ON ORBITS OF ALGEBRAIC GROUPS
AND LIE GROUPS

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In this paper we will be concerned with orbits of a closed subgroup Z of an algebraic group (respectively Lie group) G on a homogeneous space X for G. More precisely, let D be a closed subgroup of G and let X denote the coset space G/D. Let S be a subgroup of G and let Z denote $\left(G^S\right)^0$, the identity component of G^S, the centralizer of S in G. We consider the orbits of Z on X^S, the set of fixed points of S on X. We also treat the more general situation in which S is an algebraic group (respectively Lie group) which acts on G by automorphisms and acts on X compatibly with the action of G; again we consider the orbits of $\left(G^S\right)^0$ on X^S.

Our main results are the following theorems:

THEOREM A. *Let G be a closed normal connected subgroup of the affine algebraic group H. Let H act morphically on an algebraic variety X and assume that the induced action of G on X is transitive. Let S be a closed subgroup of H and let $Z = \left(G^S\right)^0$. Assume that the following two conditions are satisfied:*

(i) the adjoint representation of S on $\underline{\underline{g}}$, the Lie algebra of G, is semisimple; and

(ii)   the Lie algebra of Z is $\underline{g}^S$.

Then   Z acts transitively on each connected component of X^S . In particular, Z has only a finite number of orbits on X^S and each such Z-orbit is closed.

THEOREM B. *Let G be a closed connected normal subgroup of the Lie group H . Let H act differentiably on a differentiable manifold X such that the induced action of G on X is transitive. Let S be a subgroup of H such that the adjoint representation of S on $\underline{g}$ is semisimple. Then X^S is a closed differentiable submanifold of X and $Z = \left(G^S\right)^0$ acts transitively on each connected component of X^S .*

THEOREM C. *Let G be a closed connected normal subgroup of the affine algebraic group H and assume that G is reductive. Let S be a linearly reductive, closed subgroup of H and let $Z = \left(G^S\right)^0$. Let H act morphically on the affine algebraic variety X and let $x \in X^S$. Let C denote the unique closed G-orbit in the closure of the orbit G·x . Then there exists a (multiplicative) one-parameter subgroup λ of Z such that $\lim\limits_{t \to 0} \lambda(t)\cdot x$ exists and is a point of C . In particular, G·x is closed if and only if Z·x is closed.*

The formulation and proof of Theorem A were worked out in several conversations with V. Kǎc and this theorem should be regarded as joint work. The proof uses an idea from [13]. This theorem seems to be quite useful in a number of rather technical problems involving orbits of algebraic groups. We remark that conditions *(i)* and *(ii)* of Theorem A hold whenever S is a linearly reductive group (see Section 4). In Theorem A the Z-orbits on X^S do not necessarily all have the same dimension, so that the geometric picture is more complicated than it might appear at first glance.

Theorem B is a Lie group version of Theorem A. The idea of the proof is the same, although the technical details are a bit more complicated.

The proof of Theorem C depends on a strengthened form, due to Kempf [8], of the Hilbert-Mumford Theorem. If the base field is of

characteristic zero, then Theorem C follows from a result of Luna [9], so
that Theorem C is new only for the case of prime characteristic.

Several applications of our theorems to involutions of algebraic
groups are given in Section 9. These results will be used in an essential
way in a forthcoming paper by the author on involutions of reductive
algebraic groups. A few other applications are discussed in Section 10.

1. Preliminaries

Our basic reference for algebraic groups is the book by Borel [1]. In
general, we will follow the notation and terminology therein. All
algebraic groups and algebraic varieties are taken over an algebraically
closed field k . We will usually denote the Lie algebra of an algebraic
group (respectively Lie group) G, H, K , and so on, by the corresponding
lower case letter with a double underline $\underline{g}$, $\underline{h}$, $\underline{k}$, and so on. All
algebraic groups considered will be affine algebraic groups.

Let G be a group and let X be a G-set. If $g \in G$ and $x \in X$,
then $g \cdot x$ denotes the image of x under the action of g , $G \cdot x$ denotes
the G-orbit of x and G_x denotes the isotropy subgroup of G at x .

If S is a subset of G , we write X^S for the set of fixed points of S
on X : $X^S = \{x \in X \mid g \cdot x = x \text{ for every } g \in S\}$. For $s \in G$, we write
X^s instead of $X^{\{s\}}$. We frequently consider G as a G-set, where G
acts by inner automorphisms. Thus if S is a subset of G , G^S denotes
the centralizer of S in G . If L is a subset of G and $g \in G$, we
often write ^gL for gLg^{-1} .

If the algebraic group G acts morphically on the algebraic variety
X , we say that X is a G-*variety*. If the algebraic group S acts
morphically on the algebraic group G and if, for every $s \in S$, the
morphism $x \mapsto s \cdot x$ is an automorphism of G , then we say that G is an
S-*group*. Let G be an algebraic group, let X be a G-variety, let
$a \in X$ and let C denote the orbit $G \cdot a$. For $x \in C$, let $\varphi_x : G \to C$
denote the orbit map: $\varphi_x(g) = g \cdot x$. We say that G acts separably on the
orbit C if φ_a is a separable morphism; if φ_a is separable, then φ_x

is separable for every $x \in C$.

2. Actions of semi-direct products

In this paper we consider the action of a closed subgroup Z of an algebraic group (respectively Lie group) G on a coset space $G/D = X$. However, we can extend the scope of our results by considering an algebraic group (respectively Lie group) H which contains G as a closed normal subgroup and is such that the action of G on X extends to an action of H . This is essentially equivalent to considering the action of a semi-direct product $G \ltimes S$ on X , where G is an S-group. In this section, we discuss several elementary constructions involving actions of semi-direct products which will be used in the sequel. We confine our discussion to algebraic groups. However, all of our constructions carry over to the case of Lie groups with no essential changes.

Let S and G be algebraic groups and let G be an S-group. We denote by $G \ltimes S$ the semi-direct product of G and S ; the underlying algebraic variety of $G \ltimes S$ is $G \times S$ and multiplication is defined by $\left(g_1, s_1\right)\left(g_2, s_2\right) = \left(g_1\left(s_1 \cdot g_2\right), s_1 s_2\right)$. We consider G and S as closed subgroups of $G \ltimes S$ in the obvious way. Assume that G and S act morphically on an algebraic variety X . We say that the actions of G and S on X are *compatible* if

(2.1) $s \cdot (g \cdot x) = (s \cdot g) \cdot (s \cdot x)$ for $s \in S$, $g \in G$ and $x \in X$.

Let G and S act compatibly on X . We define a morphism

$$\eta : (G \ltimes S) \times X \to X$$

by $\eta\big((g, s), x\big) = g \cdot (s \cdot x)$. Set $\eta\big((g, s), x\big) = (g, s) \cdot x$. Then an easy computation using (2.1) shows that

$$\left(g_1, s_1\right) \cdot \left(\left(g_2, s_2\right) \cdot x\right) = \left(g_1\left(s_1 \cdot g_2\right), s_1 s_2\right) \cdot x .$$

Hence η defines a morphic action of $G \ltimes S$ on X . This is the unique action of $G \ltimes S$ on X which extends the actions of G and S . Conversely, the restrictions to G and S of an action of $G \ltimes S$ on X are compatible.

