Can. J. Math. Vol. 47 (3), 1995 pp. 606–640

LOCAL CHARACTER EXPANSIONS FOR SUPERCUSPIDAL REPRESENTATIONS OF $U(3)$

FIONA MURNAGHAN

ABSTRACT. The topic of this paper is the relationship between characters of irreducible supercuspidal representations of the p-adic unramified 3×3 unitary group and Fourier transforms of invariant measures on elliptic adjoint orbits in the Lie algebra. We prove that most supercuspidal representations have the property that, on some neighbourhood of zero, the character composed with the exponential map coincides with the formal degree of the representation times the Fourier transform of a measure on one elliptic orbit. For the remainder, a linear combination of the Fourier transforms of measures on two elliptic orbits must be taken. As a consequence of these relations between characters and Fourier transforms, the coefficients in the local character expansions are expressed in terms of values of Shalika germs. By calculating which of the values of the Shalika germs associated to regular nilpotent orbits are nonzero, we determine which irreducible supercuspidal representations have Whittaker models. Finally, the coefficients in the local character expansions of three families of supercuspidal representations are computed.

1. **Introduction.** Let F be a p-adic field of characteristic zero. Suppose π is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$. Let Θ_π and $d(\pi)$ be the character and the formal degree of π, respectively. In [Mu2], under the assumption that the residual characteristic p of F is greater than n, it was shown that $d(\pi)^{-1}\Theta_\pi$ coincides with the Fourier transform of an elliptic $\mathrm{Ad}\,\mathrm{GL}_n(F)$-orbit on some neighbourhood of zero. More precisely, there exists a regular elliptic element X_π in the Lie algebra such that if $\hat{\mu}_{O(X_\pi)}$ is the Fourier transform of the orbital integral associated to the orbit $O(X_\pi)$,

$$\Theta_\pi(\exp X) = d(\pi)\hat{\mu}_{O(X_\pi)}(X) \tag{1.1}$$

for X regular and close to zero. It is natural to ask whether (1.1), or some similar result, holds for irreducible supercuspidal representations of $G = \mathbf{G}(F)$, where $\mathbf{G}$ is a connected reductive group defined over F. Detailed information about the inducing data for supercuspidal representations (the explicit realization of supercuspidal representations as representations induced from open compact mod centre subgroups) was required to prove (1.1) for $\mathrm{GL}_n(F)$. Thus we consider those groups G for which inducing data for supercuspidal representations has been found. For general G, it is conjectured that all irreducible supercuspidal representations are induced from representations of open compact mod centre subgroups.

Research supported in part by NSERC.
Received by the editors May 3, 1993; revised October 20, 1994.
AMS subject classification: 22E50, 22E35.

In the case $G = \mathrm{SL}_n(F)$, $p > n$, it was found ([Mu3]) that (1.1) holds for most supercuspidal representations of G. However, if n is prime and divides $q-1$, q being the order of the residue class field of F, then there exist irreducible supercuspidal representations π of G such that (1.1) does not hold for any X_π. Such a representation π is a component of a reducible supercuspidal representation for which (1.1) holds, but there does not appear to be a natural way to relate Θ_π to Fourier transforms of elliptic Ad G-orbits.

Let $G = \mathbf{G}(F)$, where $\mathbf{G}$ is the 3×3 unitary group defined relative to an unramified quadratic extension of F. The residual characteristic of F will be assumed to be odd. Moy ([Mo]) proved that the irreducible supercuspidal representations of G are induced from open compact mod centre subgroups and Jabon ([J]) obtained explicit inducing data using Moy's results. Filtrations of parahoric subgroups by open normal subgroups are used to construct inducing data for supercuspidal representations. A fundamental difference between G and $\mathrm{GL}_n(F)$ is that the types of filtrations of parahoric subgroups occurring in the inducing data are more general for G than for $\mathrm{GL}_n(F)$. For $\mathrm{GL}_n(F)$, the filtrations arise from powers of the Jacobson radical of the hereditary order which stabilizes the lattice chain given by powers of the prime ideal in some degree n extension of F. For G, the filtrations do not always arise this way. Also, one of the filtrations is not a canonical filtration defined by height functions on affine roots. That is, a non-standard filtration $\{I_i^b\}_{i \geq 1}$ (see Section 4) of the Iwahori subgroup of G occurs in the inducing data for certain supercuspidal representations of G.

In this paper, we determine which irreducible supercuspidal representations π of G have the property that there exists an elliptic, not necessarily regular, X_π in $\mathfrak{g}$ such that (1.1) holds. Furthermore, the remaining irreducible supercuspidal representations are equivalent up to twisting by a one-dimensional representation of G, and we show that there exist regular elliptic elements $X_{u,1}$ and $X_{u,2}$ such that, for any of these representations,

$$(1.2) \qquad \Theta_\pi(\exp X) = d(\pi)\big(q^{-1}(q+1)^2\hat{\mu}_{O(X_{u,1})}(X) - q^{-1}(q^2-q+1)\hat{\mu}_{O(X_{u,2})}(X)\big)/3,$$

if X is regular and close to zero.

Let $(\mathcal{N}_G)$ be the set of nilpotent Ad G-orbits. Harish-Chandra's local character expansion of π at the identity is the equality

$$\Theta_\pi(\exp X) = \sum_{O \in (\mathcal{N}_G)} c_O(\pi)\hat{\mu}_O(X),$$

where X is regular and in some neighbourhood of zero. (1.1) and (1.2) can be used to relate the coefficients $c_O(\pi)$ to values of Shalika germs. Given O in $(\mathcal{N}_G)$, let Γ_O be the Shalika germ associated to O. If (1.1) holds and X_π is regular, then

$$c_O(\pi) = d(\pi)\Gamma_O(X_\pi), \quad O \in (\mathcal{N}_G)$$

and, if (1.2) holds, then

$$c_O(\pi) = d(\pi)\big(q^{-1}(q+1)^2\Gamma_O(X_{u,1}) - q^{-1}(q^2-q+1)\Gamma_O(X_{u,2})\big)/3, \quad O \in (\mathcal{N}_G).$$

The paper begins with a summary of some of the notation used throughout the paper (Section 2) and information about elliptic Cartan subgroups and subalgebras (Section 3).

Properties of certain integrals which are related to Fourier transforms and to the inducing data for supercuspidal representations are proved in Section 4.

In Section 5 properties of the inducing data for π are used to define the X_π of (1.1), and the $X_{u,1}$ and $X_{u,2}$ of (1.2). Proposition 5.1, which relates certain integrals of matrix coefficients of π to the integrals considered in Section 4, is an essential part of the proof of Theorem 6.4. The main results of the paper are Theorem 6.4 and Corollary 6.6, in which we prove (1.1), (1.2), and the above results expressing values of the coefficients in the local character expansion in terms of values of Shalika germs.

Section 7 is devoted to determining which irreducible supercuspidal representations have a Whittaker model. This is done by finding out whether the associated values of Shalika germs are nonzero.

For certain π, we compute all of the coefficients $c_{\mathcal{O}}(\pi)$ in the local character expansion. This appears in Section 8.

Results of the type obtained in Sections 4–6 of this paper have also been proved in a later paper ([Mu4]) for supercuspidal representations of classical (symplectic, orthogonal and unitary) groups, using inducing data for those families of supercuspidal representations obtained by Morris ([M1-2]). Therefore there is some overlap between the results of this paper and those of [Mu4]. It is worth noting that in this paper we deal with all supercuspidal representations of G. In [Mu4], for technical reasons, some supercuspidal representations were excluded. In particular, we did not deal with those representations whose inducing data involved cuspidal unipotent representations of reductive groups over finite fields. Also, it is not known whether the constructions of Morris yield all supercuspidal representations of classical groups. There is no analogue of the results of Sections 7 and 8 in [Mu4].

2. **Notation.** Let F be a a p-adic field of characteristic zero and $\bar{F}$ the algebraic closure of F. If L is a finite extension of F, let $\mathcal{O}_L$ and $\mathfrak{p}_L$ denote the ring of integers and maximal ideal in the ring of integers. If q_L is the order of $\mathcal{O}_L/\mathfrak{p}_L$ and ϖ_L is a prime element in $\mathfrak{p}_L$, a choice of norm $|\cdot|_L$ on L is fixed by the requirement that $|\varpi_L|_L = q_L^{-1}$. In the case $L = F$, the subscript may be dropped, that is, the notation q, ϖ and $|\cdot|$ may be used. $\mathrm{N}_{L/F}$ denotes the norm map from L to F, and $\mathrm{Res}_{L/F}$ restriction of scalars. Throughout the paper, we assume that q is odd.

Choose an element ε in $\mathcal{O}_F^\times$ whose image in $\mathcal{O}_F/\mathfrak{p}_F \simeq \mathbf{F}_q$ generates $\mathbf{F}_q{}^\times$. Let $E = F(\sqrt{\varepsilon})$. Set $E^1 = \{x \in E^\times \mid \mathrm{N}_{E/F}(x) = 1\}$. If $x = a + b\sqrt{\varepsilon}, a, b \in F$, define $\bar{x} = a - b\sqrt{\varepsilon}$. If $x = (x_{ij})$ is a matrix with entries in E, $\bar{x} = (\bar{x}_{ij})$. The notation tr will be used for the trace map on 3×3 matrices with entries in E. Fix a character ψ_F on F which is trivial on $\mathcal{O}_F$ but non-trivial on $\varpi^{-1}\mathcal{O}_F$. Define $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$, where $\mathrm{tr}_{E/F}(x) = x + \bar{x}$ for $x \in E$.

Let $\mathbf{G} = \mathbf{U}(3)$ be the 3×3 unitary group defined relative to the quadratic extension E of F. Then $G = \mathbf{G}(F)$ can be realized as $\{x \in \mathrm{GL}_3(E) \mid xJ{}^t\bar{x} = J\}$, where ${}^tx$ is the transpose of x, and

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In other words, G is the group of fixed points of the automorphism $\sigma_\varepsilon(x) = J^t\bar{x}^{-1}J$ of $GL_3(E)$. If L is a finite extension of E, $\mathbf{G}(L) = GL_3(L)$. There is one isomorphism class of 3×3 unitary groups with respect to E/F ([R2], Section 1.9).

The isomorphism classes of 2×2 unitary groups with respect to E/F are parametrized by $F^\times / N_{E/F}(E^\times)$ ([R2], Section 1.9). Let $\mathbf{H}_{qs}$ be the 2×2 unitary group defined relative to $J_{qs} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. That is $H_{qs} = \mathbf{H}_{qs}(F) = \{x \in GL_2(E) \mid xJ_{qs}{}^t\bar{x} = J_{qs}\}$. Let $\mathbf{H}_{an}$ be the 2×2 unitary group defined relative to $J_{an} = \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix}$, and let $H_{an} = \mathbf{H}_{an}(F)$. It is easily verified that $\mathbf{H}_{qs}$ is quasi-split over F and $\mathbf{H}_{an}$ is anisotropic over F. Thus these groups represent the two isomorphism classes of 2×2 unitary groups.

