FREE FINITE GROUP ACTIONS ON RATIONAL HOMOLOGY 3-SPHERES

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Abstract

We use methods from the cohomology of groups to describe the finite groups which can act freely and homologically trivially on closed 3-manifolds which are rational homology spheres.

1. Introduction

Cooper and Long [10] have shown that every finite group can act freely and smoothly on some closed, oriented 3-manifold M with the rational homology of the 3-sphere (for brevity, we shall call such an object *a rational homology 3-sphere*). However, under the natural condition that the action must induce the identity on the integral homology of M, new group theoretic restrictions arise. In this article, we apply group cohomology to establish necessary conditions for such *homologically trivial* actions and use this information to construct some new examples.

If a finite group G acts freely on an integral homology 3-sphere, then G must have periodic cohomology of period two or four. Those which act freely on the standard 3-sphere are the finite fundamental groups of closed 3-manifolds. A basic example of a free action on a rational homology sphere is given by the free action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on the real projective space $\mathbb{R}\mathbf{P}^3$, which arises from

the free action of the quaternion group $Q(8)$ on the 3-sphere. This can be seen by considering the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Q(8) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 1.$$

We will show that all finite groups acting freely and homologically trivially on rational homology spheres can be described in this way.

If π denotes a finite set of prime numbers and A is a finite abelian group, we let $A_{(\pi)} =: \bigoplus_{p \in \pi} A_p$, where A_p is the p-primary subgroup of A. Our main result is the following.

THEOREM A. *Let G denote a finite group acting freely and homologically trivially on a rational homology 3-sphere M. Let π denote the set of precisely those primes which divide both $|G|$ and $|H_1(M; \mathbb{Z})|$. Then there exists an extension*

$$1 \to H_1(M; \mathbb{Z})_{(\pi)} \to Q_\pi \to G \to 1,$$

where $H_1(M; \mathbb{Z})_{(\pi)}$ is a central, cyclic subgroup and Q_π is a finite group with periodic cohomology of period two or four.

To investigate the corresponding existence problem, we raise the following question.

Question. Let G be a finite group with periodic cohomology of period four. Does G act freely and homologically trivially on some rational homology 3-sphere?

A complete list of such groups is given in Milnor [**26**, Section 3], and those which can act freely and orthogonally on $\mathbb{S}^3$ were listed by Hopf [**18**]. Perelman [**20**] showed that the remaining groups in Milnor's list do not arise as the fundamental group of any closed, oriented 3-manifold (some families were earlier eliminated by Lee [**19**]). For those with quaternion 2-Sylow subgroups $Q(2^k)$, $k \geqslant 4$, we have a nonexistence result in our setting.

THEOREM B. *Let G be a finite group of period four which acts freely and homologically trivially on a rational homology 3-sphere. Then either (i) G is a quotient of a finite 3-manifold group by a central cyclic subgroup or (ii) the 2-Sylow subgroup of G is quaternion of order 8.*

Remark. In Proposition 6, we show that the quotient examples in part (i) include periodic groups with noncentral elements of order two (which is impossible for groups acting freely on any sphere [**26**, Corollary 1]). Proposition 8 establishes part (ii) by ruling out the groups with larger quaternion 2-Sylow subgroups in

Milnor's list. Moreover, our result rules out all quotients of these groups by central cyclic subgroups.

Among the groups in Milnor's list, the groups $Q(8n, k, l)$ with 2-Sylow subgroup $Q(8)$ (that is, those with n odd) have been much studied, and it is known that some (but not all) can act freely on integral homology 3-spheres (see Madsen [22]). This work gives some new examples of existence in the setting of Theorem A via quotients by the action of central cyclic subgroups (see Proposition 9). The results of Pardon [28] provided free actions of period four groups on rational homology 3-spheres with some control on the torsion but did not address the homological triviality requirement (see Proposition 5). More information about the actions of the groups $Q(8n, k, l)$ is given in Theorem 3.

This article is organized as follows: in Section 2, we apply methods from group cohomology to actions on rational homology spheres; in Section 3, we consider the restrictions arising in the homologically trivial case; in Section 4, we discuss the existence of homologically trivial actions, and finally Section 5 deals with applications of our approach to finite quotients of more general fundamental groups of closed 3-manifolds.

2. Application of cohomological methods

Let G denote a finite group acting freely and smoothly and preserving orientation on a closed 3-manifold M that is a rational homology sphere. In dimension three, free actions of finite groups by homeomorphisms are equivalent to smooth actions, and the quotient manifolds are homotopy equivalent to finite CW complexes (see [7]).

We denote by $\Omega^r(\mathbb{Z})$ the $\mathbb{Z}G$ module uniquely defined in the stable category (where $\mathbb{Z}G$-modules are identified up to stabilization by projectives) as the r-fold dimension shift of the trivial module $\mathbb{Z}$. Note the isomorphism of $\mathbb{Z}G$-modules $H_1(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z})$; we may use either version depending on the context. We refer to [2, 6] for background on group cohomology and to [8, Ch. 2, Sections 5–6] for a quick introduction to the stable category.

PROPOSITION 1. *If a finite group G acts freely on a rational homology 3-sphere M, then there is a short exact sequence of $\mathbb{Z}G$-modules in the stable category of $\mathbb{Z}G$-modules of the form*

$$0 \to \Omega^{-2}(\mathbb{Z}) \to \Omega^2(\mathbb{Z}) \to H_1(M; \mathbb{Z}) \to 0.$$

Proof. We will assume that M is a G-CW complex with cellular chains $C_*(M)$. Then we have exact sequences of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \to C_3(M) \to C_2(M) \to B_1 \to 0$$
$$0 \to Z_1 \to C_1(M) \to C_0(M) \to \mathbb{Z} \to 0$$
$$0 \to B_1 \to Z_1 \to H_1(M;\mathbb{Z}) \to 0,$$

where B_1 denotes the module of boundaries and Z_1 the module of cycles. The result follows from (stably) identifying Z_1 with $\Omega^2(\mathbb{Z})$ and B_1 with $\Omega^{-2}(\mathbb{Z})$. $\quad\square$

Note that the stable map $\Omega^{-2}(\mathbb{Z}) \to \Omega^2(\mathbb{Z})$ defines an element

$$\sigma \in \underline{\mathrm{Hom}}_{\mathbb{Z}G}(\Omega^{-2}(\mathbb{Z}), \Omega^2(\mathbb{Z})) \cong \underline{\mathrm{Hom}}_{\mathbb{Z}G}(\mathbb{Z}, \Omega^4(\mathbb{Z})) \cong \widehat{H}^{-4}(G, \mathbb{Z}).$$

This class appears when applying Tate cohomology to the sequence in Proposition 1.

COROLLARY 1. *The short exact sequence in Proposition 1 yields a long exact sequence in Tate cohomology*

$$\cdots \to \widehat{H}^{i+2}(G,\mathbb{Z}) \xrightarrow{\cup \sigma} \widehat{H}^{i-2}(G,\mathbb{Z}) \to \widehat{H}^i(G, H_1(M;\mathbb{Z})) \to \widehat{H}^{i+3}(G,\mathbb{Z}) \to \cdots$$

determined by the class $\sigma \in \widehat{H}^{-4}(G,\mathbb{Z})$ *which is the image of* $1 \in \widehat{H}^0(G,\mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$.

Next, we identify the class σ geometrically.

