18

Integrability in four-dimensional gauge dynamics

Integrability was discussed in Chapter 5 in the context of two-dimensional models. In particular solutions of spin chain models based on the Bethe ansatz approach were described in some detail. Integrable models are characterized by having the same number of conserved charges as the number of physical degrees of freedom. Furthermore, the scattering processes of those models always involve conservation of the number of particles.

A natural question at this point is whether integrability is a property of only two-dimensional models or whether one can also identify systems in four dimensions that admit integrability. Four-dimensional gauge theories have generically infinite numbers of degrees of freedom and their interactions do not conserve the number of particles. Thus four-dimensional gauge theories like the YM theory are not integrable theories. However, it turns out, as will be shown in this chapter, that various sectors of certain four-dimensional gauge theories, which are derived upon imposing certain limits, do admit integrability.

The two-dimensional integrable models discussed in Chapter 5 were non-conformal ones and were characterized by a scale and hence also with particles and an S-matrix. On the other hand the integrable sectors of four-dimensional gauge theories that we are about to describe are conformal invariant. The main idea is that these special conformal invariant sectors can be mapped into two-dimensional spin chains that were described in Section 5.14.

The investigation of this issue is far from complete. Nevertheless, a large body of knowledge has already been accumulated. In recent years this has followed the lines of the AdS/CFT duality [158] which is not covered in this book.[1] The purpose of this section is just to demonstrate the idea of the map between gauge theories and in particular QCD and integrable spin chain models. This will be done by describing the following cases:

(i) $\mathcal{N} = 4$ super YM theory in four dimensions.
(ii) Scale dependence of composite operators in QCD.

$\mathcal{N} = 4$ super YM theory is known to be the maximal global supersymmetric theory in four dimensions. Since supersymmetry is beyond the scope of this book we will not discuss it in the context of the $\mathcal{N} = 4$ SYM. Thus the description

[1] For a review of the AdS/CFT the reader can refer to [10].

of the integrable aspects of the theory will be incomplete and will be missing certain essential parts. However, since $\mathcal{N}=4$ SYM is the simplest interacting four-dimensional non-abelian gauge theory we start with this and then proceed to a certain limit in non-supersymmetric QCD.

There are several review papers on integrability in four-dimensional gauge dynamics. In this chapter we follow [31] about the integrability of $\mathcal{N}=4$ SYM theory and [34] for the scale dependence of composite operators of QCD.

18.1 Integrability of large N four-dimensional $\mathcal{N}=4$ SYM

The Lagrangian of $\mathcal{N}=4$ SYM is given by,

$$\mathcal{L}_{\mathcal{N}=4} = -\frac{1}{4}\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \frac{1}{2}\mathrm{Tr}\left[D_\mu\Phi_n D^\mu\Phi^n\right] - \frac{1}{4}g^2\mathrm{Tr}\left[\left[\Phi^m,\Phi^n\right]^2\right]$$
$$+\mathrm{Tr}\left[\dot{\psi}^a_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_\mu D^\mu\psi_{\beta a}\right] - \frac{i}{2}g\mathrm{Tr}\left[\psi_{\alpha a}\sigma^{ab}_m\epsilon^{\alpha\beta}\left[\Phi^m,\psi_{\beta b}\right]\right] - \frac{i}{2}g\mathrm{Tr}\left[\dot{\psi}^a_{\dot{\alpha}}\sigma^m_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}\left[\Phi_m,\dot{\psi}^b_{\dot{\beta}}\right]\right], \tag{18.1}$$

where $F_{\mu\nu}$ is the field strength associated with an $SU(N)$ gauge group, Φ^m is a set of six $m=1,\ldots,6$ scalar fields, and ψ and $\dot{\psi}$ are doublets of $SU(2)\times SU(2)$. Both the scalars and the spinors are in the adjoint representation of $SU(N)$. The matrices σ^μ and σ^m are the chiral projections of the gamma matrices in four and six dimensions, respectively and ϵ is the totally antisymmetric tensor of $SU(2)$. It is convenient to write the corresponding action as,

$$S = N\int\frac{\mathrm{d}^4x}{4\pi^2}\mathcal{L}_{\mathcal{N}=4}, \tag{18.2}$$

where the coupling constant is taken to be $g^2 \equiv \frac{g^2_{YM}N}{8\pi^2}$.

