ON THE 'THERMODYNAMICS' OF SELF-GRAVITATING N-BODY SYSTEMS

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Abstract. Results of some simple 'thermodynamic' experiments on self-gravitating n-body systems are reported for a variety of boundary conditions. Systems placed in specularly reflecting enclosures did not show any unusual behavior, even though a variety of conditions was tried in an attempt to start a 'gravothermal castastrophe'. Similarly, there was no tendency to transfer energy between 'hot' and 'cool' subclosures within a given cluster. However, systems in 'isothermal' enclosures gave up energy to the enclosure at a surprisingly high rate, and sustained the energy-transfer rate as long as the experiment was continued. An explanation of these different behaviors was sought and found in an examination of the premises that underlie certain attempts to construct a thermodynamics for self-gravitating systems. Conventional application of the H-theorem implies violations of the n-body equations of motion and predictions not consistent with observation. Both the 'gravothermal catastrophe' and the experiments in an 'isothermal' enclosure share this violation of the equations of motion. A new formulation that allows for all the interactions in an n-body system shows that isolated n-body systems need not form binaries or condense into other subaggregates. The virial theorem follows as an ensemble average over the microcanonical ensemble.

1. Introduction

The work reported here represents a different approach to the study of systems of self-gravitating mass points from the mathematical methods you have heard so nicely described at this symposium by Moser and Pollard, among others. The mathematical approach is an attempt to describe *all* solutions of the n-body equations of motion; it must be capable of treating anything an n-body system can do, and it founders on the technicalities presented by singular cases. The methods of statistical mechanics provide an alternative approach by means of which it is possible to sidestep measure-zero effects like the singularities that plague the mathematical approach. This is accomplished by averaging over the parameter space (or phase space) in such a way that measure-zero effects are unimportant. The methods promise an overview without requiring detailed solutions. But the usual price is paid – equilibrium situations are stressed, or for nonequilibrium systems, the best that can be done is to describe the approach to equilibrium. Many of the obvious questions cannot be answered within this framework.

Not only does statistical mechanics seem to be a tool that should be useful in stellar dynamics, but stellar dynamics can provide a useful testing-ground for some unsolved problems in statistical mechanics. Galaxies, clusters of galaxies, and star clusters, are not uniform; they are groups of stars or galaxies that stand well separated from their neighbors. Under the presumption that what we see is some kind of equilibrium state, it is clear that equilibrium of a stellar dynamical system is *not* the uniform state of maximum entropy that characterizes equilibrium in most cases that we know how to treat by statistical mechanics. But many naturally occurring systems are nonuniform; when we look about us, very few of the systems that we see in nature

Y. Kozai (ed.), The Stability of the Solar System and of Small Stellar Systems, 261–272. *All Rights Reserved.*

are uniform. Biological systems, especially, are nonuniform. (For an interesting discussion of the statistical mechanics of biological systems, see Prigogine *et al.*, 1972). Thus stellar systems provide a challenge: they represent what is quite likely the simplest kind of naturally occurring system for which the equilibrium state is not uniform; they are certainly the simplest one-component system that is known to display this property. The study of stellar dynamical systems promises to provide clues, not only to the understanding of the beautiful objects we see in the sky, but also of things closer to home in which nonuniformity seems to be one of the principal attributes.

Unfortunately, the statistical approach turns out (for systems of negative total energy) to emphasize an uninteresting state in which the energy is concentrated in a binary with all the remaining $(n-2)$ particles at infinite separation, a situation that is not of much interest for stellar dynamics. The statistical method founders on a different technicality from the mathematical approach. The arguments leading to this conclusion will be reviewed in this paper.

2. *N*-Body Calculations

This program started from an attempt to study the 'thermodynamics' of a self-gravitating n-body system by using that system as the 'thermodynamic fluid' in a Carnot engine. The technique of numerical experiments was used, in which n-body systems were simulated in a computer. There are many respects in which the concepts of thermodynamic systems and of stellar dynamical systems are incompatible; these gave rise to technical difficulties that will not be gone into here. The essence of a Carnot engine is that the system must be placed in an enclosure whose walls can be made adiabatic or isothermal.