PROPOSITION 2. *If* $[M/G] \in H_3(M/G, \mathbb{Z})$ *denotes the fundamental class of the quotient manifold, then* σ *is the image of* $c_*[M/G] \in H_3(BG, \mathbb{Z})$, *under the natural isomorphism* $H_3(BG, \mathbb{Z}) \cong \widehat{H}^{-4}(G, \mathbb{Z})$, *where* $c: M/G \to BG$ *is the classifying map of the covering.*

Proof. The description due to MacLane [21, Ch. V.8] of $\mathrm{Tor}_3^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong H_3(G; \mathbb{Z})$ via chain complexes shows that the image of the fundamental class $c_*[M/G] \in H_3(G; \mathbb{Z})$ is represented by the chain complex $C_*(M)$ of finitely generated free $\mathbb{Z}G$-modules. We can apply dimension-shifting in the 'complete' Ext-theory to the formula

$$\mathrm{Tor}_3^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) = \widehat{H}^{-4}(G; \mathbb{Z}) = \mathrm{Ext}_{\mathbb{Z}G}^{-4}(\mathbb{Z}; \mathbb{Z}) = \underline{\mathrm{Hom}}_{\mathbb{Z}G}(\mathbb{Z}, \Omega^4\mathbb{Z})$$

to identify $c_*[M/G]$ with the extension class of the sequence in Proposition 1 (see Wall [32, Section 2] for more background). $\quad\square$
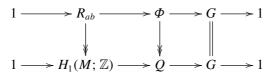
Similarly, the map $\Omega^2(\mathbb{Z}) \to H_1(M, \mathbb{Z})$ defines an extension class

$$\mathcal{E}_M \in H^2(G, H_1(M;\mathbb{Z})),$$

which appears in the long exact sequence from Corollary 1 as the image of the generator under the map $\widehat{H}^0(G, \mathbb{Z}) \to \widehat{H}^2(G, H_1(M; \mathbb{Z}))$. This algebraic map arises geometrically as follows. Let $X \subset M/G$ denote a connected one-dimensional $G - CW$ subcomplex such that $\pi_1(X) \to \pi_1(M/G)$ is onto. If we denote $F = \pi_1(X)$, then we have a diagram of extensions

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(M/G) & \longrightarrow & G & \longrightarrow & 1
\end{array}
$$

Abelianizing kernels gives rise to the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & R_{ab} & \longrightarrow & \Phi & \longrightarrow & G & \longrightarrow & 1 \\
& & \| & & \downarrow & & \| & & \\
1 & \longrightarrow & H_1(M; \mathbb{Z}) & \longrightarrow & Q & \longrightarrow & G & \longrightarrow & 1
\end{array}
$$

where Φ is the associated free abelianized extension. This extension realizes the universal class of highest exponent in $\widehat{H}^2(G, \Omega^2(\mathbb{Z}))$; note that R_{ab} is a free abelian group which, as a $\mathbb{Z}G$-module, is stably equivalent to $\Omega^2(\mathbb{Z})$. By construction, the bottom extension represents the class $\mathcal{E}_M \in H^2(G, H_1(M, \mathbb{Z}))$ (see [17, page 207]).

It is known (see [4]) that free group actions can be fruitfully analyzed using exponents. For actions on rational homology 3-spheres, the analysis can be done quite explicitly. We recall the following.

DEFINITION 1. If A is a finite abelian group, the exponent of A, $\exp(A)$ is defined as the smallest integer $N > 0$ such that $Na = 0$ for all $a \in A$. For an element $a \in A$, its exponent $\exp(a)$ is defined as the exponent of the subgroup generated by a.

COROLLARY 2. *Let G denote a finite group acting freely on a rational homology 3-sphere M, then $|G| = \exp(\sigma) \cdot \exp(\mathcal{E}_M)$. The element $\sigma \in \widehat{H}^{-4}(G, \mathbb{Z})$ is invertible in the graded ring $\widehat{H}^*(G, \mathbb{Z})$ (and the group G has periodic cohomology) if and only if the extension Q representing $\mathcal{E}_M$ is split.*

Proof. Consider the exact sequence in Corollary 1 at $i = 2$:

$$
\cdots \to \widehat{H}^4(G; \mathbb{Z}) \xrightarrow{\cup \sigma} \widehat{H}^0(G; \mathbb{Z}) \to \widehat{H}^2(G; H_1(M; \mathbb{Z})) \to \cdots.
$$

Recall that by Tate duality (see Brown [6, Ch. VI.7]), there is an element $\sigma^* \in \widehat{H}^4(G, \mathbb{Z})$ such that $\sigma \cup \sigma^* = |G|/\exp(\sigma) \in \mathbb{Z}/|G|\mathbb{Z}$. This implies that the

exponent of the subgroup of $\widehat{H}^0(G; \mathbb{Z})$ generated by the image of $\cup \sigma$ is precisely $\exp(\sigma)$. Furthermore, from the definition of $\mathcal{E}_M$, we see that $\exp(\operatorname{coker} \cup \sigma) = \exp(\mathcal{E}_M)$. Using the fact that for cyclic groups the exponent and the order are equal, we obtain the formula $|G| = \exp(\sigma) \cdot \exp(\mathcal{E}_M)$.

A finite group has periodic cohomology if and only if $\widehat{H}^*(G, \mathbb{Z})$ has an element of nonzero degree with exponent equal to $|G|$ (see [6, Ch. VI, Theorem 9.1]). This element is invertible in Tate cohomology, and the cup product by this element is an isomorphism in all degrees. From our formula, we see that the class σ has this highest exponent if and only if $\mathcal{E}_M = 0$, which is equivalent to the splitting of the extension Q. □

REMARK 1. If σ is invertible, in fact, it can be shown that $\widehat{H}^i(H, H_1(M, \mathbb{Z})) = 0$ for every subgroup $H \subset G$ and all $i \in \mathbb{Z}$. In other words, the module $H_1(M, \mathbb{Z})$ is cohomologically trivial. This holds, for example, if $|G|$ is relatively prime to $|H_1(M; \mathbb{Z})|$.

REMARK 2. On the other extreme, if σ is trivial, then the extension class $\mathcal{E}_M$ has highest exponent equal to $|G|$. Using the stable isomorphism

$$\underline{\operatorname{Hom}}_{\mathbb{Z}G}(\Omega^2(\mathbb{Z}), H_1(M, \mathbb{Z})) \cong \underline{\operatorname{Hom}}_{\mathbb{Z}G}(\mathbb{Z}, \Omega^{-2}(H_1(M, \mathbb{Z}))),$$

we can represent $\mathcal{E}_M$ by a rank-one trivial submodule in $\Omega^{-2}(H_1(M, \mathbb{Z}))$ (note that any finitely generated $\mathbb{Z}G$-module is stably equivalent to a $\mathbb{Z}$-torsion-free module via dimension-shifting). By [1, Theorem 1.1] and its proof, the short exact sequence

$$0 \to \mathbb{Z} \to \Omega^{-2}(H_1(M, \mathbb{Z})) \to \Omega^{-5}\mathbb{Z} \to 0$$

in the stable category is split exact. After shifting back, we obtain a stable decomposition $H_1(M, \mathbb{Z}) \cong \Omega^2(\mathbb{Z}) \oplus \Omega^{-3}(\mathbb{Z})$. This will occur for rational homology spheres with a free G-action where $H^4(G, \mathbb{Z}) = 0$. An example of this phenomenon is given by the Mathieu group M_{23} (see [25]).

REMARK 3. For $G = (\mathbb{Z}/p\mathbb{Z})^r$ we have that $p \cdot \widehat{H}^k(G, \mathbb{Z}) = 0$ for $k \neq 0$, by the Künneth Theorem, so we see that the exponent of $\mathcal{E}_M$ must be at least p^{r-1}. In particular, the module $H_1(M; \mathbb{Z})$ must have p^{r-1}-torsion.