It is well known that the theory, on top of being invariant under $SU(N)$ gauge symmetry and $SO(6)$ global symmetry, is also conformal invariant and in fact superconformal invariant. The β function of the theory which vanishes to all orders in perturbation theory is believed to vanish also non-perturbatively and hence the theory is assumed to be conformal also in the quantum level. In Section 17.1 we have described the conformal symmetry algebra in four dimensions. Recall the $SO(2,4)$[2] conformal transformations (see 17.7) which are being generated by P^μ, $S^{\mu\nu}$, $\mathcal{D}$, K^μ, the generators of space-time translations, Lorentz transformations, dilation and special conformal transformation, respectively.

A major player in the structure of the $\mathcal{N}=4$ is the *dilatation operator* $\mathcal{D}$. Whereas the generators of the Poincare group do not get quantum corrections, the dilatation operator does so that in fact,

$$\mathcal{D} = \mathcal{D}_0 + \delta\mathcal{D}(g), \tag{18.3}$$

[2] In fact the $\mathcal{N}=4$ SYM admits a superconformal algebra of $psu(2,2|4)$ which we do not discuss here.

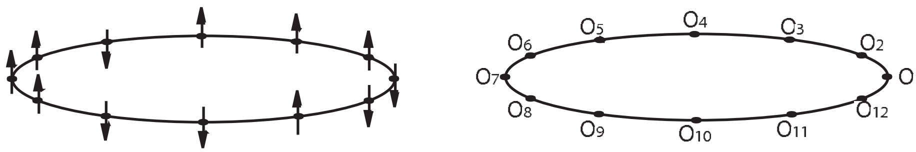

Fig. 18.1. A single trace operator as a spin chain.

where D_0 is the classical operator and δD is the anomalous dilation operator which obviously depends on the gauge coupling g.

The "*states*" of the theory take the form of multi-trace gauge invariant operators,

$$\mathrm{Tr}\,[\mathcal{W}\ldots.\mathcal{W}]\ldots\mathrm{Tr}\,[\mathcal{W}\ldots.\mathcal{W}] \quad \mathcal{W} \in \{D^k\Phi, D^k\Psi, D^k\dot{\Psi}, D^k F\}, \tag{18.4}$$

where D stands for the covariant derivative and $F \equiv F_{\mu\nu}$ is the field strength. The Hilbert space of states is built, as for any conformal field theory, from Verma modules each characterized by a *highest weight state* or a *primary state*, which were defined in (2.8). An example of a highest weight state is $|\mathcal{K}\rangle = \eta^{mn}\mathrm{Tr}[\Phi_m\Phi_n]$. The rest of the Verma module includes the descendant states which are derived by acting with lowering operators on primary states. Needless to say the general structure of correlation functions of four-dimensional conformal field theories discussed in Section 17.6 applies also for the case of the $\mathcal{N} = 4$ SYM. In particular recall (2.8) that the two-point function of two operators is given by,

$$<\mathcal{O}(x_1)\mathcal{O}(x_2)> = \frac{M(g)}{|x_1 - x_2|^{2\mathcal{D}(g)}}. \tag{18.5}$$

The anomalous dimension can be computed perturbatively as a power series in g. As was discussed in Chapter 7 the perturbation expansion becomes much more tractable in the large N limit, namely, in the planar limit. In this limit the dominant diagram has a vanishing Euler number $\chi = 2C - 2G - T = 0$ where C, G, T stand for the number of components, genus, namely the number of handles, and the number of traces, respectively. Since each component requires two traces, one incoming and one outgoing, it implies that the planar limit projects onto diagrams with $G = 0$ and $T = 2C$. This means that only single trace operators are relevant.