The system was placed in a box, which worked as follows. After each integration step, each particle was checked to see if it were outside the box. Any particle outside the box was reprojected into the box with a new kinetic energy randomly chosen to have an exponential distribution with mean value T. Changes in kinetic energy (and the change in angular momentum) were tallied. The particle was reprojected from the location it had when it was discovered to be outside the box. A cubic box, with boundaries at ± 1 in all three directions, was used. The box was endowed with a 'temperature', T, that governed the mean kinetic energy of the reprojected particles. It was also endowed with a 'heat capacity', according to which its temperature can change in proportion to the net energy interchange between the stellar system and the box. Most experiments were run with a rather large heat capacity, and the resultant temperature changes (for the box) were negligible. This approximated the isothermal enclosure.

It would be possible to design an adiabatic enclosure for these systems; however, adiabatic enclosures were not used for the experiments reported here.

A particularly simple condition results if the box temperature is zero. Any particle that ventures outside the boundary is stopped, its kinetic energy removed, and the particle is then released to fall from the point at which it was caught. This practically assures that the particle will fall back through the most dense part of the cluster, where it can interact very strongly with the remaining cluster members. This 'cold box' condition produced the interesting results shown in Figure 1, where the energy delivered to the box by the stellar system is shown as a function of time. The kinetic

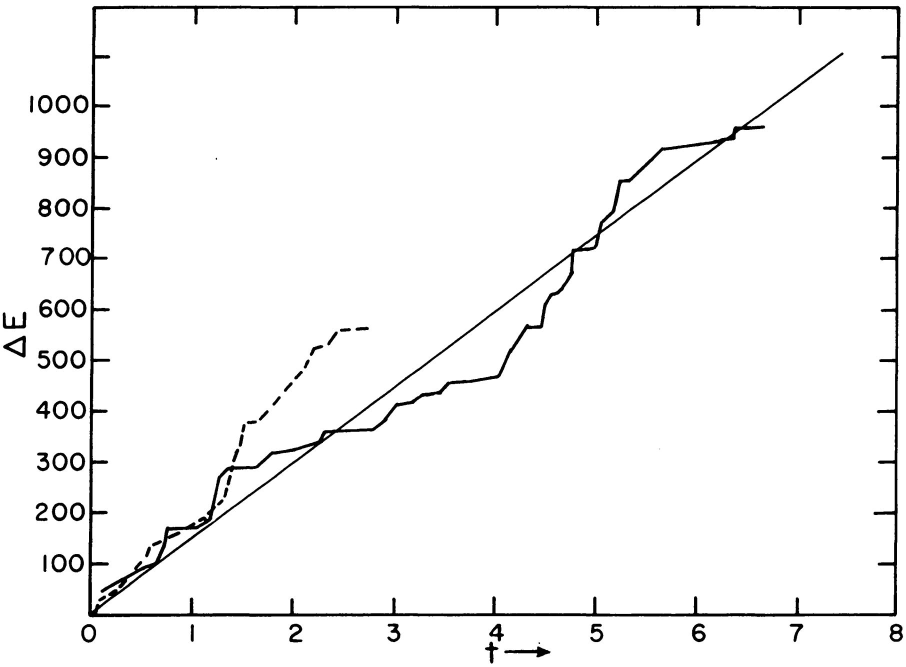

Fig. 1. Energy transfer from a 32-body system to a cold enclosure. The total initial kinetic energy was about 250. The time units on the abscissa were nearly a crossing time of the initial state. The two tracks represent two distinct calculations.

energy of the 32-body initial state was about 250, the potential energy about -500, and the total energy about -250. The time units of the abscissa were about one crossing time of the initial cluster.

The remarkable features of Figure 1 are the rate at which energy is given up to the box and the fact that the rate does not diminish appreciably as the process continues, even though the total amount of energy transferred is quite large (four times the total kinetic energy in the initial state). The solid curve represents one calculation – the dashed curve was another calculation from similar starting conditions for comparison purposes. The two curves give some idea of the reproducibility of the results from experiment to experiment.