The notation G_{reg} and $\mathfrak{g}_{\mathrm{reg}}$ will be used to denote the regular subsets of G and the Lie algebra $\mathfrak{g}$ of G, respectively. For definitions, see [HC2].

Let $\mathcal{N}_G$ be the nilpotent subset of $\mathfrak{g}$, and $(\mathcal{N}_G)$ the set of nilpotent Ad G-orbits in $\mathfrak{g}$.

The bilinear form

$$\langle X, Y\rangle = \mathrm{tr}_{E/F}\big(\mathrm{tr}(XY)\big)$$

is a non-degenerate bilinear form on $\mathfrak{g}$. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, let $\mathfrak{h}^\perp$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$.

Suppose X in $\mathfrak{g}$ is such that $\det(1 + X)$ is nonzero. Then the Cayley transform $c(X)$ of X is the element of G defined by:

$$c(X) = (1 - X)(1 + X)^{-1}.$$

3. **Elliptic Cartan subgroups and subalgebras.**

LEMMA 3.1 ([R2] SECTION 3.6). *An elliptic Cartan subgroup of G is isomorphic to one of the following:*

(1) $\mathrm{Res}_{EL/F}(\ker N_{EL/L})$, *where L is a cubic extension of F*
(2) $E^1 \times E^1 \times E^1$
(3) $E^1 \times \mathrm{Res}_{EL/F}(\ker N_{EL/L})$, *where L is a ramified quadratic extension of F*

Let T_{unr} be a Cartan subgroup of G which splits over an unramified cubic extension of E and is contained in $\mathbf{G}(O_F)$. Let $\mathcal{T}_{\mathrm{unr}}$ be the Lie algebra of T_{unr}.

To a ramified cubic extension L of F, we associate the Cartan subgroup $T_{\mathrm{ram},\zeta}$ having Lie algebra

$$\mathcal{T}_{\mathrm{ram},\zeta} = \left\{ \begin{pmatrix} a\sqrt{\varepsilon} & b & c\sqrt{\varepsilon} \\ \varpi\varepsilon\zeta c & a\sqrt{\varepsilon} & -b \\ \varpi\zeta b\sqrt{\varepsilon} & -\varpi\varepsilon\zeta c & a\sqrt{\varepsilon} \end{pmatrix} \,\middle|\, a, b, c \in F \right\},$$

where $\zeta \in O_F^*$ is chosen so that $T_{\mathrm{ram},\zeta}$ is isomorphic to $\mathrm{Res}_{EL/F}(\ker N_{EL/L})$.

Next we define two Cartan subgroups $T_{E,1}$ and $T_{E,2}$ which split over E. Their Lie algebras are, respectively:

$$\mathcal{T}_{E,1} = \left\{ \begin{pmatrix} a\sqrt{\varepsilon} & 0 & b\sqrt{\varepsilon} \\ 0 & c\sqrt{\varepsilon} & 0 \\ b\sqrt{\varepsilon} & 0 & a\sqrt{\varepsilon} \end{pmatrix} \middle| \, a,b,c \in F \right\}$$

$$\mathcal{T}_{E,2} = \left\{ \begin{pmatrix} a\sqrt{\varepsilon} & 0 & \varpi^{-1}b\sqrt{\varepsilon} \\ 0 & c\sqrt{\varepsilon} & 0 \\ \varpi b\sqrt{\varepsilon} & 0 & a\sqrt{\varepsilon} \end{pmatrix} \middle| \, a,b,c \in F \right\}.$$

Let θ be one of ϖ and $\varepsilon\varpi$. Given θ, fix $\lambda \in O_E$ such that $\lambda\bar{\lambda} = \theta\varepsilon/2\varpi$. (Such λ's exist because $|\theta\varepsilon/2\varpi| = 1$ ($p \neq 2$).) Let $T_{\theta,1}$ and $T_{\theta,2}$ be Cartan subgroups which split over $E(\sqrt{\theta})$ and have Lie algebras:

$$\mathcal{T}_{\theta,1} = \left\{ \begin{pmatrix} a\sqrt{\varepsilon} & 0 & b\sqrt{\varepsilon} \\ 0 & c\sqrt{\varepsilon} & 0 \\ \theta b\sqrt{\varepsilon} & 0 & a\sqrt{\varepsilon} \end{pmatrix} \middle| \, a,b,c \in F \right\}$$

and

$$\mathcal{T}_{\theta,2} = \left\{ \begin{pmatrix} (a+c)\sqrt{\varepsilon}/2 & \lambda b & \varpi^{-1}(c-a)\sqrt{\varepsilon}/2 \\ \varpi\bar{\lambda}b & a\sqrt{\varepsilon} & -\bar{\lambda}b \\ \varpi(c-a)\sqrt{\varepsilon}/2 & -\varpi\lambda b & (a+c)\sqrt{\varepsilon}/2 \end{pmatrix} \middle| \, a,b,c \in F \right\}.$$

Fix a, b and c in F such that both b and $(a-c)^2 - b^2$ are nonzero. Let $X_{E,1}$ and $X_{E,2}$ be the corresponding elements of $\mathcal{T}_{E,1} \cap \mathfrak{g}_{\mathrm{reg}}$ and $\mathcal{T}_{E,2} \cap \mathfrak{g}_{\mathrm{reg}}$, respectively (given in the definitions of $\mathcal{T}_{E,1}$ and $\mathcal{T}_{E,2}$). Define two additional elements $X_{E,3}$ and $X_{E,4}$ in $\mathcal{T}_{E,2} \cap \mathfrak{g}_{\mathrm{reg}}$ by:

$$X_{E,3} = \begin{pmatrix} (a-b+c)\sqrt{\varepsilon}/2 & 0 & \varpi^{-1}(-a+b+c)\sqrt{\varepsilon}/2 \\ 0 & (a+b)\sqrt{\varepsilon} & 0 \\ \varpi(-a+b+c)\sqrt{\varepsilon}/2 & 0 & (a-b+c)\sqrt{\varepsilon}/2 \end{pmatrix}$$

$$X_{E,4} = \begin{pmatrix} (a+b+c)\sqrt{\varepsilon}/2 & 0 & \varpi^{-1}(-a-b+c)\sqrt{\varepsilon}/2 \\ 0 & (a-b)\sqrt{\varepsilon} & 0 \\ \varpi(-a-b+c)\sqrt{\varepsilon}/2 & 0 & (a+b+c)\sqrt{\varepsilon}/2 \end{pmatrix}.$$

Now fix a, b, and c in F such that b is nonzero. Let $X_{\theta,1}$ and $X_{\theta,2}$ be the corresponding elements of $\mathcal{T}_{\theta,1} \cap \mathfrak{g}_{\mathrm{reg}}$ and $\mathcal{T}_{\theta,2} \cap \mathfrak{g}_{\mathrm{reg}}$, respectively.

Two elements x_1 and x_2 of G are stably conjugate ([R2], Section 3) if there exists $y \in \mathrm{GL}_3(\bar{F})$ such that $y^{-1}x_1y = x_2$. The same terminology will be used for elements of $\mathfrak{g}$. That is, elements X_1 and X_2 in $\mathfrak{g}$ are stably conjugate whenever $\mathrm{Ad}\, y^{-1}(X_1) = y^{-1}X_1y = X_2$ for some $y \in \mathrm{GL}_3(\bar{F})$. Given X in $\mathfrak{g}$, the set of elements in $\mathfrak{g}$ which are stably conjugate to X will be called the *stable orbit* of X.

LEMMA 3.2. *Let T be a Cartan subgroup of G.*

(1) Let $X \in \mathfrak{g}_{\mathrm{reg}}$. If $X \in \mathcal{T}_{\mathrm{unr}}$ or a Cartan subalgebra of the form $\mathcal{T}_{\mathrm{ram},\zeta}$, the stable orbit of X consists of the Ad *G-orbit of X.*

https://doi.org/10.4153/CJM-1995-032-x Published online by Cambridge University Press

(2) *If T is isomorphic to* $\mathrm{Res}_{EL/F}(\ker \mathrm{N}_{EL/L})$ *for some cubic extension L of F, then T is conjugate to T_{unr} if L is unramified, and T is conjugate to $T_{\mathrm{ram},\zeta}$ for some $\zeta \in O_F^\times$, if L is ramified over F.*

(3) *$X_{E,1}$, $X_{E,2}$, $X_{E,3}$ and $X_{E,4}$ are stably conjugate. Their* Ad *G-orbits are distinct, and make up a stable orbit.*

(4) *If T is isomorphic to $E^1 \times E^1 \times E^1$, then T is conjugate to one of $T_{E,1}$ and $T_{E,2}$.*

(5) *For a fixed θ (ϖ or $\varepsilon\varpi$), $X_{\theta,1}$ and $X_{\theta,2}$ are stably conjugate. Their* Ad *G-orbits do not coincide, and these two orbits make up a stable orbit.*

(6) *If T is isomorphic to $E^1 \times \mathrm{Res}_{EL/F}(\ker \mathrm{N}_{EL/L})$, $L = F(\sqrt{\theta})$ then T is conjugate to one of $T_{\theta,1}$ and $T_{\theta,2}$.*

PROOF. (1) and (2) follow from Proposition 3.5.2 of [R2].

That the elements $X_{E,j}$, $1 \le j \le 4$ are stably conjugate is immediate, because they have the same eigenvalues. A simple calculation shows that the Weyl groups $W(T_{E,1}) \simeq S_3$ and $W(T_{E,2}) \simeq \mathbf{Z}/2\mathbf{Z}$, and no two of the $X_{E,j}$'s are conjugate. Apply Proposition 3.5.2 and remarks on p. 29 of [R2] to get (3) and (4).

$X_{\theta,1}$ and $X_{\theta,2}$ have the same eigenvalues and so are stably conjugate. By Proposition 3.5.2 of [R2], their stable orbit consists of two Ad G-orbits, so it suffices two show that $X_{\theta,1}$ and $X_{\theta,2}$ do not lie in the same Ad G-orbit. (5) and (6) now follow. ∎

4. **Filtration subgroups and vanishing of certain integrals.** The topic of this section is properties of integrals of the form

$$\mathcal{J}(X,Y;C) = \int_C \psi_E\Big(\mathrm{tr}\big(X\,\mathrm{Ad}\,x^{-1}(Y)\big)\Big)\,dx \tag{4.1}$$

for various semisimple elements X and open compact subsets C of G, where Y is in $\mathcal{N}_G$. These types of integrals appear in formulas for Fourier transforms of measures on elliptic adjoint orbits in $\mathfrak{g}$. The results of this section will be used in Section 5 to relate these integrals to character values of inducing data for supercuspidal representations of G.

To begin, parahoric subgroups and filtrations are defined as in [Mo] and [J]. Let $K = \mathbf{G}(O_F)$. The Iwahori subgroup I of K consists of those matrices in K whose entries below the diagonal lie in $\mathfrak{p}_E$. The remaining conjugacy class of parahoric subgroups of G contains the normalizer L of I in G. To each of K and L there is associated one filtration, and there are two filtrations associated to I.