Now let G be an S-group and let D be an S-stable closed subgroup of G . If $s \in S$ and $g \in G$, we set $s \cdot (gD) = (s \cdot g)D$; equivalently

$s \cdot (gD)$ is the s-image of the left coset gD under the action of S on G. Clearly this defines an action of S on (the set) G/D. A standard argument using the properties of quotient morphisms (see [1, pp. 172-186]) shows that the map

$$S \times (G/D) \to G/D \ , \quad (s, \ gD) \longmapsto s \cdot (gD) \ ,$$

is a morphism of algebraic varieties. Hence we have defined a morphic action of S on G/D. We let G act on G/D by left translations. One checks immediately that the actions of G and S on G/D are compatible and hence determine a morphic action of the semi-direct product $G \ltimes S$ on G/D.

REMARK. One needs to be a bit careful in the above construction. Let S be a closed subgroup of G and let S act on G by inner auto-morphisms. This defines an S-group structure on G. Let D be a closed subgroup of G normalized by S. Then the action of S on G/D by left translations is *not* the same as the above defined action of S on G/D unless S is contained in D.

3. Proof of Theorem A

We will use the following lemma, which is an abstraction of a method used in [13] (see also [18, p. 469]):

LEMMA 3.1. *Let the algebraic group G act morphically on the algebraic variety X. Let Z be a closed connected subgroup of G and let Y be a closed Z-stable subvariety of X. Assume that, for every $y \in Y$, $T_y(G \cdot y) \cap T_y(Y) \subset \left(d\varphi_y\right)_e(\underline{z})$, where $\varphi_y : G \to G \cdot y$ is the orbit map. Then, for every $y \in Y$, $(G \cdot y) \cap Y$ is a finite union of Z-orbits and Z acts transitively on each irreducible component of $(G \cdot y) \cap Y$. Moreover, for $y \in Y$, Z acts separably on the orbit $Z \cdot y$.*

Proof. Let $y \in Y$ and let C be an irreducible component of $(G \cdot y) \cap Y$ which contains y. Clearly $Z \cdot y$ is contained in C. We have

$$T_y(C) \subset T_y(G \cdot y) \cap T_y(Y) \subset \left(d\varphi_y\right)_e(\underline{z}) \subset T_y(Z \cdot y) \subset T_y(C)$$

and hence $T_y(C) = T_y(Z \cdot y)$. Since $Z \cdot y$ is a smooth variety, it follows that

$$\dim C \leq \dim T_y(C) = \dim T_y(Z \cdot y) = \dim Z \cdot y .$$

Consequently $Z \cdot y$ is an open neighbourhood of y in C. It follows immediately that Z acts transitively on C. It is also clear that Z acts separably on the orbit $Z \cdot y$.

Proof of Theorem A. Let the notation be as in the statement of Theorem A. First we consider the case in which G acts separably on the orbit X. Let $Y = X^S$. Since Z centralizes S, Y is Z-stable. Let $y \in Y$ and let $D = G_y$. Then the orbit map $\varphi_y : G \to G \cdot y = X$ induces a G-isomorphism of the coset space G/D onto X. Let ψ denote the differential $(d\varphi_y)_e : \underline{g} \to T_y(X)$. Then ψ is surjective. Since $S \subset H_y$, S acts on the tangent space $T_y(X)$. If $s \in S$ and $g \in G$, then

$$\varphi_y \left(sgs^{-1} \right) = \left(sgs^{-1} \right) \cdot y = s \cdot (g \cdot y) = s \cdot \varphi_y(g) .$$

Hence φ_y is an S-morphism. It follows easily that $\psi : \underline{g} \to T_y(X)$ is an S-module homomorphism (here S acts on $\underline{g}$ *via* the adjoint representation). Consequently $\underline{d} = \text{kernel}(\psi)$ is an S-submodule of $\underline{g}$. Since $\underline{g}$ is a semisimple S-module, there exists an S-submodule $\underline{q}$ of $\underline{g}$ such that $\underline{g} = \underline{d} \oplus \underline{q}$. We clearly have $\underline{z} = \underline{g}^S = \underline{d}^S \oplus \underline{q}^S$. Hence we see that ψ determines an S-module isomorphism of $\underline{q}$ onto $T_y(X)$ and that $\psi(\underline{z}) = \psi(\underline{q}^S) = T_y(X)^S$. Clearly $\psi(\underline{z}) \subset T_y(Z \cdot y)$. But since $Y = X^S$, we have $T_y(Y) \subset T_y(X)^S$. Thus we have $T_y(G \cdot y) \cap T_y(Y) \subset T_y(X)^S = \psi(\underline{z})$. The conclusion of Theorem A now follows from Lemma 3.1.

Now for the case where the action of G on X is not necessarily separable. We may assume that X^S is non-empty. Let $y \in X^S$ and let $D = G_y$. Then S normalizes D. Since S normalizes G, we may consider G as an S-group with S acting on G by conjugation. As in Section 2, we define a morphic action of the semi-direct product $G \ltimes S = K$ on G/D. Since G acts separably on G/D, it follows from the case of Theorem A just proved (with H replaced by K) that Z acts transitively

on each irreducible component of $(G/D)^S$. The orbit map $\varphi_y : G \to G \cdot y = X$ determines a bijective G-morphism $\alpha : G/D \to X$. Since G/D and X are normal varieties, it follows from Zariski's Main Theorem (see [5, p. 138, Corollary 3]) that α is a finite morphism. In particular α is a closed map. Consequently α is a homeomorphism. We have

$$\alpha\big(s \cdot (gD)\big) = \alpha\big(sgs^{-1}D\big) = \big(sgs^{-1}\big) \cdot y = s \cdot (g \cdot y) = s \cdot \alpha(gD) .$$

Hence α is an S-morphism. Since Z acts transitively on each irreducible component of $(G/D)^S$, we see immediately that Z acts transitively on each irreducible component of X^S . This proves Theorem A.

We record for later use the following result, which was proved in the course of the above proof.

COROLLARY 3.2. *Let the notations and assumptions be as in Theorem* A *and assume that* G *acts separably on* X . *Then, for* $y \in X^S$, Z *acts separably on the orbit* $Z \cdot y$.

3.3. **An example.** Let $G = H = \mathrm{GL}_3(k)$, let X denote the projective plane $P_2(k)$ and let G act on X in the usual way. Let $S = \{\mathrm{diag}(t, t, 1) \mid t \in k^*\}$; S is a one-dimensional algebraic torus in G . Clearly S satisfies conditions *(i)* and *(ii)* of Theorem A. Now X^S consists of two connected components, one a projective line and one a point. Consequently we see that in Theorem A the irreducible components of X^S do not necessarily all have the same dimension.

4. A lemma on linearly reductive groups

We recall that an affine algebraic group S is *linearly reductive* if every rational representation of S is semisimple. In characteristic zero, S is linearly reductive if and only if the identity component S^0 is a reductive algebraic group. In prime characteristic p , it has been shown by Nagata [11] that S is linearly reductive if and only if S^0 is a torus and the order of S/S^0 is relatively prime to p .

Let G, H and S be as in the statement of Theorem A. We wish to

show that conditions *(i)* and *(ii)* of Theorem A hold if S is linearly reductive. It follows immediately from the definition of linearly reductive groups that condition *(i)* holds. Condition *(ii)* will follow from the following lemma:

LEMMA 4.1. *Let S and G be algebraic groups with S a linearly reductive group and G an S-group. Let $Z = \left(G^S\right)^0$. Then $\underline{z} = \underline{g}^S$.*

For the proof of Lemma 4.1, we will use the following lemma.

