} - 1\bigg).$$

Thus, we have temporal long-range dependence governed by the spatial dimension and the exponent γ. We also recall that, in the bounded domain case, the temporal covariance decayed exponentially fast, which is another important difference between the bounded and unbounded domain cases. This is also important in view of the fact that we have started with an infinite-dimensional Ornstein–Uhlenbeck process and have shown that, for a fixed point in space, the solution exhibits long-range dependence.

Proposition 5. *Let $c(t, x)$ be the asymptotic stationary solution given in Proposition 4. For a fixed spatial location $x_0 \in \mathbb{R}^d$, the spectral density of $c(x_0, t)$ satisfies the following limits at $|\omega| = 0$, depending on the value of d/γ:*

$$\lim_{|\omega|\to 0} |\omega|^{\beta(2-d/\gamma)} f_{x_0}(\omega) = \frac{S_d}{2\pi} \int_0^\infty (1 + 2\cos(\tfrac{1}{2}\pi\beta)u^\gamma + u^{2\gamma})^{-1} u^{d-1}\,\mathrm{d}u, \qquad (2 - 1/\beta)\gamma < d < 2\gamma;$$

$$\lim_{|\omega|\to 0} \frac{f_{x_0}(\omega)}{\ln|\omega|} = -\frac{S_d\beta}{2\pi\gamma}, \qquad d = 2\gamma;$$

$$\lim_{|\omega|\to 0} \frac{f_{x_0}(0) - f_{x_0}(\omega)}{|\omega|^{\beta(d/\gamma-2)}} = \frac{S_d}{2\pi} \int_0^\infty \frac{2\cos(\frac{1}{2}\pi\beta)u^{d-1-\gamma} + u^{d-1-2\gamma}}{1 + 2\cos(\frac{1}{2}\pi\beta)u^\gamma + u^{2\gamma}}\,\mathrm{d}u, \qquad 2\gamma < d < 3\gamma;$$

$$\lim_{|\omega|\to 0} \frac{f_{x_0}(0) - f_{x_0}(\omega)}{|\omega|^\beta \ln|\omega|} = \frac{-S_d\beta\cos(\frac{1}{2}\beta\pi)}{\pi\gamma}, \qquad d = 3\gamma;$$

$$\lim_{|\omega|\to 0} \frac{f_{x_0}(0) - f_{x_0}(\omega)}{|\omega|^\beta} = \frac{S_d\cos(\frac{1}{2}\pi\beta)}{\pi} \int_0^\infty \frac{u^{d-1-3\gamma}}{(1+u^2)^{3\alpha/2}}\,\mathrm{d}u, \qquad d > 3\gamma.$$

Proof. Using the change to spherical coordinates, the asymptotic temporal spectral density of $c(x_0, t)$ is given by

$$f_{x_0}(\omega) = \frac{S_d}{2\pi} \int_0^\infty \frac{\rho^{d-1}\,\mathrm{d}\rho}{|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma + (1+\rho^2)^\alpha\rho^{2\gamma}}.$$

Applying the change of variable $u = \rho|\omega|^{-\beta/\gamma}$ gives

$$f_{x_0}(\omega) = \frac{S_d}{2\pi}|\omega|^{\beta(d/\gamma-2)} \int_0^\infty \frac{u^{d-1}\,\mathrm{d}u}{1 + 2\cos(\frac{1}{2}\pi\beta)(1+\omega^{2\beta/\gamma}u^2)^{\alpha/2}u^\gamma + (1+\omega^{2\beta/\gamma}u^2)^\alpha u^{2\gamma}}. \tag{30}$$

If $2\gamma > d$ then we may apply the dominated convergence theorem to the integral to conclude that

$$\lim_{|\omega|\to 0} \frac{f_{x_0}(\omega)}{|\omega|^{\beta(d/\gamma-2)}} = \frac{S_d}{2\pi} \int_0^\infty (1 + 2\cos(\tfrac{1}{2}\pi\beta)u^\gamma + u^{2\gamma})^{-1} u^{d-1}\,\mathrm{d}u.$$