Given $i \in \mathbf{Z}$, let $\tilde{\mathfrak{k}}_i$ be the set of 3×3 matrices with entries in $\mathfrak{p}_E^i$, and let $\mathfrak{k}_i = \tilde{\mathfrak{k}}_i \cap \mathfrak{g}$. Set $K_0 = K$ and $K_i = (1 + \tilde{\mathfrak{k}}_i) \cap G$, $i \ge 1$.

Define

$$\tilde{\mathfrak{l}}_0 = \left\{\begin{pmatrix} O_E & O_E & \mathfrak{p}_E^{-1} \\ \mathfrak{p}_E & O_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & O_E \end{pmatrix}\right\} \quad \tilde{\mathfrak{l}}_1 = \left\{\begin{pmatrix} \mathfrak{p}_E & O_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & O_E \\ \mathfrak{p}_E^2 & \mathfrak{p}_E & \mathfrak{p}_E \end{pmatrix}\right\}$$

and $\tilde{\mathfrak{l}}_{2i+j} = \varpi^i \tilde{\mathfrak{l}}_j$ for i any integer, and $j \in \{0,1\}$. Set $\mathfrak{l}_i = \tilde{\mathfrak{l}}_i \cap \mathfrak{g}$, $i \in \mathbf{Z}$, $L_0 = L = \tilde{\mathfrak{l}}_0 \cap G$, and $L_i = (1 + \tilde{\mathfrak{l}}_i) \cap G$, $i \ge 1$.

The first filtration associated to I (the *standard* filtration) is given by $I_0 = I$, and $I_i = (1+\tilde{\mathfrak{i}}_i)\cap G$, $i \geq 1$, where

$$\tilde{\mathfrak{i}}_0 = \left\{\begin{pmatrix} O_E & O_E & O_E \\ \mathfrak{p}_E & O_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & O_E \end{pmatrix}\right\} \quad \tilde{\mathfrak{i}}_1 = \left\{\begin{pmatrix} \mathfrak{p}_E & O_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{p}_E \end{pmatrix}\right\} \quad \tilde{\mathfrak{i}}_2 = \left\{\begin{pmatrix} \mathfrak{p}_E & \mathfrak{p}_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{p}_E^2 & \mathfrak{p}_E & \mathfrak{p}_E \end{pmatrix}\right\}$$

and $\tilde{\mathfrak{i}}_{3i+j} = \varpi^i \tilde{\mathfrak{i}}_j$ for $i \in \mathbf{Z}$ and $j \in \{0,1,2\}$. Set $\mathfrak{i}_i = \tilde{\mathfrak{i}}_i \cap \mathfrak{g}$.

The other filtration associated to I (the *non-standard* filtration) is $I_0^\flat = I$, $I_i^\flat = (1+\tilde{\mathfrak{i}}_i^\flat)\cap G$, $i \geq 1$, where

$$\tilde{\mathfrak{i}}_0^\flat = \left\{\begin{pmatrix} O_E & O_E & O_E \\ \mathfrak{p}_E & O_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & O_E \end{pmatrix}\right\} \quad \tilde{\mathfrak{i}}_1^\flat = \left\{\begin{pmatrix} \mathfrak{p}_E & O_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{p}_E \end{pmatrix}\right\}$$

$$\tilde{\mathfrak{i}}_2^\flat = \left\{\begin{pmatrix} \mathfrak{p}_E & \mathfrak{p}_E & O_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{p}_E \end{pmatrix}\right\} \quad \tilde{\mathfrak{i}}_3^\flat = \left\{\begin{pmatrix} \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{p}_E & \mathfrak{p}_E & \mathfrak{p}_E \\ \mathfrak{p}_E^2 & \mathfrak{p}_E & \mathfrak{p}_E \end{pmatrix}\right\}$$

and $\tilde{\mathfrak{i}}_{4i+j}^\flat = \varpi^i \tilde{\mathfrak{i}}_j^\flat$ for $i \in \mathbf{Z}$ and $j \in \{0,1,2,3\}$. Set $\mathfrak{i}_i^\flat = \tilde{\mathfrak{i}}_i^\flat \cap \mathfrak{g}$.

Given any lattice $\mathfrak{l}$ in $\mathfrak{g}$, let $\mathfrak{l}^* = \{X \in \mathfrak{g} \mid \operatorname{tr}(XY) \in O_E\ \forall Y \in \mathfrak{l}\}$.

LEMMA 4.2. *Let $i \in \mathbf{Z}$. For the given Cartan subalgebra $\mathcal{T}$ (notation as in Section 3) and lattice $\mathfrak{m}_i$,*

$$(\mathcal{T} + \mathfrak{m}_{i+1}) \cap (\mathfrak{m}_i - \mathfrak{m}_{i+1}) \cap \mathcal{N}_G = \emptyset.$$

(1) $\mathcal{T} = \mathcal{T}_{\mathrm{unr}}$ or $\mathcal{T}_{E,1}$, and $\mathfrak{m}_i = \mathfrak{k}_i$
(2) $\mathcal{T} = \mathcal{T}_{\theta,1}$ and $\mathfrak{m}_i = \mathfrak{i}_i^\flat$
(3) $\mathcal{T} = \mathcal{T}_{\theta,2}$ or $\mathcal{T}_{E,2}$, and $\mathfrak{m}_i = \mathfrak{l}_i$
(4) $\mathcal{T} = \mathcal{T}_{\mathrm{ram},\zeta}$ and $\mathfrak{m}_i = \mathfrak{i}_i$

PROOF. (1) Suppose $X \in (\mathcal{T} + \mathfrak{k}_{i+1}) \cap (\mathfrak{k}_i - \mathfrak{k}_{i+1})$, where $\mathcal{T} = \mathcal{T}_{\mathrm{unr}}$ or $\mathcal{T} = \mathcal{T}_{E,1}$. Then the image of $\varpi^{-i}X$ in $\mathfrak{k}_i/\mathfrak{k}_{i+1} \simeq \mathfrak{g}(\mathbf{F}_q)$ lies in an elliptic Cartan subalgebra, so is semisimple. If $X \in \mathcal{N}_G$, then the image of $\varpi^{-i}X$ in $\mathfrak{g}(\mathbf{F}_q)$, which by assumption is nonzero, is nilpotent. But a nonzero element of $\mathfrak{g}(\mathbf{F}_q)$ cannot be both semisimple and nilpotent.

(2) Suppose $X \in \mathfrak{i}_0^\flat$. Then $X \in Y + \mathfrak{i}_1^\flat$, where

$$Y = \begin{pmatrix} A & 0 & 0 \\ 0 & c\sqrt{\varepsilon} & 0 \\ 0 & 0 & -\bar{A} \end{pmatrix}, \quad A \in O_E,\ b \in O_F.$$

If $X \in \mathcal{N}_G$, then $X^3 = 0$, which implies $Y^3 \in \tilde{\mathfrak{i}}_1^\flat$, that is, $A \in \mathfrak{p}_E$ and $b \in \mathfrak{p}_F$. Thus $X \in \mathfrak{i}_1^\flat$, and $(\mathfrak{i}_0^\flat - \mathfrak{i}_1^\flat) \cap \mathcal{N}_G = \emptyset$. Since $\mathfrak{i}_{4j}^\flat = \varpi^j \mathfrak{i}_0^\flat$, (2) holds for i divisible by 4.

It is easily seen that $\mathcal{T}_{\theta,1} \cap \mathfrak{i}_1^\flat \subset \mathfrak{i}_2^\flat$ and $\mathcal{T}_{\theta,1} \cap \mathfrak{i}_3^\flat \subset \mathfrak{i}_4^\flat$. Therefore (2) holds for i of the form $4j+1$ or $4j+3$.

If $X \in (\mathcal{T}_{\theta,1} + \mathfrak{i}_3^\flat) \cap \mathfrak{i}_2^\flat$, then $X \in Y + \mathfrak{i}_3^\flat$, where

$$Y = \begin{pmatrix} 0 & 0 & a\sqrt{\varepsilon} \\ 0 & 0 & 0 \\ \varpi a\sqrt{\varepsilon} & 0 & 0 \end{pmatrix}, \quad a \in O_F.$$

https://doi.org/10.4153/CJM-1995-032-x Published online by Cambridge University Press

If $X \in \mathcal{N}_G$, then $Y^3 \in \tilde{\mathfrak{i}}_7^\flat$, that is, $a^3 \in \mathfrak{p}_F$, or $a \in \mathfrak{p}_F$, which is equivalent to $X \in \mathfrak{i}_3^\flat$. (2) now holds for $i = 4j+2, j \in \mathbf{Z}$.

The proofs of (3) and (4) are omitted as they are similar to the proof of (2). ■

Let $X_{\theta,1}, X_{\theta,2}$, and $X_{E,2}$ be defined as in Section 3.

LEMMA 4.3. *Assume $Y \in \mathcal{N}_G$.*

(1) Suppose $X \in \mathfrak{k}_{-i-1}$, $i \geq 0$, has the property that the image of $\varpi^{i+1}X$ in $\mathfrak{k}_0/\mathfrak{k}_1 \simeq \mathfrak{g}(\mathbf{F}_q)$ is regular and elliptic. Then $\mathcal{J}(X, Y; K) = 0$ whenever $Y \notin \mathfrak{k}_i$.

(2) Let $X = X_{\theta,1}$ be such that $|a|, |c| \leq q^i$ and $|b| = q^{i+1}$, $i \geq 1$. Then $\mathcal{J}(X, Y; I) = 0$ whenever $Y \notin \mathfrak{i}_{4i-2}^\flat$.

(3) Let $X = X_{\theta,2}$ be such that $|a|, |c| \leq q^i$ and $|b| = q^{i+1}$, $i \geq 1$. Then $\mathcal{J}(X, Y; L) = 0$ whenever $Y \notin \mathfrak{l}_{2i-1}$.

(4) Let $X = X_{E,2}$ be such that $|a|, |c| \leq q^{i+1}$, $i \geq 0$, and $|b| = |(a-c)^2 - b^2|^{1/2} = q^{i+1}$. Then $\mathcal{J}(X, Y; L) = 0$ whenever $Y \notin \mathfrak{l}_{2i}$.

PROOF. For each of (1)–(4), we will use the notation $\mathfrak{m}_j$, $j \in \mathbf{Z}$, for the lattices defining a particular filtration. For (1), $\mathfrak{m}_j = \mathfrak{k}_j$, for (2), $\mathfrak{m}_j = \mathfrak{i}_j^\flat$, and for (3) and (4), $\mathfrak{m}_j = \mathfrak{l}_j$.