LEMMA 4.2. *Let E be a finite dimensional vector space over k and let K be a closed subgroup of $GL(E)$ such that $\underline{gl}(E)$ is a semisimple K-module. Let M be a closed subgroup of $GL(E)$ which contains K . Then $\underline{m}^K$ is the Lie algebra of M^K .*

Proof. We will use Corollary 3.2 of Theorem A. We identify $\underline{gl}(E)$ with $\mathrm{End}_k(E)$. Now $\underline{gl}(E)^K$ is the centralizer of K in $\mathrm{End}_k(E)$; in particular, $\underline{gl}(E)^K$ is an associative subalgebra of $\mathrm{End}_k(E)$. Moreover $GL(E)^K = C$ is simply the intersection of $GL(E)$ with $\underline{gl}(E)^K$, hence C is an open subset of $\underline{gl}(E)^K$. Clearly $\underline{c} = \underline{gl}(E)^K$ is the Lie algebra of C . We can apply Corollary 3.2 to the action of C on $\left(GL(E)/M\right)^K$. Let $\pi : GL(E) \to GL(E)/M$ be the canonical map and let $y = \pi(e)$. Then $y \in \left(GL(E)/M\right)^K$ and hence, by Corollary 3.2, the orbit map $\varphi_y : C \to C{\cdot}y$, $\varphi_y(c) = c{\cdot}y$, is separable. But the kernel of $\left(d\varphi_y\right)_e : \underline{c} \to T_y\left(GL(E)/M\right)$ is just $\underline{c} \cap \underline{m} = \underline{m}^K$ and the isotropy subgroup C_y is $C \cap M = M^K$. Since φ_y is separable, $\dim M^K = \dim \underline{m}^K$ and therefore $\underline{m}^K$ is the Lie algebra of M^K .

Now for the proof of Lemma 4.1. We may embed the semi-direct product $G \ltimes S = M$ as a closed subgroup of some $GL(E)$. We consider G and S as closed subgroups of M , hence as closed subgroups of $GL(E)$. Since S is linearly reductive, $\underline{gl}(E)$ is a semisimple S-module. By Lemma 4.2, $\dim M^S = \dim \underline{m}^S$. But it is trivial to show that $M^S = G^S \times S^S$ (as an algebraic variety) and that $\underline{m}^S = \underline{g}^S \times \underline{s}^S$ (as a vector space). Since

$\dim G^S \leq \dim \underline{\underline{g}}^S$ and $\dim S^S \leq \dim \underline{\underline{s}}^S$, we must have $\dim G^S = \dim \underline{\underline{g}}^S$. If $Z = \left(G^S\right)^0$, it follows that $\underline{z} = \underline{\underline{g}}^S$.

REMARK. The case of Lemma 4.1 in which S is a diagonalizable group is treated in [16, p. 125, Corollary 4.4.7] by a different method.

5. Proof of Theorem B

For the proof of Theorem B, we will use the following lemma:

LEMMA 5.1. *Let U and V be differentiable manifolds, let n be a positive integer, let $f : U \to V$ be a differentiable map, and let $\eta_j : V \to R^n$ be differentiable maps, $j = 1, \ldots, m$. Let $a \in U$ and let $y = f(a)$. Assume that $\eta_j\bigl(f(U)\bigr) = \{0\}$ for $j = 1, \ldots, m$. Let $M = \{v \in V \mid \eta_j(v) = 0 \text{ for } j = 1, \ldots, m\}$ and let $E \subseteq T_y(V)$ denote the intersection of the kernels of the differentials $\bigl(d\eta_j\bigr)_y : T_y(V) \to R^n$, $j = 1, \ldots, m$. Assume that the differential $df_a : T_a(U) \to T_y(V)$ has image E . Then $f(U)$ is a neighbourhood of y in M .*

Lemma 5.1 is a straightforward application of the inverse function theorem. See [12, p. 91, Lemma 6.8] for a detailed proof in a slightly more general setting.

Now for the proof of Theorem B. We use the notation in the statement of the theorem. Let $n = \dim X$ and let $y \in X^S$. We wish first to show that the orbit $Z \cdot y$ is an open neighbourhood of y in X^S . Since $y \in X^S$, S acts on $T_y(X)$. We choose a finite set $\{s_1, \ldots, s_m\}$ of elements of S such that $T_y(X)^S = \bigcap_{j=1}^{m} T_y(X)^{s_j}$. Choose an open neighbourhood E of y in X and a diffeomorphism h of E onto an open subset of R^n such that $h(y) = 0$. Let V be an open neighbourhood of y in E such that $s_j \cdot V \subset E$ for $j = 1, \ldots, m$. We define differentiable maps $\eta_j : V \to R^n$, $j = 1, \ldots, m$, by

$\eta_j(x) = h(x) - h\bigl(s_j \cdot x\bigr)$. We note that $X^S \cap V \subset \eta_j^{-1}(0)$. We also note

that the differential $\left(d\eta_j\right)_y : T_y(V) \to \mathbb{R}^n$ is given by
$\left(d\eta_j\right)_y(v) = dh_y\left(v - s_j \cdot v\right)$. Since dh_y is a linear isomorphism,
$\text{kernel}\left(d\eta_j\right)_y = T_y(X)^{s_j}$. Consequently we obtain

$$\bigcap_{j=1}^{m} \text{kernel}\left(d\eta_j\right)_y = T_y(X)^S .$$

Let D denote the isotropy subgroup G_y . Then the orbit map
$\varphi_y : G \to X$ induces a G-diffeomorphism of G/D onto X . In particular
$\left(d\varphi_y\right)_e : \underline{g} \to T_y(X)$ is a surjection; let $\psi = \left(d\varphi_y\right)_e$. The argument given
in Section 3 shows that ψ is an S-module homomorphism and that
$\psi(\underline{z}) = \psi\left(\underline{g}^S\right) = T_y(X)^S$. Let $U = \{z \in Z \mid \varphi_y(z) \in V\}$ and let $f : U \to V$
denote the restriction of φ_y to U . The image of df_e is

$$\psi(\underline{z}) = T_y(X)^S = \bigcap_{j=1}^{m} \text{kernel}\left(d\eta_j\right)_y .$$ It now follows from Lemma 5.1 that

$f(U)$ is a neighbourhood of y in $\cap\, \eta_j^{-1}(0)$. Since $f(U) \subset Z \cdot y$, we see

that $Z \cdot y$ is an open neighbourhood of y in X^S .

We have shown that, for every $y \in X^S$, $Z \cdot y$ is a neighbourhood of y
in X^S . It follows that each Z-orbit on X^S is both open and closed in
X^S . Hence Z acts transitively on each connected component of X^S .
Since each Z-orbit on X^S is closed in X^S , hence closed in X , it
follows by a standard result (see [6, p. 62, 16.10.7]) that each such orbit
is a closed differentiable submanifold of X . Since each such orbit is
relatively open in X^S , we see that X^S is a closed differentiable
submanifold of X . This completes the proof of Theorem B.

REMARK. In Theorem B, one cannot necessarily conclude that Z has
only a finite number of orbits on X^S , that is that X^S has only a finite
number of connected components. However, in certain cases of Theorem B
involving real algebraic groups, one can show that X^S has only a finite
number of connected components. For example, let H be a (complex) affine

algebraic group defined over R , let $\eta : H \to GL(V)$ be a rational
representation of H defined over R and let x be an R-rational point
of V (respectively $P(V)$). Let K be a closed connected normal
subgroup of H defined over R and let G denote the identity component
of the real Lie group $K(R)$. Let S be a subgroup of $H(R)_x$ such that
the adjoint representation of S on $\underline{g}$ is semisimple and let $Z = G^S$.
Let $X = G \cdot x$. Then X is a locally closed differentiable submanifold of
V (respectively $P(V)$) and is S-stable. One can show that X^S has only
a finite number of connected components and hence that Z has only a
finite number of orbits on X . The main point of the proof is that, by a
theorem of Whitney [20], if Y is a Zariski locally-closed subset of $V(R)$
(respectively $P(V(R))$), then Y has only a finite number of components
(with respect to the topology on Y determined by the usual topology of
the real numbers). We omit further details.