The situation is qualitatively the same with the box at other temperatures: in Figure 2, the box temperature was about twice the mean kinetic energy of particles in the initial cluster. After a short dip, in which the cluster received energy from the box, the energy transfer began, and continued much as it did with the cold box, but at about half the rate. Apparently, the box prevents the potential energy of the cluster from becoming much larger (it is always negative, so a larger potential energy means that the particles are farther apart), while the hotter box increases the mean kinetic energy of the particles. After some time, the box and the cluster have about the same temperature, and then the transfer begins. It is not clear what determines the rate of energy transfer, once it has begun.

A different set of experiments was carried out in which the cluster was confined

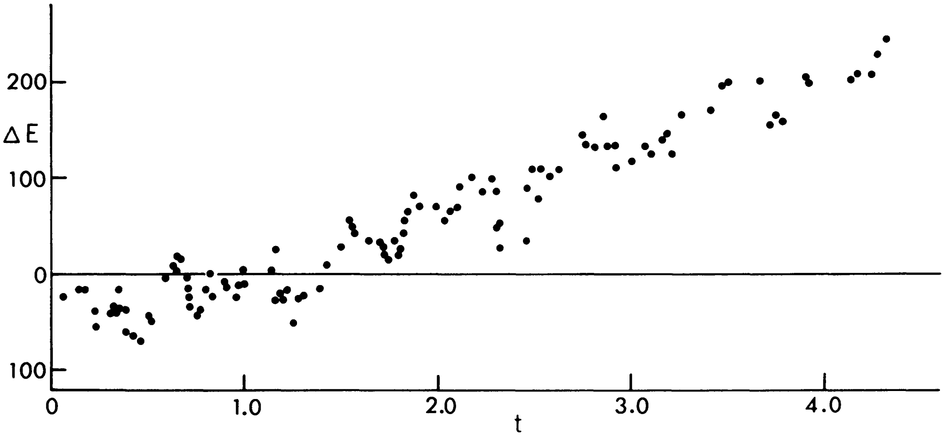

Fig. 2. Energy transfer from a 32-body system to an enclosure with nonzero temperature. The time units on the abscissa are nearly a crossing time of the initial state.

to a specularly reflecting spherical enclosure. No example tried has shown a behavior like that associated with the energy transfers of the cases run in the isothermal enclosure. The specular boundary condition prohibits energy (and angular momentum) transfer to the enclosure, but the transfers in the isothermal enclosure were associated with a shrinking of the cluster and an increase in kinetic energy that would be easy to identify if they did occur in the specular enclosure. Nothing surprising happened in the runs with the specular enclosure, except for the sharp contrast shown with the earlier runs with the isothermal box.

A third set of experiments was carried out in which the members of a given cluster were arbitrarily assigned to one of two subclusters for certain summaries such as total kinetic energy, total potential energy, and so on. The mechanism by which 'negative specific heat' is argued to lead to the formation of core-halo structures in the 'gravothermal catastrophe' (Lynden-Bell and Wood, 1968) is based on energy transfers between 'hotter' and 'cooler' subsets of particles within a given cluster. The terms 'hotter' and 'cooler' refer to particles with high and low kinetic energy respectively. One way to mimic these notions in a machine calculation is to separate the particles of an ordinary calculation into two classes according to their kinetic energy. This set of experiments makes use of a normal calculation for a star cluster, and is free of the unnatural boundary conditions of the other experiments. Whenever a fairly complete summary of cluster properties is run, the particles are sorted into two subsets according to kinetic energy. Along with the usual cluster properties, the kinetic and potential energy of each subcluster is tallied as if the rest of the particles did not exist; the potential energy of interaction of the two subclusters is also tallied. A given particle may move freely from one to the other of the two subsets. The separation applies only to the tallies; no distinction is made for the integration. This model appears to be very much in the spirit of discussions on 'negative specific heat' and of the 'gravothermal catastrophe'.

No strong tendency to transfer energy from the 'hot' to the 'cool' subcluster is evident in these experiments. The low-kinetic-energy subcluster nearly obeys the virial theorem as if the other subcluster were not present.

An alternative way of looking at these experiments that is free of some of the arbitrariness of division into two subclusters is to study the distribution of particle kinetic energy at various times. This distribution definitely did not show a tendency to segregate into extreme examples of high and low kinetic energy; high kinetic energies increased sharply during close encounters, but apart from this effect, no trend toward relatively distinct subsets of high- and low-kinetic-energy particles appeared. The experiments are consistent with the maintenance of an isothermal structure. More complete descriptions of the experimental results have been published elsewhere (Miller, 1973).