For $2\gamma = d$, we split the domain of integration of (30) into $(0, 1]$ and $(1, \infty)$. The integral over the first interval is finite for all ω. The integral over the second interval is equal to

$$\int_1^\infty u^{-1}(1 + \omega^{2\beta/\gamma}u^2)^{-\alpha}\,\mathrm{d}u \tag{31}$$

$$- \int_1^\infty \frac{u^{-1}(1+\omega^{2\beta/\gamma}u^2)^{-\alpha}(1 + 2\cos(\frac{1}{2}\pi\beta)(1+\omega^{2\beta/\gamma}u^2)^{\alpha/2}u^\gamma)\,\mathrm{d}u}{1 + 2\cos(\frac{1}{2}\pi\beta)(1+\omega^{2\beta/\gamma}u^2)^{\alpha/2}u^\gamma + (1+\omega^{2\beta/\gamma}u^2)^\alpha u^{2\gamma}}. \tag{32}$$

The integral (32) remains finite as $|\omega| \to 0$. Applying the change of variable $v = \omega^{\beta/\gamma}u$ to the integral (31) gives

$$\int_{\omega^{\beta/\gamma}}^\infty v^{-1}(1 + v^2)^{-\alpha}\,\mathrm{d}v = -\frac{\beta}{\gamma}\ln|\omega| + O(1)$$

and, so, the proposition holds for $d = 2\gamma$.

When $3\gamma > d > 2\gamma$, we may write

$$f_{x_0}(\omega) = f_{x_0}(0) + \frac{S_d}{2\pi}\int_0^\infty \bigg\{\frac{1}{|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma + (1+\rho^2)^\alpha\rho^{2\gamma}} - \frac{1}{(1+\rho^2)^\alpha\rho^{2\gamma}}\bigg\}\rho^{d-1}\,\mathrm{d}\rho$$

$$= f_{x_0}(0) - \frac{S_d}{2\pi}\int_0^\infty \frac{2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta\rho^{d-1}\,\mathrm{d}\rho}{(|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma + (1+\rho^2)^\alpha\rho^{2\gamma})(1+\rho^2)^{\alpha/2}\rho^\gamma} \tag{33}$$

$$- \frac{S_d}{2\pi}\int_0^\infty \frac{|\omega|^{2\beta}\rho^{d-1}\,\mathrm{d}\rho}{(|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma + (1+\rho^2)^\alpha\rho^{2\gamma})(1+\rho^2)^{\alpha}\rho^{2\gamma}}. \tag{34}$$

Making the change of variable $u = \rho|\omega|^{-\beta/\gamma}$ and applying the dominated convergence theorem, we have

$$\lim_{|\omega|\to 0}\frac{f_{x_0}(0) - f_{x_0}(\omega)}{|\omega|^{\beta(d/\gamma-2)}} = \frac{S_d\cos(\frac{1}{2}\pi\beta)}{\pi}\int_0^\infty \frac{u^{d-1-\gamma}\,\mathrm{d}u}{1 + 2\cos(\frac{1}{2}\pi\beta)u^\gamma + u^{2\gamma}} + \frac{S_d}{2\pi}\int_0^\infty \frac{u^{d-1-2\gamma}\,\mathrm{d}u}{1 + 2\cos(\frac{1}{2}\pi\beta)u^\gamma + u^{2\gamma}}.$$

Now, for $d = 3\gamma$, we see that

$$\lim_{|\omega|\to 0}\frac{|\omega|^{-\beta}}{\ln|\omega|}\int_0^\infty \frac{|\omega|^\beta\rho^{d-1}(1+\rho^2)^{-\alpha/2}\rho^{-\gamma}\,\mathrm{d}\rho}{|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma + (1+\rho^2)^\alpha\rho^{2\gamma}} = -\frac{\beta}{\gamma},$$

using the same approach as for $d = 2\gamma$. The integral (34) remains bounded by $c|\omega|^{2\beta}$ as $|\omega| \to 0$, which can be seen from a change of variable and application of the dominated convergence theorem.