6. Another technical result on linearly reductive groups

If the base field k is of characteristic zero, then all of the
results of this section are either well-known or follow readily from
standard results. We shall assume throughout this section that k is of
prime characteristic p . If S is a linearly reductive group, this will
imply that S^0 is a torus and that $|S/S^0|$ is relatively prime to p .

The purpose of this section is to prove the following proposition:

PROPOSITION 6.1. *Let G be a closed connected normal subgroup of an
algebraic group H and let S be a closed subgroup of H . Assume that
G is a reductive group and that S is a linearly reductive group. Let P
be a parabolic subgroup of G which is normalized by S . Then there
exists a Levi subgroup L of P which is normalized by S .*

6.2. **Non-commutative cohomology of algebraic groups.** The most
convenient way to prove Proposition 6.1 seems to be *via* the non-commutative
cohomology of algebraic groups. First we need to transpose some standard
definitions and results from Galois cohomology to this slightly different
framework. See [2] for Galois cohomology.

Let G and M be algebraic groups and assume that M is a G-group.
A morphism of algebraic varieties $\alpha : G \to M$ is a *one-cocycle* of G with

values in M if $\alpha(gh) = \alpha(g)\big(g\cdot\alpha(h)\big)$ for every $g, h \in G$. Let $Z^1(G, M)$ denote the set of all one-cocycles of G with values in M. We define an action of M on the set $Z^1(G, M)$ as follows: if $m \in M$ and $\alpha \in Z^1(G, M)$, then the one-cocycle $m\cdot\alpha$ is defined by $(m\cdot\alpha)(g) = m\alpha(g)(g\cdot m)^{-1}$ for $g \in G$. Two one-cocycles α and β are *cohomologous* if there exists $m \in M$ such that $\beta = m\cdot\alpha$. The set of orbits of M on $Z^1(G, M)$ is denoted by $H^1(G, M)$; $H^1(G, M)$ is the *first cohomology set* of G with coefficients in M. If $\alpha \in Z^1(G, M)$, then $[\alpha] \in H^1(G, M)$ denotes the orbit of α. Let $1 \in Z^1(G, M)$ be the trivial cocycle: $1(g) = e$ for $g \in G$. Then $H^1(G, M)$ is a pointed set with neutral element $[1]$. If M is abelian, then $H^1(G, M)$ is the usual first cohomology group of the algebraic group G with coefficients in the G-group M. In this case, we sometimes denote $[1]$ by 0.

LEMMA 6.2.1. *Let F be a finite group whose order is relatively prime to p and let V be an abelian unipotent F-group. Then* $H^1(F, V) = 0$.

Proof. Let $\alpha \in Z^1(F, V)$ and let $r = |F|$. Let α^r denote the cocycle defined by $\alpha^r(f) = \alpha(f)^r$ for every $f \in F$. By a standard argument (see [15, p. 5.01]), α^r is cohomologous to the trivial cocycle 1. Thus there exists $v \in V$ such that $\alpha(f)^r = v(f\cdot v)^{-1}$ for $f \in F$. Now V is a p-group and hence every element of V has a unique rth root. Let $u \in V$ be such that $u^r = v$. Then we have $\big(u(f\cdot u)^{-1}\big)^r = v(f\cdot v)^{-1} = \alpha(f)^r$ for every $f \in F$, and hence $\alpha(f) = u(f\cdot u)^{-1}$ for every $f \in F$. Thus $[\alpha] = 0$.

LEMMA 6.2.2. *Let G be an algebraic group, let M be a G-group and let N be a closed normal G-stable subgroup of M. Then there exists an exact sequence (of pointed sets)*

$$H^1(G, N) \to H^1(G, M) \to H^1(G, M/N).$$

LEMMA 6.2.3. *Let G be an algebraic group, let M be a G-group*

and let H *be a closed normal subgroup of* G . *Then there exists an exact
sequence (of pointed sets)*

$$[1] \rightarrow H^1\left(G/H, M^H\right) \rightarrow H^1(G, M) \rightarrow H^1(H, M) .$$

For the case of Galois cohomology, results similar to Lemmas 6.2.2 and
6.2.3 are proved in [2; p. 120, Proposition 1.17 and p. 123, Proposition
1.27]. The proofs in our case are essentially the same and will be
omitted.

LEMMA 6.2.4. *Let* S *be an algebraic torus and let* V *be an abelian
unipotent* S-*group. Then* $H^1(S, V) = 0$.

Proof. Let $\alpha \in Z^1(S, V)$. Corresponding to the one-cocycle α , we
define a "twisted" morphic action of S on (the algebraic variety) V ,
denoted by $(s, v) \mapsto s * v$, as follows: $s * v = \alpha(s)s \cdot v$. (See [2,
p. 115] for the general setting for this.) We note that $s * v = v$ if and
only if $\alpha(s) = v(s \cdot v)^{-1}$. Thus α is cohomologous to the trivial cocycle
1 if and only if the twisted action corresponding to α has a fixed
point. Now it is a standard fact that, for any morphic action of an
algebraic torus on an affine variety, there are only a finite number of
distinct isotropy subgroups. (This is obvious for a linear action on a
vector space, and the general case can be reduced to this case.) Let
$A_1, \ldots, A_q$ denote the set of distinct isotropy subgroups for the twisted
action of S on V . If some $A_j = S$, then S has a fixed point on V
and hence α is sohomologous to 1 . If not, there exists a finite
subgroup F of S such that F is not contained in any A_j . But $|F|$
is relatively prime to p and consequently, by Lemma 6.2.1,
$H^1(F, V) = 0$; thus F has a fixed point on V . This gives a
contradiction and proves Lemma 6.2.4.

LEMMA 6.2.5. *Let* S *be a linearly reductive group and* V *an
abelian unipotent* S-*group. Then* $H^1(S, V) = 0$.

Proof. By Lemma 6.2.3, we have an exact sequence

$$H^1\left(S/S^0, V^{S^0}\right) \rightarrow H^1(S, V) \rightarrow H^1\left(S^0, V\right) .$$

But $|S/S^0|$ is relatively prime to p and S^0 is a torus. Hence, by Lemma 6.2.1, $H^1(S/S^0, V^{S^0}) = 0$ and, by Lemma 6.2.4, $H^1(S^0, V) = 0$. Consequently $H^1(S, V) = 0$.

LEMMA 6.2.6. *Let S be a linearly reductive group and V a unipotent S-group. Then $H^1(S, V) = [1]$.*

Proof. By induction on the length n of the derived series for V. For $n = 1$, the result follows from Lemma 6.2.5. Assume that the result has been proved for the case $n - 1$. By Lemma 6.2.2, we have an exact sequence

$$H^1(S, [V, V]) \to H^1(S, V) \to H^1(S, V/[V, V]).$$

By the inductive hypothesis, $H^1(S, [V, V]) = [1]$ and, by Lemma 6.2.5, $H^1(S, V/[V, V]) = 0$. Thus $H^1(S, V) = [1]$.