Given X, let $\mathcal{T}$ be the Cartan subalgebra containing X. In (1), $\mathcal{T}$ is $\mathcal{T}_{\mathrm{unr}}$ or $\mathcal{T}_{E,1}$. Set

$$\mathfrak{m}_j' = \mathfrak{m}_j \cap \mathcal{T} \text{ and } \mathfrak{m}_j'^{\perp} = \mathfrak{m}_j \cap \mathcal{T}^{\perp}.$$

Moy ([Mo], p. 190, p. 200) has shown that

$$\mathfrak{m}_j = \mathfrak{m}_j' + \mathfrak{m}_j'^{\perp} \tag{4.4}$$

and the map induced by taking commutators is onto:

$$[X, \cdot] \colon \mathfrak{m}_j / \mathfrak{m}_{j+1} \twoheadrightarrow \varpi^{-i}\mathfrak{m}_{j+s-1}'^{\perp} / \varpi^{-i}\mathfrak{m}_{j+s}'^{\perp} \tag{4.5}$$

Here $s = 0$ in cases (1) and (3), and $s = -1$ in cases (2) and (4).

Let $d = 1, 4, 2$ and 2, in cases (1)–(4), respectively. In each case, $\varpi\mathfrak{m}_j = \mathfrak{m}_{d+j}, j \in \mathbf{Z}$. Let $P_j = c(\mathfrak{m}_j)$, for $j \geq 1$, and let P be the associated parahoric subgroup. Then we must show that $\mathcal{J}(X, Y; P) = 0$ whenever $Y \notin \mathfrak{m}_{(i-1)d-s+1}$.

Define the integer r by $Y \in \mathfrak{m}_r - \mathfrak{m}_{r+1}$. Assume that $r \leq (i-1)d - s$, that is, $Y \notin \mathfrak{m}_{(i-1)d-s+1}$. Let $\ell = (i-1)d - s + 1 - r$. The integral $\mathcal{J}(X, Y; P)$ is a nonzero multiple of

$$\int_P \int_{P_\ell} \psi_E\Big(\mathrm{tr}\big(X \operatorname{Ad}(kh)^{-1}(Y)\big)\Big)\, dh\, dk.$$

Fix $k \in P$ and set $Z = \operatorname{Ad} k^{-1}(Y)$. If $h \in P_\ell$, then $h = c(H)$ for some $H \in \mathfrak{m}_\ell$, and $\operatorname{Ad} h^{-1}(Z) - (Z - 2[Z, H]) \in \mathfrak{m}_{2\ell+r} \subset \mathfrak{m}_{(i-1)d-s+2}$. The relation ([J], p. 32)

$$\mathfrak{m}_j^* = \mathfrak{m}_{-j-d+1} \tag{4.6}$$

together with $X \in \mathfrak{m}_{-id+s-1}$, implies that $\mathrm{tr}(X\mathfrak{m}_{(i-1)d-s+2}) \subset O_E$.

https://doi.org/10.4153/CJM-1995-032-x Published online by Cambridge University Press

Using these facts, we see that the inner integral $\mathcal{J}(X,Z;P_\ell)$ equals

$$\int_{\mathfrak{m}_\ell} \psi_E\Big(\mathrm{tr}\big(X(Z-2[Z,H])\big)\Big)\,dH = \psi_E\big(\mathrm{tr}(XZ)\big)\int_{\mathfrak{m}_\ell} \psi_E\big(\mathrm{tr}(-2[X,Z]H)\big)\,dH.$$

Since $\mathfrak{m}_\ell^* = \mathfrak{m}_{-\ell-d+1}$, this integral vanishes unless $[X,Z]\in\mathfrak{m}_{-\ell-d+1}$, because otherwise the character of $\mathfrak{m}_\ell$ in the integral is non-trivial.

Now, using (4.4), write $Z = Z' + Z^\perp$, where $Z'\in\mathfrak{m}'_r$ and $Z^\perp\in\mathfrak{m}_r'^{\perp}$. Choose $n\geq r$ such that $Z^\perp\in\mathfrak{m}_n'^{\perp}-\mathfrak{m}_{n+1}'^{\perp}$. By (4.5), $[X,Z^\perp]\in\mathfrak{m}'^{\perp}_{-id+n+s-1}-\mathfrak{m}'^{\perp}_{-id+n+s}$ Assume $[X,Z] = [X,Z^\perp]\in\mathfrak{m}_{-\ell-d+1}$. Then $-id+n+s\geq-\ell-d+1$, that is, $n\geq r+1$.

Since $Z\in\mathcal{N}_G$ and $n\geq r+1$, Lemma 4.2 implies that $Z\in\mathfrak{m}_{r+1}$. But $\mathfrak{m}_{r+1}$ is $\mathrm{Ad}\,P$-invariant, so $Y\in\mathfrak{m}_{r+1}$, which is a contradiction. Thus $[X,Z]\notin\mathfrak{m}_{-\ell-d+1}$ and $\mathcal{J}(X,Z;P_\ell)=0$ for every $k\in P$. Therefore $\mathcal{J}(X,Y;P)=0$. ∎

LEMMA 4.7. *Suppose $Y\in\mathcal{N}_G$. Let $\mathcal{T}_{\mathrm{ram},\zeta}$ be as defined in Section 3. If $X\in\mathcal{T}_{\mathrm{ram},\zeta}\cap(\mathfrak{i}_{-i}-\mathfrak{i}_{-i+1})$, for some $i\geq 1$ which is not divisible by 3, then $\mathcal{J}(X,Y;I)=0$ whenever $Y\notin\mathfrak{i}_{i-3}$.*

PROOF. Argue as for Lemma 4.3, with $\mathfrak{m}_j=\mathfrak{i}_j, j\in\mathbf{Z}$, $P=I$, and $\ell=i-r-3$. Because the residual characteristic may equal 3, (4.4) does not apply. In place of (4.4), apply Lemma 3.5 of [C] to see that

$$[X,\mathrm{Ad}\,k^{-1}(Y)]\in\mathfrak{i}_\ell^*=\mathfrak{i}_{-i+r+1}\Rightarrow\mathrm{Ad}\,k^{-1}(Y)\in\mathcal{T}_{\mathrm{ram},\zeta}+\mathfrak{i}_{r+1},\quad k\in I.$$ ∎

Let $i\geq1$. Define

$$\alpha=\begin{pmatrix} a\sqrt{\varepsilon} & 0 & 0\\ 0 & c\sqrt{\varepsilon} & 0\\ 0 & 0 & a\sqrt{\varepsilon}\end{pmatrix},\quad a,c\in F,\ |a|,|c|\leq|a-c|=q^{i+1}. \tag{4.8}$$

The stabilizer G' of α in G is isomorphic to $H_{qs}\times E^1$. Let $\mathfrak{g}'$ be the Lie algebra of G'. If A is a subset of $\mathfrak{g}$, A' denotes $A\cap\mathfrak{g}'$, and $A'^{\perp}$ is $A\cap\mathfrak{g}'^{\perp}$. Next we define certain regular elliptic elements in $\mathfrak{g}'$:

$$\beta=\begin{pmatrix} a_1\sqrt{\varepsilon} & 0 & b\sqrt{\varepsilon}\\ 0 & c_1\sqrt{\varepsilon} & 0\\ b\sqrt{\varepsilon} & 0 & a_1\sqrt{\varepsilon}\end{pmatrix},\quad a_1,b,c_1\in F,\ |b|=q^{i+1},\ |a_1|,|c_1|\leq q^{i+1}, \tag{4.9i}$$

$$\beta=\begin{pmatrix} a_1\sqrt{\varepsilon} & 0 & \varpi^{-1}b\sqrt{\varepsilon}\\ 0 & c_1\sqrt{\varepsilon} & 0\\ \varpi b\sqrt{\varepsilon} & 0 & a_1\sqrt{\varepsilon}\end{pmatrix}, \tag{4.9ii}$$
$$a_1,b,c_1\in F,\ |b|=|b^2-a_1^2|^{1/2}=q^{i+1},\ |c_1|\leq q^{i+1},$$

(4.9iii)
$$\beta=\begin{pmatrix} a_1\sqrt{\varepsilon} & 0 & b\sqrt{\varepsilon}\\ 0 & c_1\sqrt{\varepsilon} & 0\\ \theta b\sqrt{\varepsilon} & 0 & a_1\sqrt{\varepsilon}\end{pmatrix},\quad a_1,b,c_1\in F,\ |b|=q^{i+1},\ |a_1|,|c_1|\leq q^{i},\ \theta\in\{\varpi,\varepsilon\varpi\}.$$

The next lemma is concerned with $\mathcal{J}(X,Y;P)$ for X of the form $X=\alpha+\beta$, where $\beta\in\mathfrak{k}_{-i}\cap\mathfrak{g}'$ is as in (4.9), and P is a parahoric subgroup. We will refer to β given by (4.9i), (4.9ii), and (4.9iii) as cases (i), (ii), and (iii), respectively.

LEMMA 4.10. *Let α be as in (4.8), β as in (4.9), and $Y \in \mathcal{N}_G$. For $r \in \mathbf{Z}$, let $\mathfrak{m}_r = \mathfrak{k}_r$, $\mathfrak{l}_r$, and $\mathfrak{i}_r^\flat$, in cases (i)–(iii), respectively. Let $P_r = c(\mathfrak{m}_r)$ for $r \geq 1$, and let P be the associated parahoric subgroup. Set $d = 1$, 2, and 4 in cases (i)–(iii), respectively. Then $\varpi \mathfrak{m}_r = \mathfrak{m}_{d+r}$ in every case.*

(1) In cases (i) and (ii), assume that $0 \leq j < i$. If $j \geq 1$, or $j = 0$ and $Y \in \mathfrak{m}_1$, set

$$P(Y,i,j) = \{k \in P \mid \mathrm{Ad}\, k^{-1}(Y) \in \mathfrak{m}'_{[dj/2]+1} + \mathfrak{m}_{[di/2]+1}\},$$

and if $j = 0$ and $Y \notin \mathfrak{m}_1$, set

$$P(Y,i,0) = \{k \in P \mid \mathrm{Ad}\, k^{-1}(Y) \in \mathfrak{m}'_0 + \mathfrak{m}_{[d(i+1)/2]}\}.$$

(2) In case (iii), assume that $0 < j < i$. Set

$$P(Y,i,j) = \{k \in P \mid \mathrm{Ad}\, k^{-1}(Y) \in \mathfrak{m}'_{[dj/2]} + \mathfrak{m}_{[di/2]+1}\}.$$

Let $X = \alpha + \beta$. Then $\mathcal{J}(X, Y; P) = \mathcal{J}\big(X, Y; P(Y, i, j)\big)$, $Y \in \mathcal{N}_G$.

PROOF.