Among the various computer experiments, that with the isothermal enclosure is clearly the anomalous case. Both the examples with the specular reflector and those in which the system was regarded as if made up of two different subsystems showed no tendency to shrink into a small, 'hot' system that coexists with another extended, 'cool' system (core-halo structure). We anticipate the logical chain of arguments of this paper to comment on the reasons for this different behavior. The essential difference between the examples run with the isothermal enclosure and the other cases seems to be that the boundary conditions for the isothermal case are not describable by a Hamiltonian. Thus, there is no Liouville theorem for this case. By contrast, both the specular reflector and the subdivision of the particles into classes yield problems that are describable by Hamiltonians, and for which there is a Liouville theorem. This is manifestly the case for the subdivision; the specular reflector can be described as a Hamiltonian system by introducing an infinite (positive) potential at the position of the wall. The condensation into a compact system does not occur for the systems describable by a Hamiltonian; there is no such constraint on the problem with the isothermal enclosure. Similarly, the 'gravothermal catastrophe' proceeds in violation of the Liouville theorem (in $6n$ dimensions).

A collapse, if it is to occur, must do so on a secular (relaxation) time scale, and not on a dynamical time scale. Unfortunately, n-body calculations are not well suited to an examination of this question since the secular and dynamical time scales are not well separated for systems that contain fewer than about 1000 particles. The timescale of energy transfer to the enclosure in the experiments with the isothermal enclosure may have been dominated either by dynamical or by relaxation time-scales, or by still another (undefined) time-scale. However, it is suggestive that the rate of energy transfer did not change with the dynamical time-scale of the cluster as the cluster shrunk. As the cluster got smaller in the configuration space, and the dynamical time-scale got shorter in the units of the calculation, the rate of energy transfer stayed about the same. It would have decreased if measured in terms of the dynamical time-scale, but it would not decrease nearly as strongly if measured in terms of a relaxation time-scale. The energy transfer, and the shrinkage of the residual cluster, seems to have been dominated by the relaxation time scale, as expected.

It was not possible, in these experiments, to define a thermodynamic temperature of a stellar system by the operational method of placing the system in contact with a foreign body, and allowing the two to reach equilibrium. No equilibrium was reached. Again, this can be attributed to violations of the Liouville theorem. It might be possible to construct the 'heat bath' in such a way that the total system is describable by a Hamiltonian; there must then be a Liouville theorem for the total system. But the phase volume accessible for the stellar component need not be conserved. It thus appears to be impossible to design a system that would permit an operational definition of 'temperature' for a stellar system.

3. *H*-Theorem

The connection between statistical and thermodynamic descriptions is usually made through identification of the Boltzmann H with entropy, with the subsequent association of the H-theorem with the increase of thermodynamic entropy in irreversible processes; of through some other equivalent assumptions. This same connection, by means of the H-theorem, has been made for stellar dynamical systems (Antonov, 1962; Lynden-Bell and Wood, 1968), although it is unlikely that there is an H-theorem for stellar dynamical systems. Prigogine and Severne (1966) have shown that there is no H-theorem for infinite, spatially uniform, self-gravitating systems, but because such systems are Jeans-unstable, the question is still open for finite systems (Prigogine and Severne made this point in their paper).

With the usual projected distribution functions normalized to unity, the functions

$$\begin{aligned} H_B &= -\int \mathrm{d}^3x\, \mathrm{d}^3v\, f_1 \log f_1\,, \\ H_G &= -\int \mathrm{d}^{3n}x\, \mathrm{d}^{3n}v\, f_n \log f_n\,, \end{aligned} \qquad (1)$$

are the same as the definitions of 'entropy' used in discussions of information theory (Shannon, 1948; Khinchin, 1957). The subscripts B and G signify that these are essentially the Boltzmann and Gibbs H-functions of statistical mechanics. The special form of Equation (1) seems to have been invented to yield systems for which a variational calculation would produce Maxwellian velocity distributions. The function, H_G, is a constant of the motion as a consequence of the Liouville theorem, and so is usually regarded as not very interesting. An essential inequality is proved in books on information theory:

$$nH_B \geqslant H_G, \tag{2}$$

with equality if and only if the n-particle distribution can be written as a product of single-particle distributions:

$$f_n(1, 2, \ldots, n) = f_1(1) \cdot f_1(2) \cdot \ldots \cdot f_1(n). \tag{3}$$

It is straightforward to extend this inequality to show that increases in H_B require increased deviation from the form of Equation (3) in the sense of requiring more correlation, in order to be consistent with the Liouville theorem (and thus with the constant value for H_G). We will not go into the details here, as the argument is presented in a forthcoming publication (Miller, 1974).

