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INTEGRALLY CLOSED CONDENSED DOMAINS ARE BÉZOUT

BY

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ABSTRACT. It is proved that an integral domain R is a Bézout domain if (and only if) R is integrally closed and $IJ = \{ij | i \in I, j \in J\}$ for all ideals I and J of R; that is, if (and only if) R is an integrally closed condensed domain. The article then introduces a weakening of the "condensed" concept which, in the context of the $k + M$ construction, is equivalent to a certain field-theoretic condition. Finally, the field extensions satisfying this condition are classified.

1. **Introduction**. Condensed (commutative integral) domains were introduced by Anderson and Dobbs in [1]. Recall that a domain R is *condensed* if $IJ = \{ij | i \in I, j \in J\}$ for all ideals I and J of R. Bézout domains provide the most obvious class of condensed domains, but as ([1], Example 2.3) illustrates, a condensed domain need not be a Bézout domain. This article is motivated by the results in [1] that characterize condensed domains within certain larger classes of domains. Specifically, a GCD-domain is condensed if and only if it is a Bézout domain ([1], Proposition 2.12); a Prüfer domain is condensed if and only if it is a Bézout domain ([1], Corollary 2.6); and a Krull domain is condensed if and only if it is a PID ([1], Proposition 2.13). Since GCD-domains, Prüfer domains, and Krull domains are integrally closed, these results are immediate consequences of our

MAIN THEOREM. *An integral domain is a Bézout domain if (and only if) it is integrally closed and condensed.*

Seeking to sharpen the above characterization of Bézout domains, we introduce in Section 3 the notion of a quasicondensed domain. The precise definition of this weakening of the "condensed" concept (given below) is motivated by the nature of the proofs and examples in [1] and Section 2. It will be shown that there exists an integrally closed quasicondensed domain which is not a Bézout domain. More generally, the question of which D + M constructions lead to quasicondensed domains is reduced in Theorem 4 to the problem of characterizing a certain type of field extension. The underlying field theory is then treated in Theorem 5.

Throughout, R denotes a domain. Any unexplained terminology is standard, as in [2] and [5].

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2. **Proof of the main theorem**. We need only prove that any integrally closed condensed domain R is a Bézout domain. By ([1], Corollary 2.6), it suffices to show that R is a Prüfer domain. We do this by showing that R_M is a valuation domain for each maximal ideal M of R ([5], Theorem 64). By ([1], Proposition 2.4), each overring of a condensed domain is again a condensed domain. Thus each R_M is a condensed domain, and we have reduced to showing that any integrally closed quasilocal condensed domain is a valuation domain.

Let R be such an integral domain, and let z be a nonzero element in the quotient field of R. We must show that either z or z^{-1} is in R. Write $z = xy^{-1}$ where $x, y \in R$, and then set $I = (x^2, y^2)$ and $J = (x, y)$. Since $x^3 + y^3 \in IJ$ and R is condensed, there exist a, b, c, d in R such that

$$x^3 + y^3 = (ax^2 + by^2)(cx + dy) = acx^3 + adx^2y + bcxy^2 + bdy^3.$$

Collecting terms and dividing by y^3 yields

$$(1 - ac)z^3 - adz^2 - bcz + (1 - bd) = 0.$$

Since R is quasilocal, it is readily seen that at least one of the coefficients of this polynomial is a unit. Thus by the u, u^{-1}-Lemma ([5], Theorem 67), either $z \in R$ or $z^{-1} \in R$. □

3. **Quasicondensed domains**. We shall say that an integral domain R is *quasi-condensed* if $I^n = \{i_1 \dots i_n | i_j \in I \text{ for } 1 \le j \le n\}$ for each positive integer n and each two-generated ideal $I = (a, b)$ of R. It is clear that each condensed domain is quasi-condensed. However, an integrally closed quasicondensed domain need not be condensed. According to the Main Theorem, this last statement means that an integrally closed quasicondensed domain need not be a Bézout domain. An example which illustrates this is $R = \mathbb{C} + X\mathbb{C}(Y)[[X]]$. A major portion of this section will be devoted to establishing (a generalization of) this assertion. First, we find consequences of "quasicondensed" in two classical settings.