REMARK 4. Given a free action of a finite group G on a rational homology sphere M, we can consider the Serre spectral sequence for the fibration $M \to M/G \to BG$. For $i \geqslant 1$, the maps $\widehat{H}^i(G, \mathbb{Z}) \to \widehat{H}^{i+2}(G, H_1(M, \mathbb{Z}))$ in the long exact sequence of Corollary 1 can be identified with the differentials $d_2: E_2^{i,3} \to E_2^{i+2,2}$. Similarly, the maps $\widehat{H}^i(G, H_1(M, \mathbb{Z})) \to H^{i+3}(G, \mathbb{Z})$ can be identified

with the differentials $d_3 \colon E_3^{i,2} \to E_3^{i+3,0}$. In particular, exactness at the term $\widehat{H}^r(G, H^2(M, \mathbb{Z}))$ reflects the fact that $E_\infty^{r,2} = 0$ for $r \geqslant 2$ given that M/G is a 3-manifold.

3. Restrictions in the homologically trivial case

In this section, we focus on the special case when the G-action on M is trivial in homology. This imposes some drastic restrictions.

PROPOSITION 3. *If G acts freely and homologically trivially on a rational homology 3-sphere M, then every elementary abelian subgroup of G has rank at most two.*

Proof. We may assume that $G = (\mathbb{Z}/p\mathbb{Z})^r$ for some prime p. Note that $\widehat{H}^k(G, H_1(M; \mathbb{Z}))$ has exponent p when $H_1(M; \mathbb{Z})$ has trivial action and $k \neq 0$, by the Künneth Theorem. From Corollary 2, we see that if $(\mathbb{Z}/p\mathbb{Z})^r$ acts freely and homologically trivially on M, then p^r divides p^2 and the result follows. □

Let us write the trivial $\mathbb{Z}G$-module $H_1(M; \mathbb{Z})$ as a direct sum of finitely generated, finite abelian p-groups $A_p = H_1(M; \mathbb{Z})_{(p)}$. Then we have the following.

LEMMA 1. *If p is a prime number dividing the order of G, then $H_1(M; \mathbb{Z})_{(p)}$ is either trivial or cyclic.*

Proof. Consider a cyclic $C \cong \mathbb{Z}/p\mathbb{Z}$ in G. Since $H^7(C; \mathbb{Z}) = 0$, from the sequence in Corollary 1 for C at $i = 4$, we see that $H^4(C, H_1(M; \mathbb{Z}))$ is a homomorphic image of $H^2(C, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. It follows that $H^4(C, H_1(M; \mathbb{Z})) \cong H^4(C, A_p) \cong A_p/pA_p$ is either trivial or $\mathbb{Z}/p$, which proves the result. □

Next, we will show that all groups acting freely and homologically trivially on rational homology spheres can be described as quotients of periodic groups.

THEOREM 1. *Let G denote a finite group acting freely and homologically trivially on a rational homology 3-sphere M. Let $\pi = p_1 \ldots p_r$ denote the product of precisely those primes which divide both $|G|$ and $|H_1(M; \mathbb{Z})|$. Then there exists an extension*

$$1 \to H_1(M; \mathbb{Z})_{(\pi)} \to Q_\pi \to G \to 1,$$

where $H_1(M; \mathbb{Z})_{(\pi)}$ is a central, cyclic subgroup and Q_π is a group with periodic cohomology.

Proof. Suppose that G acts freely on a rational homology 3-sphere M and consider the group extension

$$1 \to \pi_1(M) \to \pi_1(M/G) \to G \to 1.$$

Let L denote the kernel of the map $\pi_1(M) \to H_1(M; \mathbb{Z})_{(\pi)}$; then it is normal in both $\pi_1(M)$ and $\pi_1(M/G)$, and we can consider the associated central quotient extension:

$$0 \to H_1(M; \mathbb{Z})_{(\pi)} \to Q_\pi \to G \to 1.$$

Note that $H^2(G, H_1(M; \mathbb{Z})) \cong H^2(G, H_1(M; \mathbb{Z})_{(\pi)})$ since $H^2(G; A_p) = 0$ for $p \nmid |G|$, where A_p denotes the p-primary part of $H_1(M; \mathbb{Z})$ as before. Our construction of the extension for Q_π is the obvious quotient of the extension for Q representing the class $\mathcal{E}_M$ appearing in Corollary 2.

Suppose p is a prime that divides $|G|$ but which is relatively prime to $|H_1(M; \mathbb{Z})|$. Then $H^2(\mathrm{Syl}_p(G), H_1(M; \mathbb{Z})) = 0$ and so $\mathrm{Syl}_p(G) = \mathrm{Syl}_p(Q_\pi)$ is periodic by Corollary 2. Now suppose that p is a prime which divides π, and let $C \subset G$ denote a cyclic subgroup of order p. By naturality, we have a commutative diagram, where the rows are exact sequences:

$$
\begin{array}{ccccc}
\widehat{H}^0(G, \mathbb{Z}) & \longrightarrow & \widehat{H}^2(G, H_1(M; \mathbb{Z})) & \longrightarrow & \widehat{H}^5(G, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{H}^0(C, \mathbb{Z}) & \stackrel{\approx}{\longrightarrow} & \widehat{H}^2(C, H_1(M; \mathbb{Z})) & \longrightarrow & 0
\end{array}
$$

The isomorphism in the lower row of this diagram comes from the rest of the sequence

$$0 \to \widehat{H}^1(C; H_1(M; \mathbb{Z})) \to \widehat{H}^4(C; \mathbb{Z}) \to \widehat{H}^0(C; \mathbb{Z}) \to \widehat{H}^2(C, H_1(M; \mathbb{Z})) \to 0$$

since $\widehat{H}^1(C; H_1(M; \mathbb{Z})) \cong \mathbb{Z}/p\mathbb{Z} \cong \widehat{H}^4(C; \mathbb{Z})$.

By Lemma 1, the p-component of $H_1(M; \mathbb{Z})$ is a finite cyclic p-group with a trivial C-action. Hence, $\widehat{H}^i(C, H_1(M; \mathbb{Z})) \neq 0$ for all i. The map $\widehat{H}^0(G, \mathbb{Z}) \to \widehat{H}^0(C, \mathbb{Z})$ sends a generator to a generator, so the extension class $\mathcal{E}_M \in H^2(G, H_1(M; \mathbb{Z}))$ restricts nontrivially on all such subgroups C, and, hence, the corresponding restricted extensions of the form $0 \to H_1(M; \mathbb{Z})_{(\pi)} \to Q_\pi|_C \to C \to 1$ are all nonsplit.

If we take $H = H_1(M; \mathbb{Z})_{(\pi)}$, which we know to be cyclic by Lemma 1, then the extension expresses G as the quotient Q_π/H where H is a central, cyclic subgroup and every restricted group of the form $Q_\pi|C$ is nonsplit, where $C \cong \mathbb{Z}/p\mathbb{Z}$ and p divides π. Let $u \in Q_\pi$ denote an element of order p; if the subgroup generated by H and u is not cyclic, then it must be split abelian, a contradiction.

Therefore, all elements of order p in Q_π lie in H, a cyclic subgroup, and so Q_π has no rank-two p-elementary abelian subgroups. We have already established this for the primes which do not divide π, whence we infer that Q_π has periodic cohomology. $\qquad\square$

PROPOSITION 4. *The period of Q_π is two or four.*

Proof. Consider the central group extension $1 \to H \to Q \to G \to 1$ where $H := H_1(M, \mathbb{Z})_{(\pi)}$ and $Q := Q_\pi$. By Swan [30], the period of Q is the least common multiple of the p-periods of Q taken over all primes p dividing $|Q|$. The p-periods are determined by group cohomology with p-local coefficients. By [30, Theorem 1], the 2-period of Q is 2 or 4. Moreover, by [30, Theorem 2], the p-period of Q for p odd is twice the order of $\Phi_p(Q) \cong N_Q(\mathrm{Syl}_p(Q))/C_Q(\mathrm{Syl}_p(Q))$, the group of automorphisms of $\mathrm{Syl}_p(Q)$ induced by inner automorphisms of Q. Note that as $\mathrm{Syl}_p(Q)$ is cyclic, its automorphism group is also cyclic and hence $\Phi_p(Q)$ is cyclic of order prime to p.