There is, in general, no limit to the allowed increase in H_B through increased correlation. In the usual problems of statistical mechanics, this increase in correlation appears in momentum space along with an increase in the mean square particle momentum (or kinetic energy). Equilibrium, as defined by the H-theorem, occurs when the kinetic energy of the particles has reached the maximum value attainable, a maximum being assured by the assumption of a finite bound to the amount of energy available from the potential energy (internal energy) sources available to the system. There is no such bound for stellar dynamical systems, so other means must be used to halt the process if a finite limit is to be reached. The particle correlation does not mean that particles are necessarily close together; rather it implies that knowledge of the phase of one particle tells something about where other particles are to be found. Even if there is no upper bound to values of H_B, as is the case in stellar dynamics, evolution under the H-theorem should proceed toward states of stronger correlation, and away from uncorrelated states such as those described by Equation (3). If a maximum for H_B can be attained, it is reached by maximizing correlations.

Particle correlations make a contribution to the potential energy of a stellar system. With pair correlations written as an additive part to the two-particle distribution:

$$f_2(1, 2) = f_1(1) \cdot f_1(2) + g(1, 2), \tag{4}$$

the total potential energy breaks up into the sum of two pieces. There is the usual part obtained from the f_1's

$$V_1 = -Gm^2 \frac{n(n-1)}{n} \int d^3x^{(1)}\, d^3v^{(1)} f_1(1) \int d^3x^{(2)}\, d^3v^{(2)} \frac{f_1(2)}{|x^{(2)} - x^{(1)}|}, \tag{5}$$

and a second part due to the pair correlation:

$$V_c = -Gm^2 \frac{n(n-1)}{2} \int d^3x^{(1)}\, d^3v^{(1)} \int d^3x^{(2)}\, d^3v^{(2)} \frac{g(1, 2)}{|x^{(2)} - x^{(1)}|}. \tag{6}$$

This is not an idle exercise: real stellar systems show appreciable pair correlation, just as do n-body systems in a computer (Miller, 1971, 1972). The contribution of V_c to the total energy can become quite large (half of the total potential energy or more).

Any argument in which V_c is ignored in the bookkeeping for the energy is tantamount to the assumption of an n-particle distribution of the form of Equation (3). While it is possible to construct n-particle distributions that yield zero results for V_c, these distributions are sufficiently pathological that a rather strong argument is required to justify ignoring the contribution of V_c to the total energy of a stellar system. It is not a completely trivial exercise to construct f_n's that yield negligible values for V_c. Arguments to the effect that the H-theorem correctly indicates evolutionary trends in stellar systems, but in which V_c is ignored in calculating the total system energy, are internally inconsistent because those arguments, on the one hand, call for increased particle correlation while, on the other hand, one of the principal results of particle correlation is ignored.

Indications that correlations are important in self-gravitating systems are not new. Prigogine and Severne (1966) constructed a weak coupling kinetic theory of binary interactions which predicted an irreversible growth of correlational energy. Their model was spatially uniform and of infinite extent, and the arguments did not hinge on H_B. Similarly, Chandrasekhar has repeatedly stressed the dependence of the potential energy on pair configuration distributions (Chandrasekhar and Lee, 1968; Chandrasekhar and Elbert, 1971).

If actual stellar systems evolved toward increased H_B, they would undergo a secular trend toward states characterized by a preponderance of binaries, since this is the easiest way to increase correlation. There is no observational evidence that cluster evolution proceeds in this way. Evolution towards greater values of H_B also runs against the virial theorem. As correlation increases, the magnitude of the potential energy increases; the total kinetic energy, K, must also increase in order to conserve total energy. The ratio, $-V/K$, must approach 1, not the value of 2 appropriate to the virial theorem. Both observational data and n-body calculations seem to confirm the 2 of the virial theorem over the 1 of a variational calculation based on the H-theorem.