6.3. We are now ready to prove Proposition 6.1. Let the notation be as in the statement of the proposition. It has been shown by Borel and Tits [4] that P has a Levi subgroup and that any two Levi subgroups of P are conjugate in P. Let L be the set of all Levi subgroups of P. Then $N_H(P)$ acts on L by conjugation. Let $V = R_u(P)$ and let L be a Levi subgroup of P. Then we claim that $L = N_P(L)$. To prove this, it will be sufficient to show that $V \cap N_P(L) = \{e\}$. Let $v \in V \cap N_P(L)$. Then, for $x \in L$, we have $vxv^{-1}x^{-1} \in L \cap V = \{e\}$. Hence $v \in Z_V(L)$. But, if T is a maximal torus of L (hence a maximal torus of G), it is easy to show that $Z_V(T) = \{e\}$ and hence that $Z_V(L) = \{e\}$. Thus our claim is proved. Since P acts transitively on L by conjugation and $N_P(L) = L$, it follows that V acts simply transitively on L.

Now since S normalizes P, S normalizes V and we may consider V as an S-group, with S acting by conjugation on V. We let K denote the semi-direct product $V \ltimes S$. Now S and V act on (the set) L by conjugation and these actions are compatible; consequently they determine an action of K on L which extends the actions of S and V.

Let $R = \{(v, s) \in K \mid {}^{vs}L = L\}$; R is the stabilizer of L for the action of K on L . Since V acts simply transitively on L , we see that the morphism $\eta : V \to K/R$ defined by $\eta(v) = vR$ is a bijection. We wish to show that η is an isomorphism of varieties. It will suffice to show that the differential $d\eta_e$ has trivial kernel. The kernel of $d\eta_e$ is $\underline{v} \cap \underline{r}$. Let $x \in \underline{v} \cap \underline{r}$. Since $x \in \underline{v}$, x is a nilpotent element of $\underline{r}$. Now it is known that every nilpotent element of $\underline{r}$ is tangent to a connected unipotent subgroup of R^0 [3]. Now R^0 is contained in the connected solvable group $S^0 \ltimes V$ and V is the unipotent radical of $S^0 \ltimes V$. Consequently every unipotent subgroup of R^0 is contained in V . Hence x is tangent to a connected unipotent subgroup of $R^0 \cap V$. But since $N_P(L) = L$, $R^0 \cap V = \{e\}$. Therefore $x = 0$ and $\eta : V \to K/R$ is an isomorphism.

Define a morphism $\nu : S \to V$ by $\nu(s) = \eta^{-1}(sR)$. Thus, for $s \in S$, $\nu(s)$ is the unique element of V such that ${}^sL = {}^{\nu(s)}L$. Moreover, for $s, t \in S$, we have

$$\nu(st)_L = st_L = s\nu(t)_L = s\nu(t)s^{-1}s_L = \bigl(s\cdot\nu(t)\bigr)\nu(s)_L \ .$$

Since V acts simply transitively on L , this gives

(6.3.1) $\nu(st) = \bigl(s\cdot\nu(t)\bigr)\nu(s)$ for $s, t \in S$.

Assume now that there exists $L_1 \in L$ which is normalized by S . We may write $L_1 = {}^vL$ for some (unique) $v \in V$. Hence, for $s \in S$, we obtain

$$^vL = L_1 = {}^sL_1 = {}^sv_L = svs^{-1}s_L = (s\cdot v)\nu(s)_L \ ,$$

and consequently, since V acts simply transitively on L , we have

(6.3.2) $\nu(s) = (s\cdot v)^{-1}v$ for $s \in S$.

Conversely, if ν satisfies (6.3.2), then $^vL = L_1$ is normalized by S .

Thus we see that, in order to show that S normalizes a Levi subgroup of P , it suffices to show that if $\nu : S \to V$ is a morphism satisfying

(6.3.1), then there exists $v \in V$ such that (6.3.2) is satisfied. Let $\nu : S \to V$ satisfy (6.3.1) and let $\alpha : S \to V$ be defined by $\alpha(s) = \nu(s)^{-1}$. Then (6.3.1) implies that $\alpha \in Z^1(S, V)$. By Lemma 6.2.6, $H^1(S, V) = [1]$. Hence there exists $v \in V$ such that $\alpha(s) = v^{-1}(s \cdot v)$ for $s \in S$. But this implies that ν satisfies (6.3.2). This proves Proposition 6.1.

7. A theorem of Kempf

First we need some notation. If $\varphi : k^* \to X$ is a morphism of algebraic varieties, then we say that $\lim_{t \to 0} \varphi(t)$ is equal to $x \in X$ if there exists a morphism $\psi : k \to X$ such that $\psi(t) = \varphi(t)$ for $t \in k^*$ and $\psi(0) = x$.

Let G be a (connected) reductive algebraic group, let T be a maximal torus of G and let $W = N_G(T)/T$ be the corresponding Weyl group. A (multiplicative) one-parameter subgroup of G is a homomorphism $\lambda : k^* \to G$ of algebraic groups. We let $X_*(G)$ denote the set of one-parameter subgroups of G . Since T is abelian, the set $X_*(T)$ has a matural structure of abelian group; in fact $X_*(T)$ is a free abelian group of rank equal to the dimension of T . The group G acts on the set $X_*(G)$ *via* inner automorphisms: if $\lambda \in X_*(G)$ and $g \in G$, then the one-parameter group $g \cdot \lambda$ is defined by $(g \cdot \lambda)(t) = g\lambda(t)g^{-1}$. Similarly W acts on $X_*(T)$.

If $\lambda \in X_*(G)$, let $P(\lambda) = \left\{ g \in G \mid \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \right\}$. Then $P(\lambda)$ is a parabolic subgroup of G [10, p. 55]. If $\lambda \in X_*(T)$ and if $\Phi = \Phi(T, G)$ is the set of roots of G with respect to T , then $P(\lambda)$ is the parabolic subgroup of G containing T which corresponds to the set of roots $\{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle \geq 0\}$ (see [19, p. 320, Proposition 3]). If $\lambda \in X_*(G)$ and M is a closed subgroup of some algebraic group containing G , we often write $Z_M(\lambda)$ for $Z_M(\text{image } \lambda)$. It is well-known [4] that $Z_G(\lambda) = Z_{P(\lambda)}(\lambda)$ and that $Z_{P(\lambda)}(\lambda)$ is a Levi subgroup of $P(\lambda)$.

Now let G act morphically on an affine algebraic variety X and let $x \in X$. It is a consequence of the Mumford Conjecture that there is a

unique closed orbit C in the closure of the orbit $G \cdot x$ (see [14, Lemma 1.4]). It follows from (a minor extension of) the Hilbert-Mumford Theorem (see [8, Theorem 1.4]) that there exists $\lambda \in X_*(G)$ such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists and is a point of C. Let $\Lambda(x)$ denote the set of all $\lambda \in X_*(G)$ such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists and is a point of C. Kempf [8] has shown that there exists a (non-empty) "optimal" class $C(x) \subset \Lambda(x)$ which has the following properties:

(7.1). (i) If $\lambda, \mu \in C(x)$, then $P(\lambda) = P(\mu)$.

(ii) Let $P = P(x)$ denote $P(\lambda)$ for some (and hence every) $\lambda \in C(x)$. Then $C(x)$ is a complete P-orbit in $X_*(P)$, that is $C(x) = P \cdot \lambda$ for any $\lambda \in C(x)$.