As explained in [30, Lemma 3], the action on $\widehat{H}^{2i}(\mathrm{Syl}_p(Q), \mathbb{Z}_{(p)})$ is given by multiplication by r^i, where r is an integer prime to p that is a multiplicative generator of $\Phi_p(Q)$. Hence, this action has invariants only when i is a multiple of $|\Phi_p(Q)|$, and $\widehat{H}^*(Q, \mathbb{Z}_{(p)}) \neq 0$ only in degrees which are multiples of $2|\Phi_p(Q)|$.

If p divides $|G|$, then the projection $Q \to G$ induces an isomorphism $\Phi_p(Q) \cong \Phi_p(G)$, and hence the p-periods of Q and G are equal. Consider now the following portion of the p-local version of the long exact cohomology sequence from Corollary 2.3: $\widehat{H}^4(G; \mathbb{Z}_{(p)}) \to \widehat{H}^0(G; \mathbb{Z}_{(p)}) \to \widehat{H}^2(G; H_p)$, where $H_p = \mathrm{Syl}_p(H) = H_1(M; \mathbb{Z}_{(p)})$. As $|G|$ is divisible by p, the middle term is nonzero. Now if $\widehat{H}^2(G; H_p) = 0$, then $\widehat{H}^4(G; \mathbb{Z}_{(p)}) \neq 0$ and we conclude that G has p-period dividing four (a p-local version of Corollary 2). However, by the universal coefficient theorem applied to the trivial G-module H_p, we see that $\widehat{H}^2(G, H_p) = 0$ if the p-period of G is four or higher. Hence, we conclude that the p-periods of G and Q must both be either two or four. $\qquad\square$

Theorem A follows from Theorem 1 and Proposition 4. The structure of G is more explicit for p-groups.

COROLLARY 3. *A finite p-group G acts freely and homologically trivially on some rational homology 3-sphere M with nontrivial p-torsion in $H_1(M; \mathbb{Z})$ if and only if (i) G is cyclic or (ii) $p = 2$, $H_1(M; \mathbb{Z})_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ and G is a dihedral group.*

Proof. The finite groups of the form Q/H where Q is a periodic p-group and H is a nontrivial central cyclic subgroup are precisely the cyclic groups and

$Q(2^n)/Z(Q(2^n))$, where $Q(2^n)$ is a generalized quaternion group of order 2^n, $n \geqslant 3$, with center isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and quotient a dihedral group of order 2^{n-1}. Conversely, all the groups Q appearing above act freely on $\mathbb{S}^3$, and hence all the quotients $G = Q/H$ act freely on a rational homology sphere; note that these actions are homologically trivial as $H_1(\mathbb{S}^3/H, \mathbb{Z}) \cong H$, a central subgroup. $\quad\square$

COROLLARY 4. *Let G act freely and homologically trivially on a rational homology 3-sphere M.*

(i) *If both $|G|$ and $|H_1(M; \mathbb{Z})|$ are even, then $\mathrm{Syl}_2(G)$ is either cyclic or dihedral.*

(ii) *If p is an odd prime dividing $|G|$, then $\mathrm{Syl}_p(G)$ is cyclic.*

(iii) *If $(|H_1(M; \mathbb{Z})|, p) = 1$, then $\mathrm{Syl}_p(G)$ is either cyclic or generalized quaternion.*

REMARK 5. It is also an interesting problem to determine which groups can act homologically trivially on higher dimensional rational homology spheres. Using exponents, it can be shown that if G acts freely and homologically trivially on a simply connected rational homology n-sphere, then the rank of G can be at most $n - 2$. We expect that further group theoretic restrictions will play a role.

4. Existence of homologically trivial actions

As mentioned in Section 1, any finite group can act freely on some rational homology 3-sphere if there is no homological triviality assumption. This was first proved by Pardon [28] using local surgery theory, extending a result of Browder and Hsiang [5, page 267]. The direct three-dimensional argument of Cooper and Long [10] avoids the surgery formalism but does not give any control on the torsion in $H_1(M; \mathbb{Z})$.

PROPOSITION 5. *Let G be a finite group and p a prime such that $(p, |G|) = 1$. Then G acts freely on some p-local homology 3-sphere.*

Proof. This statement is a special case of Pardon [28, Theorem B], together with the standard remark that high-dimensional surgery existence results in dimensions $4k + 3 \geqslant 7$ imply existence results in dimension three up to homology. $\quad\square$

It appears to be much more difficult to solve the existence problem for a *given* rational homology 3-sphere M. For example, what if we consider only the space form groups but do not require homologically trivial actions?

Question. If G acts freely on $\mathbb{S}^3$, can it act freely on a given rational homology 3-sphere M with $(|G|, |H_1(M; \mathbb{Z})|) = 1$?

We will now use the information in Theorem A to make some remarks about the existence of *homologically trivial* actions on rational homology 3-spheres.

The finite groups which can act freely on $\mathbb{S}^3$ are now known (by the work of Perelman [20]): they are precisely the periodic groups in Hopf's list [18]. For any of these groups, we can obtain examples of homologically trivial actions on quotients $\mathbb{S}^3/H$, where H is a central cyclic subgroup.

EXAMPLE 1. Let H denote the quotient of the binary dihedral group $Q(4n)$ by its unique central subgroup of order two. Then H is dihedral of order $2n$ and has rank two if n is even.

Of the remaining period four groups (those which do not act freely on $\mathbb{S}^3$), we first consider those which do not satisfy Milnor's $2p$-condition, which requires that every subgroup of order $2p$, for p prime, must be cyclic (see [26, Corollary 1]).

PROPOSITION 6. *Let G be a finite group with periodic cohomology of period four containing a noncyclic subgroup of order $2p$, for some odd prime p. Then G is the product of a dihedral group by a cyclic group of relatively prime order. Any quotient of G by a central cyclic subgroup can act freely and homologically trivially on some rational homology 3-sphere.*

Proof. This follows by checking the list of periodic groups, taking into account our period four assumption. A convenient reference is Wall [33, Theorem 4.5], which states that G is an extension of a normal subgroup G_0 of odd order by a group G_1 isomorphic to one of the form $C(2^k)$, $Q(2^k)$, T_v^*, O_v^*, $\mathrm{SL}_2(p)$ or $\mathrm{TL}_2(p)$. The periods of these groups are listed in [33, Corollary 5.6]. In our case, G_0 must be cyclic (the only odd order group with period $\leqslant 4$), and $G_1 = C(2^k)$ since there is a unique element of order two in the other cases. The action of G_1 on G_0 must be faithful to violate the $2p$ condition, and G of period four implies the claimed structure for G.

Since any quotient of G by a central subgroup is again of the same form, the required actions arise by quotients of a free action on $\mathbb{S}^3$ by products of a binary dihedral group $Q(4n)$, for n odd, with a cyclic group of coprime order. $\square$

Milnor [26, Section 3] listed the period four groups which do satisfy the $2p$ conditions and identified two families of such a group which (by Perelman [20]) cannot act freely on $\mathbb{S}^3$. They are the following:

(i) $Q(8n, k, l)$, with $n > k > l \geqslant 1$ and $8n, k, l$ pairwise relatively prime;

(ii) $O(48, k, l)$, with l odd, $3 \nmid l$ and $48, k, l$ pairwise relatively prime.