Finally, when $d > 3\gamma$ we apply the dominated convergence theorem directly to the integral (33), to find that

$$\int_0^\infty \frac{\rho^{d-1}\,\mathrm{d}\rho}{(|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma + (1+\rho^2)^\alpha\rho^{2\gamma})(1+\rho^2)^{\alpha/2}\rho^\gamma} \to \int_0^\infty \frac{\rho^{d-3\gamma-1}\,\mathrm{d}\rho}{(1+\rho^2)^{3\alpha/2}}$$

as $|\omega| \to 0$, provided that $d > 3\gamma$ and $d < 3(\alpha+\gamma)$. The latter inequality is automatically satisfied whenever a solution exists (see Proposition 4). For the integral (34), we note that

$$\int_0^\infty \frac{|\omega|^{2\beta}\rho^{d-1}\,\mathrm{d}\rho}{(|\omega|^{2\beta} + 2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma + (1+\rho^2)^\alpha\rho^{2\gamma})(1+\rho^2)^{\alpha}\rho^{2\gamma}}$$
$$= |\omega|^\beta\int_0^\infty \frac{\rho^{d-1}\,\mathrm{d}\rho}{(|\omega|^{\beta} + 2\cos(\frac{1}{2}\pi\beta)(1+\rho^2)^{\alpha/2}\rho^\gamma + |\omega|^{-\beta}(1+\rho^2)^\alpha\rho^{2\gamma})(1+\rho^2)^{\alpha}\rho^{2\gamma}}.$$

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This integrand is positive, and bounded for all $|\omega|$ by

$$\frac{\rho^{d-3\gamma-1}}{2\cos(\frac{1}{2}\pi\beta)(1+\rho^2)^{3\alpha/2}}.$$

Hence, applying the dominated convergence theorem, we conclude that the proposition holds for $d > 3\gamma$.

Remark 10. From Remark 9 it is clear that, when $\beta = 1$, the temporal covariance function is a mixture of exponentials and, hence, completely monotone. It is also completely monotone in the case of $\beta < 1$. The temporal covariance function is given by

$$c(\tau) = \int_{\mathbb{R}^d} \int_0^\infty G_{(1+|\lambda|^2)^{\alpha/2}|\lambda|^\gamma}(\tau+s) G_{(1+|\lambda|^2)^{\alpha/2}|\lambda|^\gamma}(s)\,\mathrm{d}s\,\mathrm{d}\lambda.$$

From Section 4 of [6], in spherical coordinates we have

$$\begin{aligned} c(\tau) &= \frac{S_d}{\pi} \\ &\quad \times \int_0^\infty \int_0^\infty \frac{\sin(\pi\beta)\mathrm{e}^{-|\tau|\omega}\omega^\beta\rho^{d-1}\,\mathrm{d}\omega\,\mathrm{d}\rho}{(\omega^\beta+\rho^\gamma(1+\rho^2)^{\alpha/2})(\omega^{2\beta}+2\cos(\frac{1}{2}\pi\beta)\omega^\beta\rho^\gamma(1+\rho^2)^{\alpha/2}\rho^{2\gamma}(1+\rho^2)^\alpha)} \\ &=: \int_0^\infty \mathrm{e}^{-|\tau|\omega}\sigma(\omega)\,\mathrm{d}\omega. \end{aligned}$$

The asymptotic behaviour of $c(\tau)$ as $|\tau| \to \infty$ is determined by the asymptotic behaviour of $\sigma(\omega)$ as $\omega \to 0$. An analysis similar to that in Proposition 5 yields

$$\lim_{\omega\to 0} \omega^{\beta(2-d/\gamma)}\sigma(\omega) = \int_0^\infty \frac{S_d\sin(\pi\beta)u^{d-1}\,\mathrm{d}u}{\pi(1+u^\gamma)(1+2\cos(\frac{1}{2}\pi\beta)u^\gamma+u^{2\gamma})},$$
$$(2-1/\beta)\gamma < d < 3\gamma;$$
$$\lim_{\omega\to 0}\frac{\sigma(\omega)}{\omega^\beta\ln\omega} = -\frac{S_d\beta\sin(\pi\beta)}{\pi\gamma}, \qquad d = 3\gamma;$$
$$\lim_{\omega\to 0}\frac{\sigma(\omega)}{\omega^\beta} = \int_0^\infty \frac{S_d\sin(\pi\beta)\rho^{d-3\gamma-1}\,\mathrm{d}\rho}{\pi(1+\rho^2)^{3\alpha/2}}, \qquad d > 3\gamma.$$