PROPOSITION 1. *If R is a Noetherian quasicondensed domain, then depth $(R) \le 1$.*

PROOF. This follows from the proof of ([1], Theorem 3.3) using the easily proved fact that any localization of a quasicondensed domain is again quasicondensed. □

PROPOSITION 2. *Let (R, M) be an integrally closed quasilocal quasicondensed domain, with residue field k $(=R/M)$. Then either R is a valuation domain, or k is algebraically closed.*

PROOF. Suppose that R is not a valuation domain. Choose $a, b \in R$ such that $I = (a, b)$ is not principal. For $d \in R$, let $\bar{d}$ denote its image in k. We shall show that any $f \in k[X]$ of degree $n \ge 1$ has a root in k. Write $f(X) = \sum_{j=0}^{n} \bar{\alpha}_j X^j \in k[X]$, for suitable $\alpha_j \in R$. As $\sum_{j=0}^{n} \alpha_j a^j b^{n-j} \in I^n$, the fact that R is quasicondensed allows us to write $\sum_{j=0}^{n} \alpha_j a^j b^{n-j} = (\beta_0 b + \beta_1 a)(\sum_{j=0}^{n-1} \gamma_j a^j b^{n-1-j})$ for suitable $\beta_j, \gamma_j \in R$. After multiplying and collecting terms, and then dividing by b^n, we find $\sum_{j=0}^{n} \delta_j (ab^{-1})^j = 0$, where $\delta_0 = \alpha_0 - \beta_0\gamma_0$, $\delta_n = \alpha_n - \beta_1\gamma_{n-1}$, and $\delta_j = \alpha_j - \beta_1\gamma_{j-1} - \beta_0\gamma_j$, for $1 \leq j \leq n - 1$. Since neither ab^{-1} nor ba^{-1} is in R, it follows from the u, u^{-1}-Lemma ([5], Theorem 67) that $\delta_j \in M$ for each j. Thus applying $(^{-})$ yields $\bar{\alpha}_0 = \bar{\beta}_0\bar{\gamma}_0$, $\bar{\alpha}_j = \bar{\beta}_1\bar{\gamma}_{j-1} + \bar{\beta}_0\bar{\gamma}_j$ for $1 \leq j \leq n - 1$, and $\bar{\alpha}_n = \bar{\beta}_1\bar{\gamma}_{n-1}$. Accordingly, one has the factorization $f(X) = (\bar{\beta}_1 X + \bar{\beta}_0)(\bar{\gamma}_{n-1}X^{n-1} + \ldots \bar{\gamma}_0)$ in $k[X]$, and the result follows. □

We next assemble material that will lead to a partial converse of Proposition 2.

PROPOSITION 3. *Let K/k be a field extension. Let $T = K + M$ be a quasi-local domain with maximal ideal M. Let R denote the (quasilocal) domain $k + M$. Let $I = (a, b)$ be a proper ideal of R, with $ba^{-1} = \alpha + m \in T$ ($\alpha \in K$, $m \in M$). Set $W = k + k\alpha$, viewed as a k-subspace of K. Then:*

(a) $I^n = W^n a^n + Ma^n$ for all $n \geq 1$.

(b) For $n \geq 1$, $I^n = \{i_1 \ldots i_n \mid \text{each } i_j \in I\}$ if and only if $\sum_{i=0}^{n} k\alpha^i = \prod_{i=1}^{n} (k + k\alpha)$.

PROOF. The proof of (a) is straightforward (cf. [3], Lemma 3.2).

(b) $(\Rightarrow)$: One inclusion holds in general. Conversely, let $\beta \in \sum_{i=0}^{n} k\alpha^i$. By (a), $\beta a^n \in I^n$, and so the hypothesized description of I^n yields $\beta a^n = (\beta_1 a + m_1 a) \ldots (\beta_n a + m_n a)$ for some $\beta_i \in k + k\alpha = W$ and some $m_i \in M$. Since $W^n a^n \cap Ma^n = 0$, $\beta = \beta_1 \ldots \beta_n \in \prod_{i=1}^{n} (k + k\alpha)$.