(iii) If $g \in G$, then $C(g \cdot x) = g \cdot C(x)$ and hence ${}^g P(x) = P(g \cdot x)$. In particular $G_x \subset N_G(P) = P$.

Roughly speaking, the optimal class consists of those $\lambda \in \Lambda(x)$ for which $\lambda(t) \cdot x$ approaches C most rapidly as t approaches 0 . The optimal class $C(x)$ depends on the choice of a W-invariant, integer-valued, positive-definite, symmetric bilinear form on $X_*(T)$. Such a form uniquely determines a G-invariant "norm" on $X_*(G)$; see [8] for definition and details. Once such a W-invariant form on $X_*(T)$ is chosen, the optimal class $C(x)$ is uniquely determined for every $x \in X$ by the geometric data.

We wish to extend 7.1 (iii) above to a slightly more general setting. Let H be an algebraic group and G a closed, connected, normal subgroup of H such that G is a reductive group. Let T be a maximal torus of G , let $W = N_G(T)/T$ be the corresponding Weyl group and let $W_1 = N_H(T)/Z_H(T)$. By the rigidity of tori, W_1 is a finite group. Moreover, W is canonically embedded as a normal subgroup of W_1 . The group H acts on $X_*(G)$ by conjugation and W_1 acts similarly on $X_*(T)$. We let $X_*(G)/H$ (respectively $X_*(T)/W_1$) denote the set of orbits of H on $X_*(G)$ (respectively of W_1 on $X_*(T)$).

LEMMA 7.2. *The inclusion map* $X_*(T) \to X_*(G)$ *induces a bijection* β

of $X_*(T)/W_1 \to X_*(G)/H$.

Proof. Let $\lambda \in X_*(G)$. Then image(λ) is G-conjugate to a subtorus
of T . Thus there exists $g \in G$ such that $g \cdot \lambda \in X_*(T)$. Hence β is
surjective. Now let $\mu, \nu \in X_*(T)$ and assume that there exists $h \in H$
such that $h \cdot \mu = \nu$. Then T and ^hT both contain image(ν) , hence T
and ^hT are both maximal tori of $Z_G(\nu)$. Therefore, there exists
$g \in Z_G(\nu)$ such that $^{gh}T = T$. Thus $gh \in N_H(T)$. We let W denote the
image of gh in W_1 . Then $w \cdot \mu = (gh) \cdot \mu = g \cdot \nu = \nu$. This shows that β
is injective.

Since W_1 is finite, we may choose a W_1-invariant, integer-valued,
positive definite, symmetric bilinear form on $X_*(T)$. As in Kempf [8],
this extends to a G-invariant norm on $X_*(G)$. It follows trivially from
Lemma 7.2 that this norm on $X_*(G)$ is H-invariant.

Now let H act morphically on the affine variety X and let C
denote the unique closed orbit in the closure of $G \cdot x$. Considering X as
a G-variety, we may define $\Lambda(x), C(x)$ and $P(x)$ as above; $C(x)$ and
$P(x)$ are defined with respect to an H-invariant norm on $X_*(G)$
constructed as above.

LEMMA 7.3. *Let the notation be as above. Then the optimal class*
$C(x)$ *is stable under the action of the isotropy subgroup* H_x . *In*
particular, H_x *normalizes* $P(x)$.

Proof. This follows immediately from Kempf's definition of the
optimal class $C(x)$ and the fact that the norm on $X_*(G)$ is H-invariant.

8. Proof of Theorem C

Let the notation be as in the statement of Theorem C. We choose an
H-invariant norm on $X_*(G)$ as indicated in Section 7. Let $C(x)$ be the
optimal class as given by Kempf's theorem and let $P = P(x)$ be the
corresponding parabolic subgroup. Since $x \in X^S$, it follows from Lemma
7.3 that $C(x)$ is stable under the action of S and that S normalizes
P . Let $\lambda \in C(x)$. Then $P(\lambda) = P$ and $L = Z_G(\lambda)$ is a Levi subgroup of

P . Let L denote the set of all Levi subgroups of P . We define a map $F : C(x) \to L$ by $F(\mu) = Z_G(\mu)$ for $\mu \in C(x)$.

LEMMA 8.1. *F is a bijection.*

Proof. If $g \in P$ and $\mu \in C(x)$, then

$$F(g \cdot \mu) = Z_G(g \cdot \mu) = {}^gZ_G(\mu) = {}^gF(\mu) .$$

Thus F is a P-mapping of P-sets. Now P acts transitively on $C(x)$ and on L and hence F is a surjection. Assume that $g \in P$ is such that $F(g \cdot \lambda) = F(\lambda)$. Then ${}^gL = {}^gF(\lambda) = F(\lambda) = L$. But we have shown in 6.3 that $N_P(L) = L$. Hence $g \in L = Z_P(\lambda)$. Thus λ and $F(\lambda)$ have the same isotropy subgroups for the actions of P . Since F is a P-mapping of transitive P-sets, this implies that F is a bijection.

Since S normalizes P , we see by Proposition 6.1 that there exists a Levi subgroup L_1 of P which is normalized by S . We may write $L_1 = F(\mu) = Z_G(\mu)$ for some $\mu \in C(x)$. Let $s \in S$. Then, by Lemma 7.3, $s \cdot \mu \in C(x)$ and we have

$$F(s \cdot \mu) = Z_G(s \cdot \mu) = {}^sZ_G(\mu) = {}^sL_1 = L_1 = Z_G(\mu) = F(\mu) .$$

Since F is a bijection, this implies that $s \cdot \mu = \mu$. Hence we see that $s\mu(t)s^{-1} = \mu(t)$ for every $t \in k^*$. Consequently $\mu \in X_*\left(\left(G^S\right)^0\right)$. This gives the first conclusion of Theorem C. In particular, we see that if $G \cdot x$ is not closed, then $Z \cdot x$ is not closed. For the converse, assume that $G \cdot x = X_1$ is closed. Then $H_1 = GS$ is a homomorphic image of the semi-direct product $G \ltimes S$ and hence is a closed subgroup of H . Since $x \in X^S$, we see that $H_1 \cdot x = G \cdot x = X_1$. It follows from Theorem A, as applied to the action of H_1 on X_1 , that $Z \cdot x$ is closed in X_1 . Since X_1 is closed in X , we see that $Z \cdot x$ is closed in X . This completes the proof of Theorem C.

9. Applications to involutions of algebraic groups

In this section, we assume that characteristic $(k) \neq 2$.

In this section we will prove several results concerning orbits which are associated to involutions of algebraic groups. There results will be used in a forthcoming paper of the author on involutions of reductive groups.

Let G be a connected algebraic group and let θ be an automorphism of G of period two. We consider two morphic actions of G on (the algebraic variety) G :

(1) The action of G by conjugation: $g \cdot x = gxg^{-1}$ for $g, x \in G$. Here the orbit $G \cdot x$ is just the conjugacy class $C_G(x)$.

(2) The "twisted" action of G on G , denoted by $(g, x) \mapsto g * x$, where $g * x = gx\theta(g)^{-1}$. We denote the G-orbit of x for the twisted action of G by $G * x$.

Let $K = G^\theta$. We note that, for $k \in K$ and $x \in G$, we have $k \cdot x = k * x$. We let $P = G * e = \{g\theta(g)^{-1} \mid g \in G\}$ and let $Q = \{g \in G \mid \theta(g) = g^{-1}\}$. Then Q is a closed subvariety of G which is stable under the twisted action of G and P is contained in Q . Also Q is θ-stable.