Hence, the covariance function satisfies the following limits:

$$\lim_{|\tau|\to\infty} c(\tau)|\tau|^{1+\beta(d/\gamma-2)} = \int_0^\infty \frac{S_d\sin(\pi\beta)\Gamma(1+\beta(d/\gamma-2))u^{d-1}\,\mathrm{d}u}{\pi(1+u^\gamma)(1+2\cos(\frac{1}{2}\pi\beta)u^\gamma+u^{2\gamma})},$$
$$(2-1/\beta)\gamma < d < 3\gamma;$$
$$\lim_{|\tau|\to\infty}\frac{c(\tau)|\tau|^{1+\beta}}{\ln|\tau|} = \frac{S_d\sin(\pi\beta)\beta\Gamma(\beta+1)}{\pi\gamma}, \qquad d = 3\gamma;$$
$$\lim_{|\tau|\to\infty} c(\tau)|\tau|^{1+\beta} = \int_0^\infty \frac{S_d\sin(\pi\beta)\Gamma(\beta+1)\rho^{d-3\gamma-1}\,\mathrm{d}\rho}{\pi(1+\rho^2)^{3\alpha/2}}, \qquad d > 3\gamma.$$

Proposition 6. *The asymptotic variance $\sigma_x^2(\tau)$ of the time increments of the solution $c(t, x)$ given in Proposition 4 satisfies*

$$\lim_{|\tau|\to 0} \frac{\sigma_x^2(\tau)}{|\tau|^{2\beta-1-\beta d/(\alpha+\gamma)}} = \frac{S_d}{2\pi}\int_{\mathbb{R}} |e^{iu}-1|^2\,|u|^{\beta(d/(\alpha+\gamma)-2)}\,du \int_0^\infty \frac{v^{d-1}\,dv}{1+2\cos(\frac{1}{2}\pi\beta)v^{\alpha+\gamma}+v^{2(\alpha+\gamma)}},$$

for $(2\beta-1)(\alpha+\gamma) > \beta d$.

Proof. We have

$$\sigma_x^2(\tau) = \frac{S_d}{2\pi}\int_{\mathbb{R}} |e^{i\tau\omega}-1|^2 \int_0^\infty \frac{\rho^{d-1}\,d\rho\,d\omega}{|\omega|^{2\beta}+2\cos(\frac{1}{2}\pi\beta)|\omega|^\beta(1+\rho^2)^{\alpha/2}\rho^\gamma+(1+\rho^2)^\alpha\rho^{2\gamma}}.$$

Applying the changes of variable $u = |\tau|\omega$ and $v = |\tau|^{\beta/(\alpha+\gamma)}\rho$, we obtain

$$\sigma_x^2(\tau) = \frac{S_d}{2\pi}\int_{\mathbb{R}} |e^{iu}-1|^2 \times \int_0^\infty \frac{|\tau|^{2\beta-1-d\beta/(\alpha+\gamma)}v^{d-1}\,dv\,du}{|u|^{2\beta}+2\cos(\frac{1}{2}\pi\beta)|u|^\beta(|\tau|^{2\beta/(\alpha+\gamma)}+v^2)^{\alpha/2}v^\gamma+(|\tau|^{2\beta/(\alpha+\gamma)}+v^2)^\alpha v^{2\gamma}}$$

and, from the dominated convergence theorem,

$$\lim_{|\tau|\to 0}|\tau|^{2\beta-1-d\beta/(\alpha+\gamma)}\sigma_x^2(\tau) = \frac{S_d}{2\pi}\int_{\mathbb{R}}|e^{iu}-1|^2\int_0^\infty \frac{v^{d-1}\,dv\,du}{|u|^{2\beta}+2\cos(\frac{1}{2}\pi\beta)|u|^\beta v^{\alpha+\gamma}+v^{2(\alpha+\gamma)}}.$$

A final change of variable $v = wu^{\beta/(\alpha+\gamma)}$ then completes the proof.