STEP 1. Assume $Y \in \mathfrak{m}_1$. The integral $\mathcal{J}(X, Y; P)$ is a nonzero multiple of $\int_P \mathcal{J}\big(X, \mathrm{Ad}\, k^{-1}(Y); P_\ell\big)\, dk$ for any integer $\ell \geq 1$. Set $\ell = [(di+1)/2]$. Here, $[\cdot]$ denotes the greatest integer function. Fix $k \in P$ and set $Z = \mathrm{Ad}\, k^{-1}(Y)$. If $h = c(H) \in P_\ell$,

$$\mathrm{Ad}\, h^{-1}(Z) - Z + 2[Z, H] \in \mathfrak{m}_{di+1} = \mathfrak{m}^*_{-d(i+1)},$$

the last equality following from (4.6). Also, $X \in \mathfrak{m}_{-d(i+1)}$ and $\mathrm{tr}(2X[Z, H]) = \mathrm{tr}(2[X, Z]H)$. Therefore, $\mathcal{J}(X, Z; P_\ell)$ can be rewritten as

$$\psi_E\big(\mathrm{tr}(XZ)\big) \int_{\mathfrak{m}_\ell} \psi_E\big(\mathrm{tr}(-2[X, Z]H)\big)\, dH.$$

This integral vanishes unless $[X, Z] \in \mathfrak{m}^*_\ell$. A straightforward calculation shows that $\mathfrak{m}_r = \mathfrak{m}'_r + \mathfrak{m}'^{\perp}_r$, $r \in \mathbf{Z}$. Write $Z = Z' + Z^\perp$, $Z' \in \mathfrak{m}'_1$, $Z^\perp \in \mathfrak{m}'^{\perp}_1$. Define r by $Z^\perp \in \mathfrak{m}'^{\perp}_r - \mathfrak{m}'^{\perp}_{r+1}$. We remark that if d is even, then $\mathfrak{m}'^{\perp}_{2s} = \mathfrak{m}'^{\perp}_{2s+1}$, $s \in \mathbf{Z}$, so r must be even if $d = 2$ or 4. Note that $\alpha \in \mathfrak{m}_{-d(i+1)} - \mathfrak{m}_{-d(i+1)+1}$ and $\beta \in \mathfrak{m}_{-d(i+1)+1}$. It can be checked that

$$[\alpha, Z^\perp] \in \mathfrak{m}'^{\perp}_{-d(i+1)+r} - \mathfrak{m}'^{\perp}_{-d(i+1)+r+1}.$$

from which it follows, using (4.6), that $[X, Z] \in \mathfrak{m}^*_\ell = \mathfrak{m}_{-\ell-d+1}$ is equivalent to $-d(i+1) + r \geq -\ell - d + 1$, that is $r \geq [di/2] + 1$. As a result, $\mathcal{J}(X, Y; P) = \mathcal{J}\big(X, Y; P(Y, i)\big)$, where

$$P(Y, i) = \{k \in M \mid \mathrm{Ad}\, k^{-1}(Y) \in \mathfrak{g}' + \mathfrak{m}_{[di/2]+1}\}.$$

The second part of Step 1 involves writing $\mathcal{J}\big(X, Y; P(Y, i)\big)$ as a nonzero multiple of a double integral and showing that the inner integral vanishes under certain conditions.

Let $m = [(dj+1)/2]$. Observe that $P(Y,i)$ is invariant under right translation by $P_m \cap G'$. Thus $\mathcal{J}\big(X, Y; P(Y,i)\big)$ is a nonzero multiple of

$$\int_{P(Y,i)} \mathcal{J}\big(X, \operatorname{Ad} k^{-1}(Y); P_m \cap G'\big)\, dk$$
$$= \int_{P(Y,i)} \psi_E\Big(\operatorname{tr}\big(\alpha \operatorname{Ad} k^{-1}(Y)\big)\Big) \mathcal{J}\big(\beta, \operatorname{Ad} k^{-1}(Y), P_m \cap G'\big)\, dk,$$

equality holding because $\operatorname{Ad} h(\alpha) = \alpha$ for $h \in G'$. Fix $k \in P(Y,i)$ and set $Z = \operatorname{Ad} k^{-1}(Y)$, writing $Z = Z' + Z^{\perp}$ as above. Observe that

$$\operatorname{tr}\big(\beta \operatorname{Ad} h^{-1}(Z^{\perp})\big) = \operatorname{tr}\big(\operatorname{Ad} h(\beta) Z^{\perp}\big) = 0, \quad h \in G'$$
$$\Rightarrow \mathcal{J}(\beta, Z,; P_m \cap G') = \mathcal{J}(\beta, Z'; P_m \cap G').$$

Let $\mathcal{T} = \mathcal{T}_{E,1}, \mathcal{T}_{E,2}$, resp. $\mathcal{T}_{\theta,1}$, in cases (i)–(iii), respectively. Arguing as above, we find that this last integral vanishes unless $Z' \in \mathcal{T} + \mathfrak{m}'_{[dj/2]+1}$. Recall that $Z^{\perp} \in \mathfrak{m}_{[di/2]+1}$. By an easy variant of Lemma 4.2,

$$(\mathcal{T} + \mathfrak{m}'_{[dj/2]+1} + \mathfrak{m}_{[di/2]+1}) \cap \mathcal{N}_G = (\mathfrak{m}'_{[dj/2]+1} + \mathfrak{m}_{[di/2]+1}) \cap \mathcal{N}_G.$$

Thus we have shown that $\mathcal{J}(X, Y; P) = \mathcal{J}\big(X, Y; P(Y, i, j)\big)$ for $Y \in \mathcal{N}_G \cap \mathfrak{m}_1$.

STEP 2. If $j = 0$ and $Y \in \mathfrak{m}_0$, taking $\ell = [di/2] + 1$ and $m = 1$ and arguing as in Step 1 results in $\mathcal{J}(X, Y; P) = \mathcal{J}\big(X, Y; P(Y, i, 0)\big)$.

STEP 3. If $j \geq 1$ and $Y \notin \mathfrak{m}_1$ or $j = 0$ and $Y \notin \mathfrak{m}_0$, then $P(Y, i, j) = \emptyset$. The proof that $\mathcal{J}(X, Y; P) = 0$ is as for Lemma 3.9 of [Mu2]. ∎

Let $i \geq 0$. Define

$$(4.11) \quad \alpha = \begin{pmatrix} a\sqrt{\varepsilon} & 0 & \varpi^{-1} b\sqrt{\varepsilon} \\ 0 & (a-b)\sqrt{\varepsilon} & 0 \\ \varpi b\sqrt{\varepsilon} & 0 & a\sqrt{\varepsilon} \end{pmatrix}, \quad a, b \in F,\ |b| = q^{i+1},\ |a| \leq q^{i+1}.$$

An argument similar to that in [J], p. 57 shows that the stabilizer G'' of α in G is isomorphic to $H_{an} \times E^1$. Let $\mathfrak{g}''$ be the Lie algebra of G''.

LEMMA 4.12. *Suppose $r \geq 1$. Then*

$$(\mathfrak{g}'' + \mathfrak{l}_{r+1}) \cap (\mathfrak{l}_r - \mathfrak{l}_{r+1}) \cap \mathcal{N}_G = \emptyset.$$

PROOF. Suppose $X \in (\mathfrak{g}'' + \mathfrak{l}_1) \cap \mathfrak{l}_0$. Then $X \in Y + \mathfrak{l}_1$, where

$$Y = \begin{pmatrix} c\sqrt{\varepsilon} & 0 & \varpi^{-1} d\sqrt{\varepsilon} \\ 0 & e\sqrt{\varepsilon} & 0 \\ \varpi d\sqrt{\varepsilon} & 0 & c\sqrt{\varepsilon} \end{pmatrix}, \quad c, d, e \in O_F.$$

If $X \in \mathcal{N}_G$, $X^3 = 0$, which implies that $Y^3 \in \tilde{\mathfrak{l}}_1$. It is easily seen that $Y^3 \in \tilde{\mathfrak{l}}_1$ if and only if $c, d, e \in \mathfrak{p}_F$, that is, $X \in \mathfrak{l}_1$.

A similar type of argument works for $X \in (\mathfrak{g}'' + \mathfrak{l}_2) \cap \mathfrak{l}_1$. ∎

https://doi.org/10.4153/CJM-1995-032-x Published online by Cambridge University Press

LEMMA 4.13. *Suppose* $Y \in \mathcal{N}_G$. *Let* $X = \alpha+\beta$ *with* α *as in (4.11) and* $\beta \in \mathfrak{g}'' \cap \mathfrak{l}_{-2i-1}$. *Then* $\mathcal{J}(X, Y; L) = \mathcal{J}(\alpha, Y; L)$. *Furthermore* $\mathcal{J}(\alpha, Y; L) = 0$ *whenever* $Y \notin \mathfrak{l}_{2i}$.

PROOF. Suppose $Y \in \mathfrak{l}_r - \mathfrak{l}_{r+1}$ for some $r \leq 2i-1$. Otherwise there is nothing to show. Set $\ell = 2i - r$. The element α of (4.11) is slightly different from the α considered by Moy, but is conjugate to it by an element of L, so Moy's results still hold. Argue as in the proof of Lemma 4.3, using results on p. 200 of [Mo], to see that $\mathcal{J}\big(X, \operatorname{Ad} k^{-1}(Y); L_\ell\big)$ vanishes unless $\operatorname{Ad} k^{-1}(Y) \in \mathfrak{g}'' + \mathfrak{l}_{r+1}$, which, by Lemma 4.12 is equivalent to $Y \in \mathfrak{l}_{r+1}$. Thus $\mathcal{J}\big(\alpha, \operatorname{Ad} k^{-1}(Y); L_\ell\big) = 0$ for all $k \in L$. This implies, as in the proof of Lemma 4.3, that $\mathcal{J}(X, Y; L) = 0$.

We have now shown that, independent of the choice of β, $\mathcal{J}(X, Y; L) = 0$ unless $Y \in \mathfrak{l}_{2i}$. To finish the proof, note that $\beta \in \mathfrak{l}_{-2i-1} = \mathfrak{l}_{2i}^*$ ((4.6)). Thus $\mathcal{J}(X, Y; L) = \mathcal{J}(\alpha, Y; L)$ for $Y \in \mathfrak{l}_{2i}$. ∎

5. **Definition of X_π.** In [Mo], Moy defined nondegenerate representations, a set of irreducible representations of open compact subgroups of G. Up to twisting by a one-dimensional character of G, each irreducible admissible representation of G contains a nondegenerate representation. Using Hecke algebra isomorphisms, Moy classified the irreducible admissible representations of G containing a given nondegenerate representation. He identified the supercuspidal representations and proved that they are all induced from representations of open compact subgroups. Jabon ([J]) used Moy's results to explicitly determine the inducing data for each supercuspidal representation.

Suppose $\pi = \operatorname{Ind}_H^G \kappa$ for some finite-dimensional representation κ of an open compact subgroup H. Let χ_κ be the character of κ. The function $f_\pi \colon G \longrightarrow \mathbf{C}$ defined by

$$f_\pi(x) = \begin{cases} \chi_\kappa(x), & \text{if } x \in H, \\ 0, & \text{otherwise} \end{cases}$$

is a finite sum of matrix coefficients of π.