One can also take the product of any one of these groups with a cyclic group $C(r)$ of relatively prime order. We will refer to those listed in (i) as *type A* if n is odd or *type B* if $n \geqslant 2$ is even and those in (ii) as *type C* for $O(48, k, l)$. The groups of type B or C have orders divisible by 16. The direct product of one of these groups with a cyclic group of coprime order will be called a *generalized* type A, B or C periodic group. From the presentations given in [**26**, Section 3], one can check that the groups of type A, B or C all have quaternion 2-Sylow subgroup and contain no nontrivial central subgroups of odd order.

The following is a useful observation.

LEMMA 2. *Let G be a period four group with quaternion 2-Sylow subgroup. If G acts freely and homologically trivially on a rational homology 3-sphere M, then $H_1(M; \mathbb{Z})$ has odd order.*

Proof. From the extension diagram in Section 2, we have a short exact sequence

$$1 \to H_1(M; \mathbb{Z}) \to Q \to G \to 1.$$

Note that $S = \mathrm{Syl}_2(G)$ is a quaternion group. Now the restricted sequence from Corollary 1 with $i = 2$ gives an injection

$$0 \to H^1(S; H_1(M; \mathbb{Z})) \to \widehat{H}^4(S; \mathbb{Z}) \to \widehat{H}^0(S; \mathbb{Z}).$$

Since S has period four, $\widehat{H}^4(S; \mathbb{Z}) = \mathbb{Z}/|S|\mathbb{Z}$, whereas the group $H^1(S; H_1(M; \mathbb{Z}))$ contains $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ unless $H_1(M; \mathbb{Z})$ is of odd order. $\square$

We have a sharper result for the groups in Milnor's list.

PROPOSITION 7. *Suppose that G is a period four group of type A, B or C. If G acts freely and homologically trivially on a rational homology 3-sphere M, then we have $(|H_1(M; \mathbb{Z})|, |G|) = 1$.*

Proof. We start again from the short exact sequence:

$$1 \to H_1(M; \mathbb{Z}) \to Q \to G \to 1.$$

Consider the pushout sequence from Theorem 1

$$1 \to H_1(M; \mathbb{Z})_{(\pi)} \to Q_\pi \to G \to 1$$

and note that Q_π is a period four group by Proposition 4.

Since $2 \notin \pi$, it follows that Q_π is an extension of G by a central cyclic subgroup of odd order. Since G has type A, B or C, the group G contains no nontrivial central subgroups of odd order. It follows that $H_1(M; \mathbb{Z})_{(\pi)} = 0$, and hence $Q_\pi = G$. In particular, this implies that $H_1(M; \mathbb{Z})$ has order relatively prime to $|G|$. $\quad\square$

We need one more observation about the structure of the groups in Milnor's list.

LEMMA 3. *Let G be a nonperiodic quotient of a generalized type B or C period four group by a central cyclic subgroup. Then $G = C(k) \times G_1$, where G_1 is the quotient of a type B or C period four group by a central cyclic subgroup and k is coprime to the order of G_1.*

Proof. Suppose that G is a nonperiodic quotient of a generalized type B or C group by a central cyclic subgroup $C(s)$. We have an exact sequence of the form

$$1 \to C(s) \to C(r) \times Q_1 \to G \to 1,$$

where Q_1 has type B or C and r has (odd) order coprime to $|Q_1|$. Since Q_1 has a unique (central) element of order two and G is nonperiodic, $s = 2s_1$, where s_1 is odd. Since Q_1 has no nontrivial central subgroups of odd order, we have $r = ks_1$ and $G = C(k) \times G_1$, where G_1 is a nonperiodic quotient of Q_1. $\quad\square$

These results allow us to rule out the groups of (generalized) types B and C. Note that by Lemma 2, if such a group acts freely and homologically trivially on a rational homology 3-sphere M, then M must be a $\mathbb{Z}_{(2)}$-homology sphere. Theorem B follows from the following result.

PROPOSITION 8. *Let G be the quotient of a (generalized) type B or C period four group by a central cyclic subgroup. Then G cannot act freely and homologically trivially on a rational homology 3-sphere.*

Proof. The period four groups of (generalized) type B or C themselves cannot act freely on any $\mathbb{Z}_{(2)}$-homology 3-sphere (see Ronnie Lee [19, Corollary 4.15, Corollary 4.17]), so they are ruled out by Lemma 2.

By Lemma 3 and restricting the action if necessary, we may assume that G is the quotient of a type B or C group by a central cyclic subgroup.

Now suppose that G acts freely and homologically trivially on a rational homology 3-sphere M. We then have a covering space $M \to M/G \to BG$ and an induced extension $1 \to H_1(M; \mathbb{Z}) \to Q \to G \to 1$. As above, we consider the pushout $1 \to H_1(M; \mathbb{Z})_{(\pi)} \to Q_\pi \to G \to 1$, where Q_π is a period four group and $H_1(M; \mathbb{Z})_{(\pi)}$ is a central cyclic subgroup of Q_π.

Since G is nonperiodic, $2 \in \pi$ and $H_1(M; \mathbb{Z})$ has order $2d$, where d is odd. It follows from the explicit presentations in [**26**, Section 3] that Q_π must again be of type B or C and $H_1(M; \mathbb{Z})_{(\pi)}$ must contain the unique central subgroup $T = C(2)$ of order two in Q_π.

The group Q_π is constructed by a pushout from $\pi_1(M)$, and we can form a further pushout over the projection $H_1(M; \mathbb{Z}) \to T = C(2)$ to obtain the group extension

$$1 \to T \to Q' \to G \to 1$$

in which Q' is again a period four group of type B or C. The 2-fold covering $M' \to M$ given by the quotient $\pi_1(M) \to T$, followed by the G-covering $M \to M/G$, is just the Q'-covering $M' \to M/G$.

To obtain a contradiction, we will now show that M' is a $\mathbb{Z}_{(2)}$-homology sphere. From the structure of $M' \to M$ as a 2-fold covering, we have an exact sequence

$$0 \to H_0(C(2); H_1(M'; \mathbb{Z})) \to H_1(M; \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and $|H_1(M; \mathbb{Z})| = 2d$, with d odd. Hence, the coinvariants $H_0(C(2); H_1(M'; \mathbb{Z}))$ are of odd order, and $H_1(M'; \mathbb{Z})$ has no 2-torsion. We have an exact sequence of $C(2)$-modules involving the torsion subgroup and the torsion-free quotient

$$0 \to H_1(M'; \mathbb{Z})_{\text{tors}} \to H_1(M'; \mathbb{Z}) \to \mathbb{Z}^r \to 0$$

and by applying group homology $H_*(C(2); -)$ to the sequence, we see that $H_0(C(2); \mathbb{Z}^r)$ is a direct sum of r cyclic groups (each one either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, depending on whether the module action of $C(2)$ on each summand of $\mathbb{Z}^r$ is trivial or nontrivial) and conclude that $r = 0$. Hence, $H_1(M'; \mathbb{Z})$ is all odd torsion. In other words, M' is a $\mathbb{Z}_{(2)}$-homology 3-sphere and the free Q'-action cannot exist. $\qquad\square$

The remaining existence question concerns central quotients of the period four groups of type A. It is enough to consider the period four groups themselves.

PROPOSITION 9. *Let G be the quotient of a type A period four group Q by a central cyclic subgroup $T \leqslant Q$. If Q acts freely and homologically trivially on a rational homology 3-sphere M, then G acts freely and homologically trivially on M/T, which is again a rational homology 3-sphere.*

Proof. Let G be the quotient of a type A period four group Q by a central cyclic subgroup $T \leqslant Q$. If Q acts freely and homologically trivially on a rational homology 3-sphere M, then G acts freely on the rational homology 3-sphere M/T.