Evolution toward greater values of H_B does not seem to provide a useful technique for studying stellar system's on two grounds: because H_B can become infinite without local maxima, it does not predict stable equilibria; and the evolution predicted is not in agreement with observation. It also does not predict metastable equilibria, but this requires arguments not presented in this paper (Miller, 1973). At any rate, such evolution, far from being a 'catastrophe', must proceed on a secular (relaxation) timescale. Note that nothing has been said about those versions of the H-theorem that make use of coarse-graining, and that we have *not* shown that there is no H-theorem for self-gratitating systems.

4. The Microcanonical Ensemble

An alternative approach to the statistical mechanics of a stellar dynamical system is through the construction of one of the ensembles commonly used in statistical mechanics. The microcanonical ensemble is the only ensemble appropriate to stellar systems, which are presumed to obey the n-body equations of motion and to conserve total energy. The canonical and grand canonical ensembles are not suitable because the notion of a heat bath is alien to stellar systems and because the long range forces preclude treatment of subdivisions of the total system. The difficulties with numerical experiments, described in Section 2, underline these features.

The microcanonical ensemble is a useful illustration, and brings out some unexpected features of stellar dynamical systems. The volume of (the $6n$-dimensional phase space included in) the region between two neighboring hypersurfaces of total energy E and $E+\mathrm{d}E$ is $\sigma(E, R)\,\mathrm{d}E$, with

$$\begin{aligned}\sigma(E, R) &= C_N(R)\int_{-E}^{\infty} \mathrm{d}(-V)\,(-V)^{2-3N}(E-V)^{(3N-5)/2} = \\ &= \text{const.}\,(R^3)^{N-2}(-E)^{(1-3N)/2}.\end{aligned} \tag{7}$$

The coefficient $C_N(R)$ arises because this phase volume becomes infinite if infinite configuration volume is available; the R appearing in Equation (7) is a cutoff radius, inside which the entire cluster is presumed to lie. The integrand of Equation (7) may be regarded as a probability distribution function for $(-V)$. More negative values of V lead to rapidly growing phase volumes in the momentum space through the $(E-V)^{(3N-5)/2}$ term. Since $(-V)$ may become infinite by letting two or more particles come close together in the configuration space, an infinite volume (at quite a high order infinity!) opens up in the momentum space. The discussion is usually terminated at this point with the observation that there is infinite phase volume because of close binaries. However, the configuration volume available for such particle aggregates diminishes faster than the momentum space opens up, leaving a net decrease in the available phase volume. The most probable values for $(-V)$ are those in which the two terms just play off against each other. This yields a virial theorem. The details of this calculation appear elsewhere (Miller, 1973, 1974).

The integral of Equation (7) is the leading term of a sequence of similar terms that arise from the calculation of the configuration volume inside a surface $(-V)=$constant. But the volume inside such a surface can be infinite because an arbitrarily large amount of potential energy is available by letting two particles come close together; all the remaining particles can then be removed to infinity. All but three particles can be removed to large distance, but the volume inside $(-V)=$constant is infinite for three particles by leaving two close together and removing the third, and so on. The contributions of all these combinations may readily be summed. The growing infinity must soon overtake any finite contribution from interesting parts of the $(-V)$

surfaces near the origin; thus for large enough distances (R large), the volume associated with $d(-V)$ tends asymptotically to the form given in Equation (7). The remaining terms of the sequence, which have been ignored in Equation (7) are those with three particles near each other, with two sets of binaries, with four particles, and so on. Thus the phase volume in the microcanonical ensemble is dominated by a state with a single binary having all the energy, with the other particles at rest at infinite separation, the uninteresting state referred to earlier.

The expression for phase volume in Equation (7) can be taken into one of the usual definitions of 'entropy' through the relation $\exp(S)=\sigma(E)$ (see, for example, Landau and Lifshitz, 1969; but also note the admonitions in their Section 8 on the inapplicability of statistical methods to problems involving gravitation). From this, a 'thermodynamics' can be constructed, yielding a (positive) 'temperature', $T=2(-E)/(3N-1)$, related to the total system energy and not to the kinetic energy alone. The specific heat associated with this temperature is negative. Other thermodynamic functions may be worked out as well, but that does not seem to be a fruitful undertaking.