We will prove the following propositions:

PROPOSITION 9.1. *Let G act on Q via the twisted action. Then G acts transitively on each connected component of Q . Hence Q is a finite union of twisted G-orbits and each twisted G-orbit on Q is closed in G . In particular, P is closed in G .*

PROPOSITION 9.2. *Let $y \in Q$. Then K^0 acts transitively on each connected component of $Q \cap C_G(y)$. In particular, $Q \cap C_G(y)$ is a finite union of K^0-orbits, each of which is closed in $Q \cap C_G(y)$.*

PROPOSITION 9.3. *Let G be reductive.*

(a) The following conditions on $y \in P$ are equivalent:

  (i) y is semisimple;

  (ii) $C_G(y)$ is closed in G ; and

(iii) $K^0 \cdot y = C_K(y)$ *is closed in* P .

(b)   The following conditions on $y \in P$ *are equivalent:*

 (iv) y *is unipotent;*

  (v)   e   belongs to the closure of $C_G(y)$;   *and*

 (vi)   e   belongs to the closure of $K^0 \cdot y$.

Proof of Proposition 9.1. We define an action of $G \times G$ on (the algebraic variety) G by $(g_1, g_2) \cdot x = g_1 x \theta(g_2)^{-1}$. Let $\Delta(G)$ denote the diagonal subgroup of $G \times G : \Delta(G) = \{(g, g) \mid g \in G\}$. Then the $\Delta(G)$-orbit of $x \in G$ is just the twisted orbit $G * x$. Define an involutive automorphism s of $G \times G$ by $s(g_1, g_2) = (g_2, g_1)$. Let S denote the two-element subgroup of $\mathrm{Aut}(G \times G)$ consisting of the identity auto-morphism of $G \times G$ and s . Let H denote the semi-direct product of $G \times G$ and S . We identify $G \times G$ and S with closed subgroups of H in the usual way. Since   characteristic$(k) \neq 2$, S is a linearly reductive group. Clearly $\left(H^S\right)^0 = \Delta(G)$. We define an action of S on the algebraic variety G by $s \cdot x = \theta(x)^{-1}$ for $x \in G$. Then, for $g_1, g_2, x \in G$, we have

$$
\begin{aligned}
s \cdot \left((g_1, g_2) \cdot x \right) &= s \cdot \left[g_1 x \theta(g_2)^{-1} \right] \\
&= g_2 \theta(x)^{-1} \theta(g_1)^{-1} \\
&= (g_2, g_1) \cdot (s \cdot x) \\
&= \left(s \cdot (g_1, g_2) \right) \cdot (s \cdot x) .
\end{aligned}
$$

Thus the actions of $G \times G$ and S on G are compatible and determine an action of the semi-direct product H on G . We note that G^S (the set of fixed points of S on G by the action defined above) is equal to Q . Hence, by Theorem A (with the X of Theorem A equal to G above and with $\dot{G}$ and H of Theorem A equal to H^0 and H above), we see that $\Delta(G)$ acts transitively on each connected component of Q . Since $\Delta(G) \cdot x = G * x$ for $x \in G$, the proposition is proved.

Proof of Proposition 9.2. For the proof of this proposition, we let
S denote the two-element subgroup $\{1_G, \theta\}$ of Aut(G) and let H denote
the semi-direct product of G and S . Let G act on G by
conjugation. Thus $G \cdot x = C_G(x)$ for $x \in G$. We define an action of S
on the algebraic variety G by $\theta \cdot x = \theta(x)^{-1}$ for $x \in G$. With this
action of S on G , the set of fixed points of S on G is exactly Q .
We have

$$
\begin{aligned}
\theta \cdot (g \cdot x) &= \theta \cdot \left(gxg^{-1}\right) \\
&= \theta(g)\theta(x)^{-1}\theta(g)^{-1} \\
&= \theta(g) \cdot (\theta \cdot x) \ .
\end{aligned}
$$

Thus the actions of G and S on G are compatible and determine an
action of the semi-direct product H on G . Let $y \in Q$ and let
$X = G \cdot y = C_G(y)$. Then $\theta \cdot (G \cdot y) = G \cdot (\theta \cdot y) = G \cdot y$ and hence X is
H-stable. Now $K^0 = \left(G^S\right)^0$ and $X^S = C_G(y) \cap Q$. Since
characteristic$(k) \neq 2$, the action of S on $\underline{g}$ is semisimple. The
conclusion of Proposition 9.2 now follows from Theorem A.

Proof of Proposition 9.3. Let H and the action of H on G be
defined as in the proof of Proposition 9.2. To conform with the notation
of Theorem C, let $G = X$ (as the algebraic variety on which $H = G \ltimes S$
acts). Then $X^S = Q$ and $K^0 = \left(G^S\right)^0$. It follows from Theorem C that
$K^0 \cdot y$ is closed if and only if $G \cdot y = C_G(y)$ is closed. This proves the
equivalence of (ii) and (iii) . Theorem C also implies the equivalence of
(v) and (vi) . The equivalence of (i) and (ii) and the equivalence of (iv)
and (v) are well known [17]. This proves Proposition 9.3.

9.4. Now let G be a connected Lie group and let θ be an
involutive automorphism of G . Let P, Q and the twisted action of G
on G be defined as above. Then we have:

PROPOSITION 9.4.1. *The twisted action of G is transitive on each*
connected component of Q . In particular, Q is a closed differentiable
submanifold of G and each twisted G-orbit on Q is closed. Hence P
is closed.

The proof is a precise Lie group analogue (using Theorem B) of

Proposition 9.1. We omit the details. One can, in fact, show that Q has only a finite number of connected components, but we shall not give the proof here.

10. Some more applications

10.1. Affine orbits and reductive subgroups. We will use the following characterization of reductive subgroups of reductive groups:

10.1.1. Let H be a closed subgroup of the reductive algebraic group G . Then H^0 is a reductive group if and only if the coset space G/H is an affine variety.

See [14] for a proof.

Before giving our applications, we need a few elementary lemmas.

LEMMA 10.1.2. *Let* S *be a subgroup of* $\mathrm{GL}(E)$ *such that* E *is a semisimple* S-module. *Then* $\mathrm{GL}(E)^S$ *is a reductive group.*

Proof. Let A denote the commuting algebra of S in $\mathrm{End}_k(E)$. Then A is a direct product of matrix algebras. (This is a standard result in the theory of semisimple associative algebras. See, for example, [7, p. 79].) Hence $\mathrm{GL}(E)^S$ is isomorphic to a product of the form $\mathrm{GL}_{m_1}(k) \times \ldots \times \mathrm{GL}_{m_r}(k)$. In particular $\mathrm{GL}(E)^S$ is reductive.

LEMMA 10.1.3. *Let the algebraic group G act morphically on the affine algebraic variety X and let $x \in X$ be such that the orbit $G \cdot x$ is closed. Then G/G_x is an affine variety.*

Proof. The orbit map $\varphi_x : G \to G \cdot x$ determines a bijective morphism $\psi : G/G_x \to G \cdot x$. Since G/G_x and $G \cdot x$ are normal varieties, it follows from Zariski's Main Theorem that ψ is a finite morphism (see [5, p. 138, Corollary 3]). Since $G \cdot x$ is an affine variety, this implies that G/G_x is an affine variety.