It remains to show that the G-action on M/T is homologically trivial. Since T is central, the covering space $M \to M/T$ is preserved by Q, and we have an exact sequence

$$0 \to H_1(M; \mathbb{Z}) \to H_1(M/T; \mathbb{Z}) \to H_1(T; \mathbb{Z}) \to 0$$

since $H_2(T; \mathbb{Z}) = 0$ and T acts homologically trivially on M. Looking at the associated long exact sequence in group cohomology and noting that Q acts trivially on both $H_1(M; \mathbb{Z})$ and $H_1(T; \mathbb{Z})$, we obtain

$$0 \to H_1(M; \mathbb{Z}) \to H_1(M/T; \mathbb{Z})^Q \to H_1(T; \mathbb{Z}) \to H^1(Q; H_1(M; \mathbb{Z})).$$

By Proposition 7, we have $(|H_1(M; \mathbb{Z})|, |Q|) = 1$, and hence $H^1(Q; H_1(M; \mathbb{Z})) = 0$. Therefore, $H_1(M/T; \mathbb{Z})^Q = H_1(M/T; \mathbb{Z})$, and the G-action on M/T is homologically trivial. $\square$

The period four groups $G = Q(8n, k, l)$ of type A cannot act freely on $\mathbb{S}^3$ (by Perelman), but some members of this family do act freely on integral homology 3-spheres. For the existence of such actions, there are two obstacles: a finiteness obstruction and a surgery obstruction. Swan [31] showed that for every period four group, there exists a finitely dominated Poincaré 3-complex X with $\pi_1(X) = G$ and universal covering $\widetilde{X} \simeq \mathbb{S}^3$. Such a complex is called a *Swan complex* of type $(G, 3)$.

We recall that the homotopy types of $(G, 3)$-complexes are in bijection (via the first k-invariant) with the invertible elements in $\widehat{H}^4(G; \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$.

LEMMA 4. *Let G be a period four group which acts freely and homologically trivially on a rational homology 3-sphere M. Then there exists a $(G, 3)$-complex X, unique up to homotopy, and a degree 1 map $f : M/G \to X$ compatible with the classifying maps of the G-fold coverings.*

Proof. The classifying map $c : M/G \to BG$ of the covering $M \to M/G$ gives a class $c_*[M] \in H_3(G, \mathbb{Z})$. By Proposition 2, this class corresponds to a generator

$$\sigma^* \in \widehat{H}^4(G; \mathbb{Z}) \cong \widehat{H}^{-4}(G; \mathbb{Z}) \cong H_3(G; \mathbb{Z}).$$

Let X be the $(G, 3)$-complex defined (up to homotopy) by this k-invariant. Since the classifying map $c : M/G \to BG$ is surjective on fundamental groups, it follows that c lifts to a map $f : M/G \to X$. Since the images of the fundamental classes of M/G and X agree in $H_3(G; \mathbb{Z})$, it follows that f has degree 1. $\square$

REMARK 6. Any degree 1 map $f : N \to X$ from a closed oriented 3-manifold to a $(G, 3)$-complex provides a degree 1 normal map by pulling back a framing of the trivial bundle over X.

The Wall finiteness obstruction $\sigma(X) \in \widetilde{K}_0(\mathbb{Z}G)$ vanishes if and only if there exists a finite Swan complex of type $(G, 3)$. This is the first obstruction to existence. By varying the homotopy type of X, Swan defined an invariant $\sigma(G) \in \widetilde{K}_0(\mathbb{Z}G)/S(G)$, depending only on G, where $S(G) \subseteq \widetilde{K}_0(\mathbb{Z}G)$ is the *Swan subgroup* generated by projective ideals of the form $\langle r, \Sigma \rangle \subset \mathbb{Z}G$, where $(r, |G|) = 1$ and Σ denotes the norm element. Then $\sigma(G) = 0$ if and only if $\sigma(X) \in S(G)$ for every Swan complex X of type $(G, 3)$.

PROPOSITION 10. *Let $G = Q(8n, k, l)$, with n odd, be a period four group of type* A. *If G acts freely and homologically trivially on a rational homology 3-sphere, then $\sigma(G) = 0$.*

Proof. Under the given assumptions, G acts freely and homologically trivially on a rational homology 3-sphere M such that $(|H_1(M; \mathbb{Z})|, |G|) = 1$. By Lemma 4, the classifying map $M/G \to BG$ of the covering $M \to M/G$ lifts to a degree 1 map $f : M/G \to X$, to a uniquely defined $(G, 3)$-complex X. Since the map f induces a surjection on fundamental groups, the argument of Mislin [27, Theorem 3.3] shows that $\sigma(M/G) = \sigma(X) + \langle d, \Sigma \rangle \in \widetilde{K}_0(\mathbb{Z}G)$, where $d = |H_1(M; \mathbb{Z})|$, and hence $\sigma(X) \in S(G)$. Since varying the homotopy type of X changes $\sigma(X)$ only by an element of the Swan subgroup (see Swan [31, Lemma 7.3]), we conclude that $\sigma(G) = 0$. $\square$

The secondary obstruction comes from surgery theory (and is defined only if the finiteness obstruction is zero). It can be computed in some cases to show existence (see Madsen [22]). For the type A groups, a $(G, 3)$-complex X has *almost linear k-invariant* $e_0 \in H^4(G; \mathbb{Z})$ if the restriction of e_0 to each Sylow subgroup of G is the k-invariant of a standard free orthogonal action on $\mathbb{S}^3$ (see [22, page 195]).

DEFINITION 2. We will say that a free homologically trivial action of a type A group G on a rational homology 3-sphere M has *almost linear k-invariant* if there exists a degree 1 map $f : M/G \to X$ to a *finite* $(G, 3)$-complex with almost linear k-invariant $e_0 \in H^4(G; \mathbb{Z})$.

REMARK 7. If G acts freely and smoothly on an integral homology 3-sphere Σ, then the quotient manifold $\Sigma/G = X$ is a finite $(G, 3)$-complex with almost linear k-invariant (see [15, Corollary C] and the discussion of [22, Conjecture D]). By Proposition 9, any quotient of such a group G by a central cyclic subgroup would act freely and homologically trivially on a rational homology 3-sphere.

Conversely, we expect that the following existence statement holds.

CONJECTURE 1. *Let $G = Q(8n, k, l)$, with n odd, be a period four group of type* A. *Then G acts freely and homologically trivially on a rational homology 3-sphere with almost linear k-invariant if and only if G acts freely on an integral homology 3-sphere.*

In the remainder of this section, we prove this conjecture under some additional assumptions. If $f \colon N \to X$ denotes a degree 1 normal map to a finite $(G, 3)$-complex, with $\pi_1(X) = G$, then there is *weakly simple* surgery obstruction $\lambda'(f) \in L'_3(\mathbb{Z}G)$. This is defined since every finite Poincaré 3-complex with finite fundamental group is weakly simple (meaning that its Poincaré torsion lies in $SK_1(\mathbb{Z}G)$). We let $\lambda^h(f) \in L^h_3(\mathbb{Z}G)$, the image of $\lambda'(f)$ under the natural map, denote the obstruction to surgery on f up to homotopy equivalence.

Let $H = Q(4ab)$ denote the index two subgroup of G, containing the subgroup $C(4) \leqslant Q(8)$ which acts by inversion on the normal subgroup of order ab.