3.3. General fractional-in-time operators

In the previous two subsections, the extension of the results of [2] depends mainly on finding bounds on the integral of the squared Green function of the fractional differential equation (16), in terms of λ. If we were to replace the fractional-in-time derivative of (5) with the more general linear fractional-in-time operator

$$A_n\frac{\partial^{\beta_n}}{\partial t^{\beta_n}}+\cdots+A_1\frac{\partial^{\beta_1}}{\partial t^{\beta_1}}, \qquad 0<\beta_1<\cdots<\beta_n\le 1,$$

then we would have to bound the integral of the squared Green function of the fractional differential equation

$$A_n\frac{\partial^{\beta_n}}{\partial t^{\beta_n}}f(t)+\cdots+A_1\frac{\partial^{\beta_1}}{\partial t^{\beta_1}}f(t)+\lambda f(t)=\delta(t). \tag{35}$$

The Green function of (35) has a series representation in terms of the two-parameter Mittag-Leffler function and its derivatives (see [42]). Following [8], we note that the integral over $[0,\infty)$ of the squared Green function is given by

$$\int_{\mathbb{R}}\biggl(\sum_{j=0}^{n}\sum_{k=0}^{n}A_jA_k|\omega|^{\beta_j+\beta_k}\cos(\tfrac{1}{2}\pi(\beta_j-\beta_k))\biggr)^{-1}d\omega, \tag{36}$$

where $A_0 = \lambda$ and $\beta_0 = 0$. Clearly, it is necessary to have $\beta_n > \frac{1}{2}$ for (36) to be finite. Indeed, by a simple change of variable $u = \omega\lambda^{-1/\beta_n}$ we see that (36) equals

$$\lambda^{1/\beta_n - 2} \int_{\mathbb{R}} \left(\sum_{j=0}^{n} \sum_{k=0}^{n} A_j A_k \lambda^{(\beta_j+\beta_k)/\beta_n - 2} |u|^{\beta_j+\beta_k} \cos(\tfrac{1}{2}\pi(\beta_j - \beta_k)) \right)^{-1} \mathrm{d}u. \tag{37}$$

This is clearly finite for λ finite while, if we let $\lambda \to \infty$, the integral in (37) converges to

$$\int_{\mathbb{R}} (|u|^{2\beta_n} + 2\cos(\tfrac{1}{2}\pi\beta_n)|u|^{\beta_n} + 1)^{-1} \, \mathrm{d}u.$$

Hence, for the bounded rectangular domain case, Propositions 1 and 2, and (24) of Proposition 3, hold with β being replaced by β_n. In (23), the asymptotic, stationary temporal spectral density is given by

$$f_x(\omega) = \sum_{k \in \mathbb{N}_+^d} \phi_k^2(x) \left(\sum_i \sum_j A_i A_j |\omega|^{\beta_i+\beta_j} \cos(\tfrac{1}{2}\pi(\beta_i - \beta_j)) \right)^{-1},$$

where $A_0 = \lambda_k$ and $\beta_0 = 0$. Also, we note that Remark 3 still holds for $\beta_n \leq 1$. To see this, it is necessary to replace Theorem 1.3-5 of [19] with Theorem 1 of [6]. The role of β in (26) is now taken by β_1.

In the case of the unbounded spatial domain $\mathbb{R}^d$, for the existence of a solution it will be sufficient to require that

$$(2\beta_n - 1)(\alpha + \gamma) > \beta_n d.$$

This condition is derived from an argument similar to that in the proof of Proposition 4, using (36). For the solution to be asymptotically stationary, it is sufficient to impose the additional condition that $\gamma(2\beta_1 - 1) < \beta_1 d$, as can be seen by applying the change of variable $u = \omega\lambda^{-1/\beta_1}$ to (35). The spectral density function of the stationary solution is then given by

$$\left(\sum_{j=0}^{n} \sum_{k=0}^{n} A_j A_k |\omega|^{\beta_j+\beta_k} \cos(\tfrac{1}{2}\pi(\beta_j - \beta_k)) \right)^{-1},$$

where $A_0 = (1 + |\lambda|^2)^{\alpha/2}|\lambda|^{\gamma}$ and $\beta_0 = 0$. We note that (29) no longer holds under this more general model. For a fixed spatial location, the temporal covariance function is completely monotone by Theorem 1 of [6] and the arguments of Remark 9. The temporal covariance function satisfies Propositions 5 and 6 with β being replaced by β_1 and β_n in the respective propositions.