Let ${}^0\mathcal{E}(G)$ be the set of irreducible supercuspidal representations of G. Suppose κ_u is an irreducible cuspidal unipotent representation of $\mathbf{G}(\mathbf{F}_q)$ (see Lemma 5.2). The representation π_u obtained by inflating κ_u to K and then inducing to G is irreducible ([Mo]). Let ${}^0\mathcal{E}_u(G)$ be the subset of ${}^0\mathcal{E}(G)$ consisting of those representations which are equivalent to $\pi_u \otimes \chi$ for some one-dimensional representation χ of G. Since any two choices for κ_u differ by a one-dimensional representation of $\mathbf{G}(\mathbf{F}_q)$ ([Mo]), ${}^0\mathcal{E}_u(G)$ is independent of the choice of κ_u. Given X and Y in $\mathfrak{g}$, and an open compact subgroup K_c, let $\mathcal{J}(X, Y; K_c)$ be defined by (4.1). The goal of this section is to prove the following analogue of Proposition 3.10 of [Mu2]:

PROPOSITION 5.1. *Let* $\pi \in {}^0\mathcal{E}(G)$.

(1) If $\pi \notin {}^0\mathcal{E}_u(G)$, *there exists an* $X_\pi \in \mathfrak{g}$ *and an open compact subgroup* K_π *such that*

$$f_\pi(1)^{-1} \int_{K_\pi} f_\pi\big(k^{-1} c(-Y/2) k\big)\, dk = \mathcal{J}(X_\pi, Y; K_\pi), \quad Y \in \mathcal{N}_G.$$

(2) *If* $\pi \in {}^0\mathcal{E}_u(G)$, *there exist* $X_{u,1}$ *and* $X_{u,2} \in \mathfrak{g}_{\mathrm{reg}}$ *such that, if* $Y \in \mathcal{N}_G$,

$$f_\pi(1)^{-1}\int_K f_\pi\big(k^{-1}c(-Y/2)k\big)\,dk = \frac{(q+1)^2}{3q}\mathcal{J}(X_{u,1},Y;K) - \frac{(q^2-q+1)}{3q}\mathcal{J}(X_{u,2},Y;K).$$

REMARKS. (a) In every case, the centralizer of X_π in G is compact, but X_π may not be regular.

(b) Note that $c(-Y/2)=\exp(Y)$ if $Y\in\mathcal{N}_G$.

(c) In (2), $X_{u,1}$ and $X_{u,2}$ are independent of the choice of $\pi \in {}^0\mathcal{E}_u(G)$.

There are three general types of π to be considered, according to the properties of the nondegenerate representations Ω which they contain. The first type (Lemmas 5.3 and 4) contains a nondegenerate representation Ω of K or L which factors to an irreducible cuspidal representation of K/K_1 or L/L_1. The second type (Lemma 5.6) contains an Ω which is represented by a regular element α (see (5.5)). Finally, for the third type (Lemmas 5.8, 5.11, 5.12), Ω is represented by a singular semisimple element α of the form (4.8) or (4.11).

LEMMA 5.2. *Let* κ *be an irreducible cuspidal representation of* $\mathbf{G}(\mathbf{F}_q)$. *Then* κ *has degree* $(q-1)(q+1)^2$, $(q-1)(q^2-q+1)$, *or* $q(q-1)$. *Let* $\bar{Y}$ *be a nilpotent element of* $\mathfrak{g}(\mathbf{F}_q)$, *and let* $\bar{\psi}$ *be a nontrivial character of* $\mathbf{F}_{q^2}$. *Given a regular element* $\bar{X}\in\mathfrak{g}(\mathbf{F}_q)$, *let* $\bar{T}$ *be the Cartan subgroup of* $\mathbf{G}(\mathbf{F}_q)$ *such that* $X\in\bar{\mathcal{T}}=\mathrm{Lie}(\bar{T})$. *Define*

$$Q(\bar{X},\bar{Y}) = q^{-3}|\bar{T}|^{-1}\sum_{x\in\mathbf{G}(\mathbf{F}_q)}\bar{\psi}\Big(\mathrm{tr}\big(\bar{X}\,\mathrm{Ad}\,x^{-1}(\bar{Y})\big)\Big),$$

where $|\bar{T}|$ *denotes the order of* $\bar{T}$.

(1) *If* κ *has degree* $(q-1)(q+1)^2$, *then* $\chi_\kappa\big(c(\bar{Y})\big) = Q(\bar{X}_{\mathrm{unr}},\bar{Y})$ *for any* $\bar{X}_{\mathrm{unr}}$ *which is regular in* $\mathfrak{g}(\mathbf{F}_q)$ *and belongs to the image of* $\mathcal{T}_{\mathrm{unr}}\cap\mathfrak{k}_0$.

(2) *If* κ *has degree* $(q-1)(q^2-q+1)$, *then* $\chi_\kappa\big(c(\bar{Y})\big) = Q(\bar{X}_E,\bar{Y})$, *for any* $\bar{X}_E$ *which is regular in* $\mathfrak{g}(\mathbf{F}_q)$ *and belongs to the image of* $\mathcal{T}_{E,1}\cap\mathfrak{k}_0$.

(3) κ *is unipotent if and only if* κ *has degree* $q(q-1)$. *In that case,* $\chi_\kappa\big(c(\bar{Y})\big) = \big(Q(\bar{X}_{\mathrm{unr}},\bar{Y}) - Q(\bar{X}_E,\bar{Y})\big)/3$, *where* $\bar{X}_{\mathrm{unr}}$ *and* $\bar{X}_E$ *are as in (1) and (2).*

PROOF. For the definition of cuspidal and unipotent representations of a reductive group over a finite field, see [DL]. Suppose $\bar{T}$ is the image of T_{unr} or $T_{E,2}$ in $\mathbf{G}(\mathbf{F}_q)$. Let θ be a regular character of $\bar{T}$, that is, a character which is not fixed by any nontrivial element of the Weyl group of $\bar{T}$ in $\mathbf{G}(\mathbf{F}_q)$. The virtual character $R_{\bar{T}}(\theta)$ ([DL]) is, up to sign, the character of an irreducible cuspidal representation of $\mathbf{G}(\mathbf{F}_q)$. The values of the character χ_θ of this representation on the unipotent set are independent of the choice of character θ. Kazhdan ([K]) proved that, if $u\in\mathbf{G}(\mathbf{F}_q)$ is unipotent, then $\chi_\theta(u) = Q(\bar{X},\log u)$, where $\bar{X}$ is any regular element of $\bar{T}$. Note that, if $\bar{Y}$ is nilpotent, then $\log c(-\bar{Y}/2)=\bar{Y}$.

Ennola ([E]) computed the characters of $\mathbf{G}(\mathbf{F}_q)$. The cuspidal representations can be identified using the properties of their characters on the unipotent set. There are three families of irreducible cuspidal representations, of dimension $(q-1)^2(q+1)$, $(q-1)(q^2-q+1)$ and $q(q-1)$. The members of each family take the same values on the unipotent set. If $\bar{T}$ is the image of T_{unr}, then $|\bar{T}| = q^3+1$ because $\bar{T}$ is the set of norm one elements in a cubic unramified extension of $\mathbf{F}_{q^2}$. χ_θ is the character of an irreducible cuspidal representation of dimension $Q(\bar{X}_{\mathrm{unr}}, 0) = (q-1)(q+1)^2$. Each member κ of the family of cuspidal representations having dimension $(q-1)(q+1)^2$ therefore has the property that $\chi_\kappa(u) = \chi_\theta(u)$ for u unipotent. (1) now follows.

(2) also holds by the same argument, using the fact that if $\bar{T}$ is the image of $T_{E,1}$, $|\bar{T}| = (q+1)^3$.

Suppose κ_j, $1 \leq j \leq 3$, are cuspidal representations of $\mathbf{G}(\mathbf{F}_q)$ having degrees $(q-1)(q+1)^2$, $(q-1)(q^2-q+1)$ and $q(q-1)$, respectively. From the character tables in [E], we find that $\chi_{\kappa_3}(u) = \big(\chi_{\kappa_1}(u) - \chi_{\kappa_2}(u)\big)/3$ for every unipotent $u \in \mathbf{G}(\mathbf{F}_q)$. Thus the second part of (3) follows from (1) and (2). That the unipotent cuspidal representations are those of degree $q(q-1)$ is implied by [L], Section 9. ∎

LEMMA 5.3. *Suppose $\pi \in {}^0\mathcal{E}(G)$ contains a nondegenerate representation Ω of K, Ω being trivial on K_1 and factoring to a cuspidal representation of $\mathbf{G}(\mathbf{F}_q) \simeq K/K_1$. Let $X_{u,1}$, resp. $X_{u,2}$, be any element of $\mathcal{T}_{\mathrm{unr}} \cap \mathfrak{k}_{-1}$, resp. $\mathcal{T}_{E,1} \cap \mathfrak{k}_{-1}$, such that the image of $\varpi X_{u,1}$, resp. $\varpi X_{u,2}$, in $\mathfrak{g}(\mathbf{F}_q)$ is regular. Then*

(1) If Ω has degree $(q-1)(q+1)^2$, Proposition 5.1(1) holds with $X_\pi = X_{u,1}$ and $K_\pi = K$.

(2) If Ω has degree $(q-1)(q^2-q+1)$, Proposition 5.1(1) holds with $X_\pi = X_{u,2}$ and $K_\pi = K$.

(3) $\pi \in {}^0\mathcal{E}_u(G)$ if and only if the degree of Ω is $q(q-1)$, and in that case Proposition 5.1(2) holds.

PROOF. Let $Y \in \mathcal{N}_G$. If $Y \notin \mathfrak{k}_0$, then the left sides of the equalities are zero, because f_π is supported on $K = K_\pi$. The right sides vanish as a consequence of Lemma 4.3(1). Thus we may assume that $Y \in \mathfrak{k}_0$. Observe that (3) is a consequence of (1) and (2). If $X \in \mathfrak{k}_0$ and $k \in K$, let $\bar{X}$ and $\bar{k}$ denote the images of X and k in $\mathfrak{g}(\mathbf{F}_q)$ and $\mathbf{G}(\mathbf{F}_q)$, respectively. In cases (1) and (2), for $Y \in \mathcal{N}_G \cap \mathfrak{k}_0$,

$$
\begin{aligned}
f_\pi(1)^{-1} &\int_K f_\pi\big(k^{-1}c(-Y/2)k\big)\,dk \\
&= \Omega(1)^{-1} \int_K \Omega\big(k^{-1}c(-Y/2)k\big)\,dk \\
&= \int_K |\mathbf{G}(\mathbf{F}_q)|^{-1} \Big(\sum_{x \in \mathbf{G}(\mathbf{F}_q)} \bar{\psi}\Big(\mathrm{tr}\big(\overline{\varpi X_\pi}\,\mathrm{Ad}\,x^{-1}\,\mathrm{Ad}\,\bar{k}^{-1}(\bar{Y})\big)\Big)\Big)\,dk \\
&= \int_K \psi_E\Big(\mathrm{tr}\big(X_\pi\,\mathrm{Ad}\,k^{-1}(Y)\big)\Big)\,dk = \mathcal{J}(X_\pi, Y; K).
\end{aligned}
$$