We have seen above that in the planar limit we deal with single trace operators. Pictorially, (see Fig. 18.1) a single trace operator looks like a cyclic spin chain. This map can be made precise. Spin chain as integrable models were discussed in Section 5.14. Recall that a spin chain includes a set of L spins with cyclic adjacency property.[3] The spin at each site is a module of the symmetry algebra of the system. The Hilbert space of the whole system is the tensor product of L modules. In Section 5.14 we discussed only chains with a fixed number

[3] In Section 5.14 we denoted the number of spins by N. Here to avoid confusion with the rank of the gauge group we will refer to the number of spins as L.

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Table 18.1. *$\mathcal{N} = 4$ SYM theory to spin chain dictionary*

planar $\mathcal{N} = 4$ SYM	spin chain
Single trace operator	Cyclic spin chain
Field operator	Spin at a site
Anomalous dilatation operator $g^{-2}\delta\mathcal{D}$	Hamiltonian
Anomalous dimension	Energy eigenvalue
Cyclicity constraint	Zero momentum condition $U = 1$

of spins. One can generalize this situation to incorporate also a *dynamic* spin chain with an unfixed number of spins. In this case the Hilbert space is a tensor product of all Hilbert spaces of a fixed length. In the Heisenberg model each spin has two possible states and the Hilbert space is therefore $\mathcal{C}^{(2^{(L)})}$. In general the spin in the chain can point in more than two directions and in particular also in infinitely many directions, as is the case for the spin chain of the $\mathcal{N} = 4$ SYM theory. In the latter case the spin is mapped into a field operator and the possible spin states to the components of the gauge symmetry multiplet. The cyclicity of the single trace operators maps into a constraint on the spin chain so that states that differ by a trivial shift are identified and hence states with non-trivial momentum are unphysical. In the language of Section 5.14 we have to impose $U = 1$ as a constraint. In the Heisenberg model this renders the Hilbert space into $\frac{\mathcal{C}^{(2^{(L)})}}{Z_L}$. The Hamiltonian of the spin chain model translates into the dilatation operator and the energy eigenvalues to the anomalous dimensions. The full correspondence between the spin chain and the planar limit of the $\mathcal{N} = 4$ SYM theory is summarized in Table 18.1.

Once the correspondence with a spin chain model has been established, one can proceed in a similar way as for the Heisenberg spin chain model. The next step is to write down the algebraic Bethe ansatz which now corresponds to an $SO(6)$ symmetry if one considers operators constructed only from the fields Φ_m or in general the $psu(2,2|4)$ for the full $\mathcal{N} = 4$ SYM theory. The algebraic Bethe ansatz, the analog of (18.6) now reads as follows,

$$\left(\frac{\lambda_k + i/2V_{jk}}{\lambda_k - i/2V_{jk}}\right)^L = \prod_{l\neq k}^{K} \frac{\lambda_k - \lambda_l + iM_{j_k,j_l}}{\lambda_k - \lambda_l - iM_{j_k,j_l}}, \tag{18.6}$$

where L is the size of the chain (N in (5.224)), the total number of excitation is K (l in (5.224)) and where for each of the corresponding Bethe roots λ_k one specifies which of the simple roots is excited by j_k which takes the values of $1, \ldots, \#_{\text{sr}}$ with $\#_{\text{sr}}$ being the number of simple roots which for the $SO(6)$ case is three and for the $psu(2,2|4)$ is seven. M is the Cartan matrix of the algebra (1 in (5.224)) and V are the Dynkin labels of the representation (s in (5.224)).

The condition of zero momentum now reads,

$$1 = U = \prod_{k=1}^{K} \frac{\lambda_k + \frac{i}{2} V_{jk}}{\lambda_k - \frac{i}{2} V_{jk}}. \tag{18.7}$$

The energy of a configuration of roots that satisfies the Bethe equation is,

$$E = \sum_{k=1}^{K} \frac{V_{jk}}{\lambda_k^2 + \frac{1}{4} V_{jk}^2}. \tag{18.8}$$

This is of course the analog of (5.225) and the higher conserved charges are,

$$Q_r = \frac{i}{r-1} \sum_{k=1}^{K} \left(\frac{1}{\left(\lambda_k^2 + \frac{i}{2} V_{jk}\right)^{r-1}} - \frac{1}{\left(\lambda_k^2 - \frac{i}{2} V_{jk}\right)^{r-1}} \right). \tag{18.9}$$