4. Non-Gaussian random fields

In this section, we consider equation (5) driven by infinitely divisible noise. We first recall some properties of these measures from [43]. Let S be an arbitrary nonempty set and $\mathcal{S}$ a σ-ring of subsets of S with the property that there exists an increasing sequence $\{A_n\}$ of sets in $\mathcal{S}$ with $\bigcup_n A_n = S$. Let $\{\Lambda(A) \colon A \in \mathcal{S}\}$ be a real stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$. We call Λ an *independently scattered random measure* if, for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$, the random variables $\Lambda(A_n)$, $n = 1, 2, \ldots$, are independent, and if $\Lambda(\bigcup_n A_n) = \sum_n \Lambda(A_n)$ almost surely, with $\bigcup_n A_n \in \mathcal{S}$. If $\Lambda(A)$ is infinitely divisible, then

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Λ is called an *infinitely divisible, independently scattered random measure*. The characteristic function of an infinitely divisible random measure $\Lambda(A)$ can be written in the Lévy form as

$$\hat{\phi}(\Lambda(A))(t) = \exp\left[\mathrm{i}t v_0(A) - \tfrac{1}{2}t^2 v_1(A) + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t\tau(x)) F_A(\mathrm{d}x)\right],$$

where v_0 is a signed measure, v_1 is a measure, and F_A is a Lévy measure for all $A \in \mathcal{S}$. Here, $\tau(x)$ is the centring function

$$\tau(x) = \begin{cases} x & \text{if } |x| \le 1, \\ x/|x| & \text{if } |x| > 1. \end{cases}$$

We will only be interested in the case where v_0, v_1, and F_A depend on A through its measure $\mu(A)$.

The stochastic integral of a deterministic function, with respect to Λ, has the usual construction, i.e. beginning with a sequence of simple functions $f_n \to f$, μ almost everywhere, the integral is defined as the limit in probability of $\int f_n \, \mathrm{d}\Lambda$. Conditions on f for the limit to exist are detailed in [43]. We will assume, in this section, that $\mathrm{E}|\Lambda(A)|^p < \infty$ for some $p > 2$. Hence, the integral can be interpreted as the limit in the mean-square sense, and a sufficient condition on f for the integral to exist is $f \in L^2$.

From the definition of the stochastic integral, a generalized random function, which we will call infinitely divisible noise, can be defined. In fact, let ε now be the generalized random function defined by

$$\varepsilon = (f, \varepsilon) = \iint f(t, x) \, \mathrm{d}\Lambda(t, x)$$

for $f \in C_0^\infty(\mathbb{R}^{d+1})$. As in the Gaussian case, ε is a mean-square continuous linear function with respect to the L^2-norm over $C_0^\infty(\mathbb{R}^{d+1})$ and, following [44], we may treat it as a random Schwartz distribution. Formally, ε is identified with the derivative of the infinitely divisible, independently scattered random measure and is called infinitely divisible noise.

Assuming finite second-order moments, the conditions for the existence of a solution to (5) driven by an infinitely divisible noise are given in Proposition 4. The solution can then be written as

$$c(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) \, \mathrm{d}\Lambda(s, y),$$

where $G(t, x)$ is the usual Green function. For $c(t, x)$ to have finite pth-order moments, it is sufficient that Λ have finite pth-order moments and that $G(t, x) \in L^p$. The following theorem gives the sufficient conditions on the Green function.

Proposition 7. *The Green function of (5) on $\mathbb{R}^d$ is in $L^p([0, T] \times \mathbb{R}^d)$ provided that $(q\beta - 1)(\alpha + \gamma) > \beta d$, where $1/p + 1/q = 1$.*

Proof. The Green function is given by the inversion of the Fourier transform, i.e.

$$G(t, x) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle \lambda, x \rangle} t^{\beta - 1} E_{\beta,\beta}(-(1 + |\lambda|^2)^{\alpha/2} |\lambda|^\gamma t^\beta) \, \mathrm{d}\lambda,$$

which is the Fourier transform of

$$t^{\beta - 1} E_{\beta,\beta}(-(1 + |\lambda|^2)^{\alpha/2} |\lambda|^\gamma t^\beta).$$

Assuming that $p \geq 2$, we have

$$\int_0^T \int_{\mathbb{R}^d} |G(t,x)|^p \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle \lambda, x\rangle} t^{\beta-1} E_{\beta,\beta}(-|\lambda|^\gamma (1+|\lambda|^2)^{\alpha/2} t^\beta)\,\mathrm{d}\lambda \right|^p \mathrm{d}x\,\mathrm{d}t.$$