LEMMA 10.1.4. *Let H and K be closed subgroups of an algebraic group G , let $\pi_H : G \to G/H$ and $\pi_K : G \to G/K$ be the canonical morphisms*

*and let $a = \pi_H(e)$, $b = \pi_K(e)$. Let G act on G/H and G/K by left
translations. If the orbit $K \cdot a$ is closed in G/H , then $H \cdot b$ is closed
in G/K .*

Proof. If $K \cdot a$ is closed in G/H , then $KH = \pi_H^{-1}(K \cdot a)$ is closed in

G and therefore $HK = (KH)^{-1}$ is closed in G . But the topology on G/K

is the quotient topology. Consequently, since $HK = \pi_K^{-1}(H \cdot b)$ is closed in

G , we see that $H \cdot b$ is closed in G/K .

If S is a reductive subgroup of a reductive algebraic group G and

if k is of characteristic zero, then it is known that $Z_G(S)^0$ and

$N_G(S)^0$ are reductive groups. This result is not true in prime

characteristic p . However, the following result shows that it does hold

if S is linearly reductive.

PROPOSITION 10.1.5. *Let S be a linearly reductive group and let G
be a connected S-group. Then G/G^S is an affine variety. In particular,
if G is reductive, then $(G^S)^0$ is reductive. If S is a closed
linearly reductive subgroup of G acting on G by inner automorphisms,
then $N_G(S)^0 = (SG^S)^0$. If, in addition, G is reductive, then $N_G(S)^0$
is reductive.*

Proof. Let H denote the semi-direct product $G \ltimes S$. We consider
G and S as closed subgroups of H in the usual way. We may embed H
as a closed subgroup of some $GL(E)$. By Lemma 10.1.2 we see that
$M = GL(E)^S$ is a reductive group. Since S normalizes G , it follows
from Section 2 that we may define a morphic action of S on $GL(E)/G = X$
by $s \cdot (gG) = sgs^{-1}G$. (Warning: this is not necessarily the same as the
action of S on $GL(E)/G$ by left translation.) We consider $GL(E)$ as an
S-group, with S acting by inner automorphisms. Let $GL(E)$ act on X by
left translations. Then the actions of $GL(E)$ and S on X are
compatible and hence define an action of the semi-direct product $GL(E) \ltimes S$
on X . Let $\pi : GL(E) \to GL(E)/G = X$ be the canonical map and let
$a = \pi(e)$. To avoid confusing notation, let S' denote the subgroup

$\{e\} \times S$ of $GL(E) \ltimes S$. We now apply Theorem A to the above situation, with the H (respectively G, S) of Theorem A replaced by $GL(E) \ltimes S$ (respectively $GL(E), S'$) above. By Theorem A, every orbit of M on $X^{S'}$ is closed. In particular $M \cdot a$ is closed. Let $\pi' : GL(E) \to GL(E)/M$ be the canonical map and let $b = \pi'(e)$. Since $M \cdot a$ is closed, we see from Lemma 10.1.4 that $G \cdot b$ is closed for the action of G on $GL(E)/M$ by left translations. Since M is reductive, it follows from 10.1.1 that $GL(E)/M$ is an affine variety. Thus, by Lemma 10.1.3, we see that G/G_b is an affine variety. But $G_b = G \cap M = G^S$ and consequently G/G^S is an affine variety. If G is reductive, then 10.1.1 implies that $\left(G^S\right)^0$ is reductive.

Assume now that the linearly reductive group S is a closed subgroup of G acting on G by inner automorphisms. We let G act on $X_1 = G/S$ by left translation and let S act on X_1 as a subgroup of G. Let $\pi_1 : G \to G/S = X_1$ be the canonical map and let $c = \pi_1(e)$. Then $c \in X_1^S$. By Theorem A, $Z = \left(G^S\right)^0$ acts transitively on the connected component Y of X_1^S which contains c. Let $N = N_G(S)^0$. Clearly Y is N-stable. Hence $N \cdot c$ is contained in $Z \cdot c = Y$. Thus we see that, for $n \in N$, we have $nS = zS$ for some $z \in Z$ and hence $n \in ZS$. Thus we have shown that $N = (ZS)^0$. Assume further that G is reductive. By the proof above, Z is reductive. Moreover, $N/Z = \left(ZS^0\right)/Z$ is a homomorphic image of the reductive group S^0 and hence is a reductive group. Since Z and N/Z are reductive groups, it follows easily that N is a reductive group. This proves Proposition 10.1.5.

COROLLARY 10.1.6. *Let S be a linearly reductive group, let G be a connected S-group and let $Z = \left(G^S\right)^0$. Let L be a closed S-stable subgroup of G such that the coset space G/L is an affine variety. Then $Z/Z \cap L$ is an affine variety.*

Proof. Let $\pi : G \to G/L$ be the canonical map and let $a = \pi(e)$. By

Theorem A, the orbit $Z \cdot a$ is closed in the affine variety G/L .
Consequently, by Lemma 10.1.3, Z/Z_a is an affine variety and clearly
$Z_a = Z \cap L$.

10.2. Complete orbits and parabolic subgroups.

PROPOSITION 10.2.1. *Let* S *be a linearly reductive group, let* G
be a connected S-*group and let* $Z = \left(G^S\right)^0$. *Let* P *be an* S-*stable*
parabolic subgroup of G *and let* $X = G/P$. *Let* G *act on* X *by left*
translation and let S *act on* X *by* $s \cdot (gP) = (s \cdot g)P$. *Then* Z *acts*
transitively on each irreducible component of X^S . *In particular,* Z *has*
only a finite number of orbits on X^S *and each such orbit is a complete*
variety. Let P *denote the set of subgroups of* G *which are* G-*conjugate*
to P . *Then* P *is stable under the natural action of* S *on subgroups of*
G . *If* $Q \in P^S$, *then* $Z \cap Q$ *is a parabolic subgroup of* Z .

Proof. Since S and G act compatibly on X , these actions
determine a morphic action of the semi-direct product $H = G \ltimes S$ on X as
in Section 2. Consequently, by Theorem A, Z acts transitively on each
irreducible component of X . The other conclusions of Proposition 10.2.1
now follow easily.

REMARKS. (i) For the case of Proposition 10.2.1 in which S is a
torus subgroup of G acting by inner automorphisms and P is Borel
subgroup of G containing S , we recover a result of Chevalley (see [1,
p. 279, 11.18]).

(ii) The example of 3.3 shows that, in Proposition 10.2.1, it is not
necessarily the case that all orbits of Z on X^S have the same
dimension.

10.3. Compact orbits and uniform subgroups. We recall that a closed
subgroup L of the Lie group G is *uniform* if the coset space G/L is
compact.

PROPOSITION 10.3.1. *Let* G *be a connected Lie group and let* S *be*
a subgroup of Aut(G) *such that the representation of* S *on* $\underline{g}$ *is*
semisimple. Let L *be an* S-*stable uniform subgroup of* G , *let* G *act*

on $X = G/L$ by left translations and let S act on X by
$s \cdot (gL) = (s \cdot g)L$. Then X^S is a closed differentiable submanifold of X
and $Z = \left(G^S\right)^0$ acts transitively on each connected component of X^S . In
particular, Z has only a finite number of orbits on X^S and each such
orbit is compact. Moreover $Z \cap L$ is a uniform subgroup of Z .

Proof. Let $N = \{a \in \text{Aut}(G) \mid a \cdot L = L\}$. We let N act on X by
$a \cdot (gL) = (a \cdot g)L$. Then the actions of N and G on X are compatible
and hence determine an action of $H = G \ltimes N$ on X . The proof of
Proposition 10.3.1 now follows immediately from Theorem B.

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