The leading order part of the transfer matrix reads,

$$T(\lambda) = \prod_{k=1}^{K} \frac{\lambda - \lambda_k + \frac{i}{2} V_{jk}}{\lambda - \lambda_k - \frac{i}{2} V_{jk}} + \ldots, \tag{18.10}$$

It was shown that these generalized Bethe equations provide a solution to the planar anomalous dimensions of the $\mathcal{N} = 4$ SYM theory [162]. This is just the tip of the iceberg. The integrable structure of the planar $\mathcal{N} = 4$ has been investigated very thoroughly in recent years. For an early review on the topic the reader can consult [31]. As an epilog let us mention that, as was shown in [106], one can identify in a similar manner with what was done in $\mathcal{N} = 4$ SYM [35], a spin chain structure in gauge theories which are confining and with less or even no supersymmetries. In that case the spin chain Hamiltonian would not correspond to the dilatation operator but was rather associated with the excitation energies of hadrons.

18.2 High energy scattering and integrability

High energy scattering is characterized by the fact that the Mandelstam parameter $s = (p_A + p_B)^2$ is the largest scale of the system, and in the limit of $s \to \infty$ the energy dependence corresponds to a renormalization group flow of the dynamical system that "resides" on the two dimensions transverse to the scattering plane.

It is convenient to study the properties of the high energy scattering amplitude $\mathcal{A}(s,t)$ using the Mellin transform,

$$\mathcal{A}(s,t) = is \int_{\delta - i\infty}^{\delta + i\infty} \frac{\mathrm{d}w}{2\pi i} s^w \tilde{\mathcal{A}}(w,t), \tag{18.11}$$

where the integration contour goes to the right of the poles of $\mathcal{A}(w,t)$ in the w complex plane. The high energy asymptotic of $\mathcal{A}(s,t)$ is determined by the poles of the partial wave amplitudes, namely if $\tilde{\mathcal{A}}(w,t) \sim \frac{1}{(w - w_0(t))}$, then $\mathcal{A}(s,t) \sim is^{1+w_0(t)}$. Poles in the w plane are referred to as *reggeons* and the position of the pole is called the *reggeon trajectory*.

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The partial wave amplitude $\tilde{\mathcal{A}}(w,t)$ can be written using the impact parameter representation as follows,

$$\tilde{\mathcal{A}}(w,t) = \int d^2b_0 e^{i(qb_0)} \int d^2b_A d^2b_B \Phi_A(\vec{b}_A - \vec{b}_0) T_w(\vec{b}_A, \vec{b}_B) \Phi_B(\vec{b}_B)$$
$$\equiv \int d^2b_0 e^{i(qb_0)} <\Phi(b_0)|\mathcal{T}_w|\Phi(0)>, \tag{18.12}$$

where the impact factors $\Phi_A(\vec{b}_A)$ and $\Phi_B(\vec{b}_B)$ are the parton distributions which are functions of the transverse coordinates $\vec{b}_A = \vec{b}_A^1, \vec{b}_A^2, \ldots, \vec{b}_A^n$ for the A colliding hadron and $\vec{b}_B = \vec{b}_B^1, \vec{b}_B^2, \ldots, \vec{b}_B^n$ for the B hadron, and $\mathcal{T}_w(\vec{b}_A, \vec{b}_B)$ is the scattering (partial wave) amplitude for a given parton configuration. The idea behind this representation of the amplitude is that the transverse coordinates of the partons can be considered as "frozen" during the interaction. It implies that the structure of the poles in the w-plane does not depend on the parton distribution in the colliding hadrons but rather on the general properties of the gluon interaction of the t-channel. It was shown [145], that the propagators of the t-channel gluons develop their own Regge trajectory due to interactions. A t-channel gluon "dressed" by the virtual corrections is referred to as *reggeized gluon*. The reggeized gluons are the relevant degrees of freedom of the high energy scattering. The partial waves $\mathcal{T}_w(\vec{b}_A, \vec{b}_B)$ can be classified according to the number of the reggeized gluons propagating in the t-channel. The minimal number required to get a colorless exchange is two gluons. We will discuss here only this case. It can be shown that the amplitude $\mathcal{T}_w(\vec{b}_A^1, \vec{b}_A^2 \vec{b}_B^1 \vec{b}_B^2)$ satisfies the so-called BFKL equation that reads [23], [145],