By the Hausdorff–Young inequality, if q is such that $1/q + 1/p = 1$, then

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |G(t,x)|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_0^T c\left(\int_{\mathbb{R}^d} |t^{\beta-1} E_{\beta,\beta}(-|\lambda|^\gamma (1+|\lambda|^2)^{\alpha/2} t^\beta)|^q \,\mathrm{d}\lambda \right)^{p/q} \mathrm{d}t \\
&\quad = c \int_0^T t^{p(\beta-1) - d\beta p/q(\alpha+\gamma)} \left(\int_{\mathbb{R}^d} |E_{\beta,\beta}(-|\omega|^\gamma (t^{2\beta/(\alpha+\gamma)} + |\omega|^2)^{\alpha/2})|^q \,\mathrm{d}\omega \right)^{p/q} \mathrm{d}t \\
&\quad \leq c \int_0^T t^{p(\beta-1) - d\beta p/q(\alpha+\gamma)} \,\mathrm{d}t \left(\int_{\mathbb{R}^d} |E_{\beta,\beta}(-|\omega|^{\gamma+\alpha})|^q \,\mathrm{d}\omega \right)^{p/q}.
\end{aligned}$$

These integrals are finite if $2q(\alpha+\gamma) > d$ and $(p(\beta-1)+1)q(\alpha+\gamma) > p\beta d$. The first inequality is automatically satisfied if $G(t,x)$ is in L^2, while the second inequality can be simplified to that contained in the statement of the proposition.

As we are interested in the asymptotic stationary solution of (5), we will need the Green function to be in $L^p(\mathbb{R}_+ \times \mathbb{R}^d)$. It follows from Propositions 4 and 7 that this will be the case if the conditions of these propositions are satisfied. Assume that these conditions are satisfied and that $\mathrm{E}\,\Lambda(A) = 0$, this latter condition being necessary for processes displaying long-range dependence. Let $\kappa = \mathrm{E}(\Lambda^3(A))/\mu(A)$. As $\Lambda(A)$ is an independently scattered random measure, the third-order moments of the stationary solution are given by

$$\gamma_3(s_1, s_2, x_1, x_2) = \kappa \int_0^\infty \int_{\mathbb{R}^d} G(s_1+u, x_1+y) G(s_2+u, x_2+y) G(u,y)\,\mathrm{d}u\,\mathrm{d}y. \tag{38}$$

The integral (38) is finite under the assumptions of Proposition 7. The bispectrum is determined by

$$\begin{aligned}
&f(\lambda_1, \lambda_2, \omega_1, \omega_2) \\
&= \frac{1}{(2\pi)^{2d+2}} \int_{\mathbb{R}^{d+1}} \exp\{-\mathrm{i}(\lambda_1 s_1 + \lambda_2 s_2 + \omega_1 x_1 + \omega_2 x_2)\} \gamma_3(s_1, s_2, x_1, x_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned} \tag{39}$$

By interchanging the order of integration in (38) and (39), the bispectrum is then given by

$$f(\omega_1, \omega_2, \lambda_1, \lambda_2) = \frac{\kappa}{(2\pi)^2} \hat{G}(\omega_1, \lambda_1) \hat{G}(\omega_2, \lambda_2) \hat{G}(-\omega_1 - \omega_2, -\lambda_1 - \lambda_2),$$

where

$$\hat{G}(\omega, \lambda) = ((\mathrm{i}\omega)^\beta + |\lambda|^\gamma (1+|\lambda|^2)^\alpha)^{-1}.$$

Similarly, the higher-order cumulant spectra are given by

$$f(\omega_1, \ldots, \omega_p, \lambda_1, \ldots, \lambda_p) = \frac{\kappa_{p+1}}{(2\pi)^p} \hat{G}(-\omega_1 - \cdots - \omega_p, -\lambda_1 - \cdots - \lambda_p) \prod_{i=1}^p \hat{G}(\omega_i, \lambda_i),$$

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where κ_p, $p \in \mathbb{N}$, is the pth-order cumulant, determined as

$$\kappa_p = \mathrm{i}^{-p} \frac{\mathrm{d}^p \log \hat{\phi}(\Lambda(A))(t)}{\mathrm{d}t^p}\bigg|_{t=0} \times \mu(A)^{-1}.$$

It would be of interest to investigate the long-time behaviour of higher-order moments and their dependence on the spatial and temporal operators. We leave this problem to future research.

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