$(\Leftarrow)$: Let $x \in I^n \backslash \{0\}$. By (a), write $x = \beta a^n + ma^n$, for some $\beta \in W^n$, $m \in M$. We may assume that $\beta \neq 0$. Since $\beta = \beta_1 \ldots \beta_n$ for some nonzero elements $\beta_i \in W$, $x = (\beta_1 a) \ldots (\beta_{n-1} a)(\beta_n a + \gamma ma)$, where $\gamma = (\beta_1 \ldots \beta_{n-1})^{-1}$. In particular, $x \in \{i_1 \ldots i_n \mid \text{each } i_j \in I\}$. □

In view of Proposition 3, the next definition seems timely. An extension K/k of fields is said to satisfy property $(*)$ if $\sum_{i=0}^{n} k\alpha^i = \prod_{i=1}^{n} (k + k\alpha)$ for each $\alpha \in K$ and $n \geq 1$.

THEOREM 4. *Let V be a valuation domain of the form $K + M$, where K is a field and M ($\neq 0$) is the maximal ideal of V. Let k be a subfield of K, and set $R = k + M$. Then R is quasicondensed if and only if K/k satisfies $(*)$.*

PROOF. This follows easily from Proposition 3(b) in view of the following two observations. First, for nonzero a, $b \in R$, either ab^{-1} or ba^{-1} is in V since V is a valuation domain. Secondly, since M is nonzero, each nonzero $\alpha \in K$ may be written as $\alpha = ba^{-1}$ for some nonzero a, $b \in M$. (Choose $0 \neq m \in M$; let $a = m$ and $b = \alpha m$.) □

Thus, we have reduced the question of which $k + M$'s are quasicondensed to the question of which field extensions K/k satisfy $(*)$. Here is the answer to the latter question.

THEOREM 5. *An extension K/k of fields satisfies $(*)$ if and only if either*

(a) k is algebraically closed, or

(b) K/k is algebraic and for each $\alpha \in K$, each nonconstant $f \in k[X]$ with $\deg f < [k(\alpha):k]$ splits over k.

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PROOF. If $K = k$, then (*) and (b) hold, so we may assume that $K \neq k$. Choose $\alpha \in K \backslash k$. First, suppose that ($a$) holds. Then any $\sum_{i=0}^{n} a_i X^i \in k[X]$ factors as $b \prod_{i=1}^{n} (X - c_i)$ for some $b, c_i \in k$. Hence $\sum_{i=0}^{n} a_i \alpha^i = b \prod_{i=1}^{n} (\alpha - c_i)$. Next, suppose that ($b$) holds. Let $m = \deg(\alpha)$. As above, we then have $\sum_{i=0}^{n} k\alpha^i = \prod_{i=1}^{n} (k + k\alpha)$ for each $n < m$. For $n \geq m$, just note that $\sum_{i=0}^{n} k\alpha^i = \sum_{i=0}^{m-1} k\alpha^i$ and $\prod_{i=1}^{n} (k + k\alpha) \supset \prod_{i=1}^{m-1} (k + k\alpha)$ since $\deg(\alpha) = m$.

Conversely, suppose that K/k satisfies (*). If K/k is not algebraic, select $\alpha \in K$ transcendental over k; interpreting (*) in $k[X]$ ($\cong k[\alpha]$), we see that k is algebraically closed. Thus we may assume that K/k is algebraic. Let $\alpha \in K \backslash k$, with g the minimal polynomial of α over k; set $n = \deg g$. Let $f(X) = a_0 + a_1 X + \ldots + a_m X^m \in k[X]$, with $1 \leq m < n$. Since K/k satisfies (*),

$$a_0 + a_1\alpha + \ldots + a_m\alpha^m = (b_1 + c_1\alpha) \ldots (b_m + c_m\alpha)$$

for some $b_i, c_i \in k$. Let $h(X) = (b_1 + c_1 X) \ldots (b_m + c_m X)$. Then $f(X) - h(X)$ has α as a root, and so is divisible by g. Since $m < n$, necessarily $f(X) - h(X) = 0$, so $f = h$ splits over k. □

Combining Theorems 4 and 5 (with the Fundamental Theorem of Algebra), we see that $R = \mathbb{C} + X\mathbb{C}(Y)[[X]]$ is quasicondensed. It is well known that R is quasilocal and integrally closed, but not a Bézout domain (cf. [2], Exercise 11, page 202 and Exercise 13, page 286). While each localization of a quasicondensed domain is quasicondensed, an overring of quasicondensed domain need not be quasicondensed. (Contrast with the "condensed" case [1], Proposition 2.4.) Indeed, by Theorems 4 and 5, the overring $\mathbb{C}(Y^3) + X\mathbb{C}(Y)[[X]]$ is not quasicondensed.