THEOREM 2 (Madsen [22]). *Suppose that $G = Q(8n, k, l)$, with n odd, is a period four group of type* A *such that $\sigma(G) = 0$. Let $f \colon N \to X$ be a degree 1 normal map to a finite $(G, 3)$-complex with almost linear k-invariant. Then $\lambda'(f) = 0$ if and only if $\mathrm{Res}_H(\lambda'(f)) = 0$ for each subgroup $H \leqslant G$ of the form $H = Q(4ab)$. Furthermore, $\lambda^h(f) = 0$ if and only if $\mathrm{Res}_K(\lambda^h(f)) = 0$ for each subgroup $K \leqslant G$ of the form $K = Q(8a, b)$.*

Proof. This is a summary statement of the calculations in [22, Sections 4–5]. See in particular [22, Theorems 4.19, 4.21 and Corollary 5.12]. □

For the groups $G = Q(8a, b) = Q(8a, b, 1)$, the *top component* $S(ab) \subseteq S(G) \subset \widetilde{K}_0(\mathbb{Z}G)$ of the Swan subgroup is defined as the kernel of the restrictions to all odd index subgroups. For example, Bentzen and Madsen [3, Proposition 4.6] computed $S(Q(8p, q))$, for p, q odd primes, almost completely and showed that $S(pq) = 0$ in many cases (for example, $(p, q) \equiv (\pm 3, \pm 3) \mod 8$; or $(p, q) \equiv (1, \pm 3) \mod 8$, and 2 has odd order mod p).

THEOREM 3. *Let $G = Q(8p, q)$, for odd primes $p > q$, and assume that $S(pq) = 0$. Then G acts freely and homologically trivially on a rational homology 3-sphere with almost linear k-invariant if and only if G acts freely on an integral homology 3-sphere.*

Proof. Remark 7 explains the sufficiency part. For the converse, suppose that $G = Q(8p, q)$ with $S(pq) = 0$ acts freely and homologically trivially on a rational homology 3-sphere M with almost linear k-invariant. Then there exists a

finite $(G, 3)$-complex X with almost linear k-invariant and a degree 1 normal map $f : M/G \to X = X(G)$. By [14, Theorem 3.1], we may assume that the covering space $X(H)$ is homotopy equivalent to an orthogonal spherical space form, for $H = Q(4pq) \leqslant G$, and that the normal invariant restricts to the normal invariant of an orthogonal spherical space form over the 2-Sylow covering $X(Q(8))$. In particular, since $|H_1(M; \mathbb{Z})| = d$ is odd, we must have that $\sigma(X(Q(8)) = \langle d, \Sigma \rangle = 0 \in \widetilde{K}_0(\mathbb{Z}Q(8))$. Hence, $d \equiv \pm 1 \pmod 8$ and $\mathrm{Res}_{Q(8)}(\lambda(f)) = 0$ by [11, Theorem 5.1(ii)]). This information about the normal map $f : M/G \to X$ was extracted from the work of Madsen, Thomas and Wall (see [23, 24]).

Now, we consider the restriction of the surgery obstruction

$$\mathrm{Res}_H(\lambda^h(f)) \in L_3^h(\mathbb{Z}H).$$

Since $X(H)$ is homotopy equivalent to an orthogonal space form, $\mathrm{Res}_H(\lambda^h(f))$ is the surgery obstruction of a normal map between closed manifolds. Therefore, $\mathrm{Res}_H(\lambda^h(f))$ is detected by further restriction to the 2-Sylow subgroup $C(4)$, and hence $\mathrm{Res}_H(\lambda^h(f)) = 0$. It follows that $H_1(M; \mathbb{Z})$ stably supports a hyperbolic linking form, and hence

$$d = |H_1(M; \mathbb{Z})| \equiv r^2 \mod (8ab)$$

is a square. Since $S(pq) = 0$, it follows that $\langle r, \Sigma \rangle = 0 \in \widetilde{K}_0(\mathbb{Z}G)$. Now, by [11, Theorem 5.1(ii)] applied to $\lambda^h(f) \in L_3^h(\mathbb{Z}G)$, we see that $\lambda^h(f) = 0$. Therefore, G acts freely on an integral homology 3-sphere.   □

REMARK 8. By taking full advantage of Madsen's results as summarized in Theorem 2, we could give a statement for the groups $Q(8a, b)$ under the assumption that $S(a', b') = 0$ for all divisors $1 \neq a' \mid a$, and $1 \neq b' \mid b$.

REMARK 9. We would like to remove the almost linear k-invariant assumption. However, the group $G = Q(8)$ acts freely and homologically trivially on M with $H_1(M; \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$ since $Q(8) \times \mathbb{Z}/3\mathbb{Z}$ acts freely on $\mathbb{S}^3$. This action has nonlinear k-invariant in our sense. Indeed, by the proof of Proposition 10, there is a degree 1 map $f : M/G \to X$, where X is a Swan complex for $Q(8)$ with nontrivial finiteness obstruction.

5. Finite quotients of fundamental groups of 3-manifolds

In this section, we consider closed 3-manifolds with regular finite coverings which are rational homology spheres. The associated finite covering groups act freely on such rational homology 3-spheres, so they afford examples to which

our methods will apply. Note that according to [**10**, Theorem 2.6], every finite group, in fact, acts freely on some *hyperbolic* (hence aspherical) closed rational homology sphere. In such cases, the fundamental group determines the topology, and we are really just considering finite index normal subgroups of certain Poincaré duality groups with vanishing first Betti number (for background, see [**6**, Ch. VIII.10]).

Recall that for any group Q and integer $n \geqslant 0$, we define the *n*th term of its *derived series* as $Q^{(n+1)} = [Q^{(n)}, Q^{(n)}]$, where $Q^{(0)} = Q$. The derived series for a finite group stabilizes at a perfect normal subgroup but may not terminate for an infinite group with nonzero abelianization. In fact, an interesting open question is whether or not the derived series for the fundamental group Γ of a closed orientable hyperbolic 3-manifold stabilizes if $\Gamma/\Gamma^{(n)}$ is finite and nontrivial for all $n > 0$. If it does stabilize, that is, for some $i > 0$, $\Gamma^{(i)}$ is perfect, then $\Gamma/\Gamma^{(i)}$ is a solvable group with periodic cohomology (of period dividing four) as it acts freely on an integral homology 3-sphere, namely the covering space associated with the normal finite index subgroup $\Gamma^{(i)}$. Independently of the stability question, one can ask (as in [**9**]) about possible restrictions on the finite quotient groups $\Gamma/\Gamma^{(n)}$.

Let L denote a closed 3-manifold such that for some $n > 0$, the quotient $\pi_1(L)/\pi_1(L)^{(n)}$ is finite. Let $\Gamma = \pi_1(L)$. From the extensions

$$1 \to \Gamma^{(i)}/\Gamma^{(i+1)} \to \Gamma/\Gamma^{(i+1)} \to \Gamma/\Gamma^{(i)} \to 1$$

for $0 \leqslant i \leqslant n - 1$, we infer that all the groups $\Gamma/\Gamma^{(i)}$ and $\Gamma^{(i)}/\Gamma^{(i+1)}$ are finite in that range. Hence, the covering spaces L_i corresponding to the $\Gamma^{(i)}$ are rational homology spheres. The finite groups $\Gamma/\Gamma^{(i)}$ act freely on them, with quotient L; note that $H_1(L_i, \mathbb{Z}) \cong \Gamma^{(i)}/\Gamma^{(i+1)}$. Applying Corollary 1, we obtain the following.