Here we have applied Lemma 5.2 to obtain the second equality. As $\bar{\psi}$ can be taken to be any nontrivial character of $\mathbf{F}_{q^2}$, we can assume that $\psi_E(t) = \bar{\psi}(\overline{\varpi t})$, $t \in \mathfrak{p}_E^{-1}$. ∎

LEMMA 5.4. *Suppose $\pi \in {}^0\mathcal{E}(G)$ contains a nondegenerate representation Ω of L, Ω being trivial on L_1 and factoring to an irreducible cuspidal representation of $L/L_1 \simeq \mathbf{H}_{qs}(\mathbf{F}_q) \times \mathbf{U}(1)(\mathbf{F}_q)$. There exists $X_\pi \in \mathcal{T}_{E,2} \cap \mathfrak{l}_{-2} \cap \mathfrak{g}_{\mathrm{reg}}$ such that the image of ϖX_π in $\mathfrak{l}_0/\mathfrak{l}_1$ is regular. and Proposition 5.1(1) holds with $K_\pi = L$.*

PROOF. Using Ennola's character tables ([E]), and arguing as in the proof of Lemma 5.2, we can show that there exists a regular $\bar{X} \in \mathrm{Lie}\big(\mathbf{H}_{qs}(\mathbf{F}_q)\big)$ such that

$$\chi_\kappa\big(c(-\bar{Y}/2)\big) = \chi_\kappa(1)|\mathbf{H}_{qs}(\mathbf{F}_q)|^{-1} \sum_{x \in \mathbf{H}_{qs}(\mathbf{F}_q)} \bar{\psi}\Big(\mathrm{tr}\big(\bar{X}\,\mathrm{Ad}\,x^{-1}(\bar{Y})\big)\Big), \quad \bar{Y} \text{ nilpotent}$$

where κ is any irreducible cuspidal representation of $\mathbf{H}_{qs}(\mathbf{F}_q)$.

Given $Z \in \mathfrak{l}_0$, let $\bar{Z}$ denote the image of Z in $\mathfrak{l}_0/\mathfrak{l}_1$. Let $X_{E,2} \in \mathcal{T}_{E,2}$ be such that $\bar{X}_{E,2} = \bar{X}$. Then the entries of $X_{E,2}$ satisfy $|b| = |a^2 - b^2|^{1/2} = q$, and $|c| \leq 1$, so $X_{E,2} \in \mathfrak{g}_{\mathrm{reg}}$. If $Y \in \mathcal{N}_G \cap \mathfrak{l}_0$, then, since $\bar{Y}$ is a nilpotent element of $\mathrm{Lie}\big(\mathbf{H}_{qs}(\mathbf{F}_q) \times \mathbf{U}(1)(\mathbf{F}_q)\big)$, $\bar{Y} \in \mathrm{Lie}\big(\mathbf{H}_{qs}(\mathbf{F}_q)\big)$.

Set $X_\pi = X_{E,2}$. The remainder of the proof is much like the proof of Lemma 5.3, except that Lemma 4.3(4) is used. ∎

Suppose $\mathfrak{m}_\ell = \mathfrak{k}_\ell, \mathfrak{l}_\ell, \mathfrak{i}_\ell$, or $\mathfrak{i}_\ell^\flat$ for some $\ell \geq 1$. If $\alpha \in \mathfrak{m}_{\ell+1}^*$, the representation Ω_α of $P_\ell = c(\mathfrak{m}_\ell)$ is defined by:

$$\Omega_\alpha\big(c(X)\big) = \psi_F(\langle\alpha, -2X\rangle), \quad X \in \mathfrak{m}_\ell. \tag{5.5}$$

The nondegenerate representations appearing in the remainder of the section all have this form.

The next case to be considered is that of $\pi \in {}^0\mathcal{E}(G)$ which contains a nondegenerate representation Ω_α of P_ℓ, some $\ell \geq 1$, where $\alpha \in \mathfrak{g}_{\mathrm{reg}}$.

LEMMA 5.6. *For each α and P_ℓ given below, if $\pi \in {}^0\mathcal{E}(G)$ contains a nondegenerate representation Ω_α of P_ℓ (defined by (5.5)), then Proposition 5.1(1) holds, with $X_\pi = \alpha$ and $K_\pi = P$.*

(1) $\alpha \in \mathcal{T}_{E,1}$ or $\mathcal{T}_{\mathrm{unr}}$ such that the image of $\varpi^{i+1}\alpha$ in $\mathfrak{g}(\mathbf{F}_q)$ is regular, and $P_\ell = K_i$
(2) $\alpha = X_{\theta,1}$, with $|a|, |c| \leq q^i$, $|b| = q^{i+1}$, and $P_\ell = I_{4i-2}^\flat$
(3) $\alpha = X_{\theta,2}$, with $|a|, |c| \leq q^i$, $|b| = q^{i+1}$, and $P_\ell = L_{2i-1}$
(4) $\alpha = X_{E,2}$ with $|a|, |c| \leq q^{i+1}$, $|b| = |(a-c)^2 - b^2|^{1/2} = q^{i+1}$, and $P_\ell = L_{2i}$
(5) $\alpha \in \mathcal{T}_{\mathrm{ram},\zeta}$ with $|a|, |c| \leq q^i$, $|b| = q^{i+1}$, and $P_\ell = I_{3i-1}$
(6) $\alpha \in \mathcal{T}_{\mathrm{ram},\zeta}$ with $|a|, |b| \leq q^i$, $|c| = q^{i+1}$, and $P_\ell = I_{3i-2}$

PROOF. The inducing data for π as in (1), (2), (3) and (4) is given in Propositions 3.5, 3.8, 3.25, 3.23, and 3.27 of [J] respectively. For a description in cases (5) and (6), see [Mo], p. 201.

Let T be the Cartan subgroup containing $c(\alpha)$. Set $m = [(\ell+1)/2]$. In every case, $\pi = \mathrm{Ind}_{TP_m}^G \kappa$, where κ is an irreducible representation of TP_m such that $\kappa|P_{[\ell/2]+1}$ is a multiple of a character ρ of $P_{[\ell/2]+1}$ which has the property $\rho|P_\ell = \Omega_\alpha$.

If $X \in \mathcal{N}_G$, then $c(X) \in TP_r$, $r \geq 1$, if and only if $X \in \mathfrak{m}_r$. This is a variant of Lemma 4.2. Suppose $Y \in \mathcal{N}_G$. Let $k \in P$. Then $k^{-1}c(-Y/2)k = c\big(-\operatorname{Ad} k^{-1}(Y)/2\big) \in TP_m$ if and only if $Y \in \mathfrak{m}_m$.

Let $Y \in \mathfrak{m}_\ell \cap \mathcal{N}_G$. Since $\kappa|P_\ell$ is a multiple of Ω_α, it follows that

$$f_\pi\big(k^{-1}c(-Y/2)k\big) = f_\pi(1)\psi_E\big(\operatorname{tr}(\alpha \operatorname{Ad} k^{-1}(Y)\big)$$

Thus

$$f_\pi(1)^{-1}\int_P f_\pi\big(k^{-1}c(-Y/2)k\big)\,dk = \mathcal{J}(\alpha, Y; P), \quad Y \in \mathcal{N}_G \cap \mathfrak{m}_\ell.$$

By Lemmas 4.3 and 4.7, if $Y \in \mathcal{N}_G$ and $Y \notin \mathfrak{m}_\ell$, then $\mathcal{J}(\alpha, Y; P) = 0$. Therefore to complete the proof it suffices to show that $\int_P f_\pi\big(k^{-1}c(-Y/2)k\big)\,dk$ vanishes for $Y \in \mathcal{N}_G \cap (\mathfrak{m}_m - \mathfrak{m}_\ell)$.

Suppose $Y \in \mathfrak{m}_r - \mathfrak{m}_{r+1}$, where $\ell - 1 \geq r \geq [\ell/2]+1$. By definition of f_π, the integral $\int_P f_\pi\big(k^{-1}c(-Y/2)k\big)\,dk$ is a nonzero multiple of

$$\int_P \int_{P_{\ell-r}} \chi_\kappa\big(h^{-1}k^{-1}c(-Y/2)kh\big)\,dh\,dk.$$

Fix $k \in P$ and set $Z = \operatorname{Ad} k^{-1}(Y)$. If $h = c(H)$, $H \in \mathfrak{m}_{\ell-r}$, then after verifying that

$$c(Z/2)h^{-1}c(-Z/2)h \in c([Z,H])P_{\ell+1} \quad \text{and} \quad [Z,H] \in \mathfrak{m}_\ell,$$

the inner integral above can be rewritten as

$$\chi_\kappa(1)\rho\big(c(-Z/2)\big)\int_{\mathfrak{m}_{\ell-r}} \psi_E(-2\operatorname{tr}[\alpha, Z]H)\,dH$$

As was seen in the proofs of Lemmas 4.3 and 4.7, this last integral vanishes because $Z \notin \mathfrak{m}_\ell$. Thus

$$\int_P f_\pi\big(k^{-1}c(-Y/2)k\big)\,dk = 0, \quad Y \in \mathcal{N}_G \cap (\mathfrak{m}_{[\ell/2]+1} - \mathfrak{m}_\ell).$$

Finally, we must consider the case ℓ is even and $Y \in \mathcal{N}_G \cap (\mathfrak{m}_m - \mathfrak{m}_{m+1})$. In this case, κ is obtained by a Heisenberg construction, and $\kappa|P_m$ is the unique irreducible component of $\operatorname{Ind}_{(T\cap P_m)P_{m+1}}^{P_m} \rho$. Since the unipotent subset does not intersect $(T \cap P_m)P_{m+1} - P_{m+1}$, no P-conjugate of $c(Y)$ can lie in $(T \cap P_m)P_{m+1}$. This, together with the formula for characters of induced representations of finite groups, implies that

$$f_\pi\Big(c\big(-\operatorname{Ad} k^{-1}(Y)/2\big)\Big) = \chi_\kappa\Big(c\big(-\operatorname{Ad} k^{-1}(Y)/2\big)\Big) = 0, \quad k \in P. \qquad \blacksquare$$

Let α be as in (4.8). We shall use notation from Section 4. Lemmas 5.8–5.13 are concerned with those supercuspidal representations π which contain the representation Ω_α of K_i defined by (5.5). Before stating the lemmas, we use Ω_α to define a character of G', and discuss the parametrization of the representations π.