$$w\mathcal{T}_w = \mathcal{T}_w^{(0)} + \frac{\alpha_s N_c}{\pi} H_{BFKL} \mathcal{T}_w, \tag{18.13}$$

where $\mathcal{T}_w^{(0)}$ corresponds to the free exchange of two gluons. Formally one can write the solution as,

$$\mathcal{T}_w = \left[w - \frac{\alpha_s N_c}{\pi} H_{BFKL} \right]^{-1} \mathcal{T}_w^{(0)}, \tag{18.14}$$

so that the singularities of $\mathcal{T}_w$ are determined by the eigenvalues of the operator,

$$H_{BFKL} \Psi_\alpha(\vec{b}^1, \vec{b}^2) = E_\alpha \Psi_\alpha(\vec{b}^1, \vec{b}^2), \tag{18.15}$$

where Ψ_α is the eigenstate. The high energy behavior of the scattering amplitude is dominated by the right-most singularity of $\mathcal{T}_w$, namely on the maximal eigenvalue $(E_\alpha)_{\max}$. The equation (18.15) has the interpretation of the two-dimensional Schrödinger equation of two interacting particles. The interacting particles can be identified with reggeized gluons and $\Psi_\alpha(\vec{b}^1, \vec{b}^2)$ is the wavefunction of a colorless bound state of them. Defining the holomorphic and anti-holomorphic coordinates of the reggeized gluons as follows,

$$z_j = x_j + iy_j \quad \bar{z}_j = x_i - iy_i, \tag{18.16}$$

where $\vec{b}_j = (x_j, y_j)$, we can split the BFKL Hamiltonian into a sum of two terms, one that acts only on the holomorphic coordinates and another that acts only on the anti-holomorphic ones, as follow $H_{BFKL} = H + \bar{H}$ where,

$$H = \partial_{z_1}^{-1} \ln(z_{12}) \partial_{z_1} + \partial_{z_2}^{-1} \ln(z_{12}) \partial_{z_2} + \ln(\partial_{z_1} \partial_{z_2}) - 2\psi(1), \tag{18.17}$$

where $z_{12} = z_1 - z_2$, $\psi(x)$ is the Euler digamma function defined by $\psi(x) = \frac{\mathrm{d} \ln \Gamma(x)}{\mathrm{d}x}$ and in $\bar{H}$ we replace all the z_is by $\bar{z}_i$s.

The BFKL Hamiltonian is further invariant under $SL(2, C)$ transformations. Denoting the $SL(2, C)$ generators (see Section 2.9) by,

$$L_{j-} = -\partial_{z_j} \quad L_{j0} = z_j \partial_{z_j} \quad L_{j+} = z_j^2 \partial_{z_j} \quad L_a = L_{1a} + L_{2a}, \tag{18.18}$$

and similarly for the anti-holomorphic generators, the invariance takes the form,

$$[H_{BFKL}, L_a] = [H_{BFKL}, \bar{L}_a] = 0. \tag{18.19}$$

This implies that H_{BFKL} depends only on the Casimir operators of $SL(2, C)$ algebra of the two particles, namely, with,

$$L_{12}^2 = -(z_1 - z_2)^2 \partial_{z_1} \partial_{z_2} \quad \bar{L}_{12}^2 = -(\bar{z}_1 - \bar{z}_2)^2 \partial_{\bar{z}_1} \partial_{\bar{z}_2}, \tag{18.20}$$

the Hamiltonian must take the form,

$$H = H(L_{12}^2) \quad \bar{H} = \bar{H}(\bar{L}_{12}^2). \tag{18.21}$$

It thus follows that the eigenstates of the Hamiltonian must also be eigenstates of L_{12}^2 and of $\bar{L}_{12}^2$,