PROPOSITION 11. *Let M denote a closed 3-manifold with $\Gamma = \pi_1(L)$ such that $\Gamma/\Gamma^{(n)}$ is finite for some $n > 0$. Then for all $0 \leqslant i \leqslant n - 1$, there are long exact sequences*

$$\cdots \to \widehat{H}^{i+2}(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \xrightarrow{\cup \sigma_i} \widehat{H}^{i-2}(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \to \widehat{H}^i(\Gamma/\Gamma^{(i)}, \Gamma^{(i)}/\Gamma^{(i+1)})$$
$$\to \widehat{H}^{i+3}(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \to \cdots$$

These sequences are determined by elements $\sigma_i \in \widehat{H}^{-4}(\Gamma/\Gamma^{(i)}, \mathbb{Z})$, which are images of the respective generators in $\widehat{H}^0(\Gamma/\Gamma^{(i)}, \mathbb{Z}) \cong \mathbb{Z}/|\Gamma/\Gamma^{(i)}|\mathbb{Z}$.

COROLLARY 5. *If $\Gamma^{(i)}$ is perfect, then $\sigma_i \in \widehat{H}^{-4}(\Gamma/\Gamma^{(i)}, \mathbb{Z})$ is an invertible element in the Tate cohomology of $\Gamma/\Gamma^{(i)}$.*

As would be expected, Proposition 11 can be used to obtain restrictions on the finite groups $\Gamma/\Gamma^{(i)}$. As an application, we take the opportunity to apply our

methods to quickly sketch how to obtain some of the results in [**9**] and [**29**]. A closed hyperbolic 3-manifold L is aspherical, and hence $H_3(L, \mathbb{Z}) = H_3(\Gamma, \mathbb{Z})$ if $\Gamma = \pi_1(L)$. For any finite quotient G of Γ, we let $\sigma_G \in \widehat{H}^{-4}(G, \mathbb{Z})$ be the element determined by the fundamental class of L, as in Proposition 2.

PROPOSITION 12 (Cavendish [**9**]). *Let L be a closed hyperbolic 3-manifold and $q: \Gamma \to G$, a surjective homomorphism from $\Gamma = \pi_1(L)$ onto a finite group G inducing an isomorphism $H_1(\Gamma, \mathbb{Z}) \cong H_1(G, \mathbb{Z})$ and such that $\ker(q) \subset \Gamma^{(2)}$. Then the homomorphism $\phi_G: \widehat{H}^2(G, \mathbb{Z}) \to \widehat{H}^{-2}(G, \mathbb{Z})$ given by $x \mapsto \sigma_G \cup x$ is bijective and the cup product defines a nondegenerate pairing*

$$\widehat{H}^2(G, \mathbb{Z}) \otimes \widehat{H}^2(G, \mathbb{Z}) \to \widehat{H}^4(G, \mathbb{Z}).$$

Moreover, this pairing factors through a cyclic subgroup of $\widehat{H}^4(G, \mathbb{Z})$.

Proof. First, we observe that the condition $\ker(q) \subset \Gamma^{(2)}$ implies that G maps onto $\Gamma/\Gamma^{(2)}$; hence, it is also finite and so the commutator $S = [\Gamma, \Gamma]$ corresponds to a covering $\tilde{L}$ of L which is a rational homology 3-sphere with a free action of the finite group $Q = H_1(G, \mathbb{Z})$. Now, any group G satisfying the hypotheses necessarily maps onto Q, inducing an isomorphism $\widehat{H}^2(Q, \mathbb{Z}) \cong \widehat{H}^2(G, \mathbb{Z})$, and so by naturality, it will suffice to prove the statements for Q.

Consider the group extension $1 \to S \to \Gamma \to Q \to 1$. Due to the vanishing of $H^1(S, \mathbb{Z})$ and $H^1(Q, \mathbb{Z})$ and the fact that $H^3(\Gamma, \mathbb{Z})$ is torsion-free, the associated spectral sequence of the extension gives rise to the following exact sequence in low degrees:

$$0 \to H^2(Q, \mathbb{Z}) \to H^2(\Gamma, \mathbb{Z}) \to H^2(S, \mathbb{Z})^Q \to H^3(Q, \mathbb{Z}) \to 0.$$

However, as Q is the (finite) abelianization of Γ, we have $H^2(Q, \mathbb{Z}) \cong H^2(\Gamma, \mathbb{Z})$, and so we infer that $\widehat{H}^0(Q, H^2(S, \mathbb{Z})) \cong H^3(Q, \mathbb{Z})$ (it factors through Tate cohomology). On the other hand, from Corollary 1 applied to the Q-action on $\tilde{L}$, we have an exact sequence

$$0 \to \operatorname{coker}(\phi_Q) \to \widehat{H}^0(Q, H_1(S, \mathbb{Z})) \to H^3(Q, \mathbb{Z}) \to 0.$$

Noting the identification of Q-modules $H^2(S, \mathbb{Z}) \cong H_1(S, \mathbb{Z})$ and using the isomorphism above, we infer that ϕ_Q is surjective. As the domain and codomain of ϕ_Q have the same number of elements, this implies that it is an isomorphism. Now, given $y \in \widehat{H}^2(Q, \mathbb{Z})$, we can choose $z \in \widehat{H}^2(Q, \mathbb{Z})$ such that the Tate dual $y^* = \sigma \cup z$. Then $0 \neq z \cup y$ because $0 \neq y^* \cup y = \sigma \cup z \cup y$, showing that the pairing is nondegenerate.

Now, let $J = [G, G]$, Γ maps onto G, so S maps onto J and $H_1(S, \mathbb{Z})$ maps onto $H_1(J, \mathbb{Z})$. The condition $\ker(q) \subset \Gamma^{(2)} = S^{(1)}$ means that the kernel of $q|_S: S \to J$ is contained in $S^{(1)}$, and so the abelianized map is an isomorphism

$H_1(S, \mathbb{Z}) \cong H_1(J, \mathbb{Z})$ or equivalently $H^2(S, \mathbb{Z}) \cong H^2(J, \mathbb{Z})$. Once again applying Corollary 1 to the Q-action on $\tilde{L}$, we obtain the exact sequence

$$0 \to \widehat{H}^1(Q, H_1(S, \mathbb{Z})) \xrightarrow{d} \widehat{H}^4(Q, \mathbb{Z}) \to \widehat{H}^0(Q, \mathbb{Z}).$$

As has been observed in Remark 4, we can identify d with the differential $d_3 : E_3^{1,2} \to E_3^{4,0}$ arising from the Serre spectral sequence for the fibration $\tilde{L} \to L \to BQ$, which, in this case, is simply the Lyndon–Hochschild–Serre spectral sequence for the extension $1 \to S \to \Gamma \to Q \to 1$. Using the above identifications, this agrees with the corresponding differential arising from the spectral sequence for the group extension $1 \to J \to G \to Q \to 1$. Therefore, the image of d goes to zero under the inflation map $H^4(Q, \mathbb{Z}) \to H^4(G, \mathbb{Z})$, and so it factors through $\operatorname{coker} d \subset \widehat{H}^0(Q, \mathbb{Z}) \cong \mathbb{Z}/|Q|\mathbb{Z}$, a cyclic group. Using the isomorphism $\widehat{H}^2(Q, \mathbb{Z}) \cong \widehat{H}^2(G, \mathbb{Z})$ and naturality of the cup product completes the proof. $\qquad\square$

COROLLARY 6 (Reznikov [29]). *Let L denote a closed three-manifold such that $G = \pi_1(L)/\pi_1(L)^{(n)}$ is a finite 2-group, and $H_1(L, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then G is a generalized quaternion group.*

Proof. Clearly, $G/[G, G] \cong H_1(L, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and so G is a 2-group of maximal class and thus must be (generalized) quaternion, dihedral or semidihedral (see [13], Section 5.4). The condition that the cup product pairing be nonsingular eliminates the semidihedral groups (see [12]), and the fact that the image has rank one eliminates the dihedral groups (see [16]). $\qquad\square$

Conflict of interest declaration

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