Dynamical formation of binaries is not a preferred process in this formulation: systems do not tend to form many binaries because there is not a preponderance of phase volume accessible. The same argument applies to higher particle aggregates as well. The principle underlying these assertions is that situations with more phase volume accessible are more probable; this seems to be one principle of statistical mechanics that it should be possible to carry over into stellar dynamics with some confidence. The results quoted here apply to cases where all particles have equal mass. With unequal masses, there is considerably more phase volume available to a configuration in which the two most massive particles form a binary; the tendencies in this direction, as described at this meeting by Heggie, might profitably be considered from this viewpoint.

In the study of star clusters, we do not want a true ensemble average or (assuming ergodicity) a true time-average; either of these would only tell about the state with a single binary and infinite dispersion of the other particles. The desired averages are over systems with most of the particles near the origin. For example, the 'virial theorem' that can be obtained from this formulation actually results from the lack of interaction of the particles removed to infinite dispersion. All the energy (both kinetic and potential) resides in the binary; the other particles make no contribution. But binary systems are known to obey the virial theorem, so the stated result is not surprising. However, the formulation also yields expressions for the probability distribution of the virial ratio as a function of particle number; this is an interesting result.

The development of the microcanonical ensemble does not proceed in the direction that would cause the most rapid increase of H_B, as might be expected if the H-theorem were valid. The argument is based on the nature of the infinity in the phase volume. Larger values of H_B imply that there are conditional probabilities such that some knowledge of the state of the system allows more precise statements to be made about the total state than could be made in the absence of that knowledge. But the final state of the microcanonical ensemble is the antithesis of that condition: the preponderance of phase volume is dominated by states such that knowledge of the phase of one particle only carries information of order $(1/n)$ about which particle should be near some other particle. While H_B might become arbitrarily large with correlation order of $(1/n)$, it could become much larger still with correlations of order unity (in the particle number).

The microcanonical ensemble does not provide a valid counterexample to the H-theorem. However, the H-theorem is not useful if there is no maximum to H_B (so that no terminal equilibrium state can be predicted) and the evolution does not even proceed in such a way that the state of the system can be correctly predicted at later times by solving for a maximum of H_B while H_B is still finite.

There is a vaguely unreal feeling to the arguments based on ensemble averages over the microcanonical ensemble, and the time averages that are equivalent if the system is ergodic. The arguments leading to the assertion that larger phase volumes are more probable imply that there is some mechanism available to permit the system to evolve in that direction even if it has started out in some other direction. For example, suppose the system formed *two* binaries at some early stage. Then the system begins to dissolve. The dynamics says that if all the other particles are essentially infinitely far apart, and if the two binaries are similarly at infinite separation, there is no way that the two binaries can exchange energy to reach the more probable terminal state with only one binary. But it is equally difficult to imagine that the system, while it is still relatively compact with interactions available to redistribute energy among the constituent parts, can know that it should arrange itself to have only one binary, in order to be properly prepared for future developments. Further, the arguments, like those of the mathematical treatment of n-body systems, presuppose infinite time, and so allow states of little physical interest to dominate the ensemble or time-averages. The time-scales for development of such systems is unrealistically long from a physical point of view.

No equilibrium solution has been found by any of the methods used here. Some equilibrium solutions are known, but they represent a 'set of measure zero' relative to all possible configurations, so they might understandably escape detection by these methods. The criteria used to define equilibrium may be too strict; certainly it is not realistic to demand stability over times greatly in excess of the age of the universe. It seems likely that we are faced with a situation in which there may be no equilibrium in the mathematical sense, but in which the natural processes leading to the dissolution of clusters are so slow that clusters represent an equilibrium in a practical sense. If so, we need mathematical tools that allow us to calculate the properties of interesting subsets of all possible systems – those with all the particles near each other, but without need to be preoccupied with pathological collisions. An alternative is that other processes (dissipation due to interstellar gas, etc.) might be more important than we have believed them to be, and that some other mechanism may be responsible for the apparent equilibria observed in natura. The challenging puzzle still stands.

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