$$L_{12}^2 \Psi_{n,\nu} = h(h-1)\Psi_{n,\nu} \quad \bar{L}_{12}^2 \Psi_{n,\nu} = \bar{h}(\bar{h}-1)\Psi_{n,\nu}, \tag{18.22}$$

where the complex dimensions h and $\bar{h}$ are given by,

$$h = \frac{1+n}{2} + i\nu \quad \bar{h} = \frac{1-n}{2} + i\nu. \tag{18.23}$$

The non-negative integer n and the real parameter ν specify the irreducible representation of the $SL(2, C)$ group to which $\Psi_{n,\nu}$ belongs. The wave functions which are eigenstates of the Casimir operators take the form,

$$\Psi_{n,\nu}(\vec{b}) = \left(\frac{z_{12}}{z_{10} z_{20}}\right)^{\frac{1+n}{2} + i\nu} \left(\frac{\bar{z}_{12}}{\bar{z}_{10} \bar{z}_{20}}\right)^{\frac{1-n}{2} + i\nu}, \tag{18.24}$$

where $z_{ij} = z_i - z_j$ and $\vec{b}_0 = (z_0, \bar{z}_0)$ is the center of mass of the state. The conformal dimension of the state is $h + \bar{h} = 1 + 2i\nu$ and the spin $h - \bar{h} = n$. Upon substituting these eigenstates into (18.15) and using the explicit form of the BFKL kernel we find the following eigenvalues,

$$E_{n,\nu} = 2\psi(1) - \psi\left(\frac{1+n}{2} + i\nu\right) - \psi\left(\frac{1+n}{2} - i\nu\right). \tag{18.25}$$

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The maximal eigenvalue corresponds to $n = \nu = 0$ or $h = \bar{h} = \frac{1}{2}$ $E_{0,0} = 4\ln 2$. This maximal eigenstate defines the right-most singularity of the partial wave amplitude. This determines the asymptotic behavior of the scattering amplitude in the leading logarithmic approximation,

$$\mathcal{A}(s,t) = i s^{1+\frac{\alpha_s N}{\pi} 4 \ln 2} \tag{18.26}$$

which is referred to as the BFKL *Pomeron*. Using the explicit form of the eigenvalue one can reconstruct the operator form of H_{BFKL} acting on the representations of the $SL(2,C)$ group,

$$H_{BFKL} = \frac{1}{2}[H(J_{12}) + H(\bar{J}_{12}) \quad H(j) = 2\psi(1) - \psi(j) - \psi(1-j), \tag{18.27}$$

where $L_{12}^2 = J_{12}(J_{12}-1)$, and similarly for $\bar{L}_{12}^2$. This is a special case of the Heisenberg spin chain of spin s operators whose Hamiltonian takes the form,

$$H_s = \sum_i^L H(J_{i,i+1}) \quad J_{i,i+1}(J_{i,i+1}+1) = (\vec{S}_i + \vec{S}_{i+1})^2, \tag{18.28}$$

where $J_{i,i+1}$ is related to the sum of two spins of the neighboring sites, $\vec{S}_i^2 = s(s+1)$ and $H(x)$ is the following harmonic function,

$$H(x) = \sum_{l=x}^{2s-1} \frac{1}{1+l} = \psi(2s+1) - \psi(x+1). \tag{18.29}$$

To connect it to the analysis of Section 5.14 we check this for $s = 1/2$. For this case $J_{i,i+1}$ can take one of the two values $0, 1$ for which we have $H(0) = 1$ and $H(1) = 0$, so that the Hamiltonian is a projection into $J_{i,i+1} = 0$ subspace with $H(J_{i,i+1}) = \frac{1}{4} - \vec{S}_i \cdot \vec{S}_{i+1}$ which is identical to (5.158).

One can generalize the exchange of colorless boundstates of two reggeized gluons to exchange of multireggeon boundstates built from N_r reggeized gluons. This is beyond the scope of this book and can be found for instance in [34].