If K/k is a field extension such that each $\alpha \in K$ has $[k(\alpha):k] \leq 2$, then clearly K/k satisfies (*). In particular, if $[K:k] = 2$, then K/k satisfies (*). The next remark investigates (*) in greater detail.

REMARK 6. (a) If an algebraic field extension K/k satisfies (*), then K/k is either separable or purely inseparable. To show this, we may assume that char $(k) = p > 0$. If $\alpha \in K$ is not purely inseparable over k, then $k(\alpha^{p^n}) = k(\alpha)$ for each $n \geq 1$ since K/k satisfies (*). Hence each element of K is either separable or purely inseparable over k ([4], Exercise 9, page 288). By [4], Exercise 4, page 288, the field extension K/k is either separable or purely inseparable.

(b) Suppose that k is not algebraically closed and that K/k is separable and satisfies (*). Then K/k is finite. For if K/k is infinite, then K has elements of arbitrarily high degree ([6], Lemma 1, page 194), contradicting (*). If K/k is a proper finite Galois field extension which satisfies (*), then $[K:k]$ is a prime. This follows easily from the Fundamental Theorem of Galois Theory and the Primitive Element Theorem.

Let K/k be a purely inseparable field extension with char $(k) = p$ which satisfies (*). Then $[k(\alpha):k] = p$ for each $\alpha \in K \backslash k$. However, unlike the separable case, such a purely inseparable field extension can be infinite dimensional. For example, let $k = \mathbb{Z}/2\mathbb{Z}(\{X_n^2\}_{n=1}^{\infty})$ and $K = \mathbb{Z}/2\mathbb{Z}(\{X_n\}_{n=1}^{\infty})$. Then K/k is an infinite purely inseparable field extension which satisfies (*) since each $\alpha \in K\backslash k$ has degree two over k.

(c) We show that for each prime p there is a Galois field extension K/k of degree p which satisfies (*). Our example is modeled after Artin's proof (cf. [6], Theorem 1, page 169) of the existence of algebraic closures. Let $k_0 = \boldsymbol{Q}$. For each $n \geq 0$, we define inductively a field extension k_n. If k_n has been defined, then k_{n+1} is the field generated over k_n by all elements of $\mathbb{C}$ which have degree strictly less than p over k_n. Then let $k = \bigcup_{n=1}^{\infty} k_n$ and $K = k(2^{1/p})$. Since the degree over $\boldsymbol{Q}$ of each $\alpha \in k$ is not divisible by p, $[K:k] = p$. By construction k contains all pth roots of unity; thus K is a splitting field for $X^p - 2$ over k and hence is Galois. Finally we show that K/k satisfies (*). Let $f \in k[X]$ have $1 \leq \deg f < p$. Then $f \in k_n[X]$ for some n, and hence by construction f splits over $k_{n+1} \subset k$.

By ([1], Proposition 2.5), each condensed domain has trivial Picard group, i.e., each invertible ideal is principal. As a closing question, we ask if, analogously, each quasicondensed domain also has trivial Picard group. If this were true, then by Proposition 1, an integrally closed quasicondensed Noetherian domain would be a PID. In closing, we do have the following partial result.

PROPOSITION 7. *Let I be an ideal of a domain R with $I^n = xR$ for some $n \geq 1$ and $x \in R$. If $I^n = \{i_1 \ldots i_n \mid \text{each } i_j \in I\}$, then I is principal.*

PROOF. By hypothesis, $x = i_1 \ldots i_n$ for some elements $i_j \in I$. Since each $(i_j) \subset I$ and $I^n = xR = (i_1) \ldots (i_n) \subset I^n$, it readily follows that $I = (i_j)$ for each $1 \leq j \leq n$. □

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