We define a one-dimensional representation of G' which coincides with (5.5) on a subset of G' containing $K_i \cap G'$. This extension of Ω_α to G', though it is not unique, will also be denoted by Ω_α. Let $E_i^1 = E^1 \cap (1 + \mathfrak{p}_E^{[i/3]+1})$. Note that the Cayley transform

$c\colon t\sqrt{\varepsilon} \longmapsto (1-t\sqrt{\varepsilon})(1+t\sqrt{\varepsilon})^{-1}$ maps $\{t\sqrt{\varepsilon} \mid t \in \mathfrak{p}_E^{[i/3]+1}\}$ onto E_i^1. The map $\phi_\alpha\colon E_i^1 \times E_i^1 \longrightarrow \mathbf{C}$ defined by

$$\phi_\alpha\big(c(s\sqrt{\varepsilon}), c(t\sqrt{\varepsilon})\big) = \psi_E\big(-2\varepsilon(as+ct)\big), \quad s,t \in \mathfrak{p}_E^{[i/3]+1}$$

is a linear character of $E_i^1 \times E_i^1$. Fix an extension, also called ϕ_α, of ϕ_α to $E^1 \times E^1$.

Given $x \in G' \simeq H_{qs} \times E^1$, let x_1 be the H_{qs}-component of x, and x_2 the E^1-component. Note that $\det x_1 \in E^1$. Set

$$\Omega_\alpha(x) = \phi_\alpha(\det x_1, x_2), \quad x \in G'.$$

Suppose $X \in \mathfrak{g}'$ is such that $X^3 \in \tilde{\mathfrak{k}}_{i+1}$. Let λ_ℓ, $1 \le \ell \le 3$, be the eigenvalues of X, λ_1 and λ_3 being the eigenvalues of the Lie(H_{qs})-component of X. Let L be a finite extension of E containing λ_1 and λ_3. Note that $\lambda_2 \in E$. Extend $|\cdot|_E$ to $|\cdot|_L$ on L. Since $\varpi^{-i-1}X^3 \in \tilde{\mathfrak{k}}_0$, we have $\varpi^{-i-1}\lambda_\ell^3 \in O_L$, $1 \le \ell \le 3$. That is, $|\lambda_\ell|_L^3 \le q_E^{i+1}$. Let $x = c(X)$. Then $\det x_1 = c(\lambda_1\lambda_3)$ and $x_2 = c(\lambda_2)$. A simple argument shows that $c(\lambda_1\lambda_3) \in c(\lambda_1+\lambda_3)(1+\mathfrak{p}_E^{i+1})$. Thus

$$\phi_\alpha(\det x_1, x_2) = \psi_E\Big(-2\sqrt{\varepsilon}\big(a(\lambda_1+\lambda_3)+c\lambda_2\big)\Big) = \psi_F(\langle \alpha, -2X\rangle).$$

We have shown

$$\Omega_\alpha\big(c(X)\big) = \psi_F(\langle \alpha, -2X\rangle), \quad X \in \mathfrak{g}' \text{ such that } X^3 \in \tilde{\mathfrak{k}}_{i+1}. \tag{5.7}$$

In particular, the new definition of Ω_α on $K_i \cap G'$ coincides with the old ((5.5)).

The supercuspidal representations containing the representation Ω_α of K_i defined by (5.5) are parametrized by those supercuspidal representations π' of G' containing the trivial representation of $G' \cap I_{3i}$ ([J], p. 42). The supercuspidal representations obtained by Jabon in Theorems 3.12, 3.14, 3.17, 3.19, and 3.22 of [J] are actually those which contain Ω_α^{-1}. To get the ones containing Ω_α, it suffices to replace Ω_α^{-1} in Jabon's theorems by Ω_α.

If $\beta \in \mathfrak{g}'$ and $\mathfrak{m}'$ is a lattice in $\mathfrak{g}'$ which has the property that $\langle \beta, (\mathfrak{m}')^2\rangle \subset O_F$, let Ω_β be the representation of $c(\mathfrak{m}')$ defined by:

$$\Omega_\beta\big(c(X)\big) = \psi_F(\langle \beta, -2X\rangle), \quad X \in \mathfrak{m}'.$$

The different types of π' which must be considered are of the form $\pi' = \mathrm{Ind}_H^{G'}\kappa'$, where H and κ' are as below ([J]):

(a) $H = T_{E,1}K'_{[(j+1)/2]}$, $1 \le j < i$. The restriction of κ' to K'_j is a multiple of the representation Ω_β, β as in (4.9i).

(b) $H = K'$. κ' is trivial on K'_1 and factors to an irreducible cuspidal representation of the finite group $\mathbf{H}_{qs}(\mathbf{F}_q)$.

(c) $H = T_{E,2}L'_j$, $1 \le j < i$. The restriction of κ' to L'_{2j} is a multiple of the representation Ω_β, β as in (4.9ii).

(d) $H = L'$. κ' is trivial on L'_1 and factors to an irreducible cuspidal representation of the finite group $L/L_1 \simeq \mathbf{H}_{qs}(\mathbf{F}_q)$.

(e) $H = T_{\theta,1}I_{2j}^{\flat\prime}$, $\theta \in \{\varpi, \varepsilon\varpi\}$, $1 \le j < i$. The restriction of κ' to $I_{4j-2}^{\flat\prime}$ is a multiple of the representation Ω_β, β as in (4.9iii).

If M is a subgroup of G, let $M' = M \cap G'$.

LEMMA 5.8. *Suppose $\pi \in {}^0\mathcal{E}(G)$ contains the representation Ω_α of K_i, α as in (4.8), and the for the corresponding representation π' of G' has inducing data as in (a), (c), or (e). Then Proposition 5.1(1) holds with $X_\pi = \alpha + \beta$, β as in (4.9i), (ii), or (iii) ($j \geq 1$), and $K_\pi = K$, L or I, respectively.*

PROOF. Let $\mathfrak{m}_\ell = \mathfrak{k}_\ell, \mathfrak{l}_\ell$, and $\mathfrak{i}_\ell^\flat$, $\ell \in \mathbf{Z}$, $r = [j/2]+1, j+1$, and $2j$, $s = [(j+1)/2]$, j, and $2j$, $t = j, 2j$, and $4j-2$, $d = 1, 2$ and 4, $T = T_{E,1}$, $T_{E,2}$, and $T_{\theta,1}$, in cases (a),(c) and (e), respectively. In each case, let $\mathcal{T}$ be the Lie algebra of T.

The first step in the construction of the inducing data for π is to define a representation κ_α of $TP'_sP_{[(di+1)/2]}$. Set

$$J_i = c(\mathfrak{m}_{di} + \mathfrak{m}'^{\perp}_{[di/2]+1}) \quad \text{and} \quad \tilde{J}_i = c(\mathfrak{m}_{di} + \mathfrak{m}'^{\perp}_{[(di+1)/2]}).$$

Observe that since $\alpha \in \mathfrak{m}_{-d(i+1)} = \mathfrak{m}^*_{-di+1}$, Ω_α may be regarded as a representation of P_{di} which is trivial on P_{di+1}. To extend Ω_α from P_{di} to J_i, set $\Omega_\alpha|c(\mathfrak{m}'^{\perp}_{[di/2]+1}) \equiv 1$. Recall that Ω_α is already defined on $TP'_s \subset G'$. Note that TP'_s normalizes J_i and conjugation by TP'_s fixes $\Omega_\alpha|J_i$. Also $P'_{[di/2]+1}J_i = P_{[di/2]+1}$ and $s \leq [(di+1)/2]$. Therefore Ω_α extends to $TP'_sP_{[di/2]+1}$. If $J_i = \tilde{J}_i$, then $TP'_sJ_i = TP'_sP_{[(di+1)/2]}$ and κ_α is just Ω_α. If $J_i \neq \tilde{J}_i$, then a Heisenberg construction must be used to produce the representation κ_α (*cf.* [J]). This representation has the property that $\kappa_\alpha|\tilde{J}_i$ is the unique irreducible component of $\mathrm{Ind}_{J_i}^{\tilde{J}_i}\Omega_\alpha$. Since J_i is normal in $\tilde{J}_i$, this implies that

$$\begin{aligned} &\chi_{\kappa_\alpha}|\tilde{J}_i - J_i \equiv 0 \quad \text{and} \\ &\chi_{\kappa_\alpha}(1)^{-1}\chi_{\kappa_\alpha}|J_i = \Omega_\alpha. \end{aligned} \tag{5.9}$$

The representation $\kappa'|P'_r$ is a multiple of a character ρ of P'_r which coincides with Ω_β on P'_t. Note that $2r \geq t+1$. Because $\Omega_\beta|T\cap P_{t+1} \equiv 1$, and

$$\begin{aligned} \big((T\cap P_r)/(T\cap P_{t+1})\big)^\wedge &\simeq (\mathcal{T}\cap \mathfrak{m}^*_{t+1})/(\mathcal{T}\cap\mathfrak{m}^*_r) \\ &\simeq (\mathcal{T}\cap\mathfrak{m}_{-t-d})/(\mathcal{T}\cap\mathfrak{m}_{-r-d+1}), \quad (cf.\ (4.6)) \end{aligned}$$

there exists $\tilde{\beta} \in \mathcal{T}\cap\mathfrak{m}_{-t-d}$ such that $\rho|T\cap P_r = \Omega_{\tilde{\beta}}$. Furthermore, $\rho|c(\mathcal{T}^\perp \cap \mathfrak{m}'_r) \equiv 1$ ([J]). It then follows from

$$P'_r = (T\cap P_r)c(\mathcal{T}^\perp\cap\mathfrak{m}'_r) \quad \text{and} \quad \Omega_{\tilde{\beta}}|c(\mathcal{T}^\perp\cap\mathfrak{m}'_r) \equiv 1,$$

that $\rho = \Omega_{\tilde{\beta}}$ on all of P_r. Because $\Omega_\beta|P_t = \Omega_{\tilde{\beta}}|P_t$, we may (and do) assume that $\beta = \tilde{\beta}$.

If $s = r$, then $\kappa' = \Omega_\beta$. Otherwise $r = s+1$ and $\kappa'|P'_s$ is the unique irreducible component of $\mathrm{Ind}_{(T\cap P_s)P'_r}^{P'_s}\Omega_\beta$. Therefore, if $x \in P'_s - P'_r$ is such that no P'_s-conjugate of x lies in $(T\cap P_s)P'_r$, then $\chi_{\kappa'}(x) = 0$. In both cases $s = r$ and $s = r-1$,

$$\frac{\chi_{\kappa'}(x)}{\chi_{\kappa'}(1)} = \Omega_\beta(x), \quad x \in P'_r. \tag{5.10}$$

π is induced from the representation κ of $TP'_sK_{[(di+1)/2]} = TP'_s\tilde{J}_i$ defined by

$$\kappa|\tilde{J}_i = \kappa_\alpha \otimes 1_{\dim\kappa'} \quad \text{and} \quad \kappa|TP'_s = \kappa_\alpha\otimes\kappa'$$

Here, $1_{\dim\kappa'}$ denotes the trivial representation of dimension $\dim\kappa'$.

Now we determine the values of the character χ_κ of κ on unipotent elements in the inducing subgroup. If $Y \in \mathcal{N}_G$, then

$$\begin{aligned} c(Y) \in TP_s'P_{[(di+1)/2]} &\Longleftrightarrow c(Y) \in P_s'P_{[(di+1)/2]} \\ &\Longleftrightarrow Y \in \mathfrak{m}_s' + \mathfrak{m}_{[(di+1)/2]}. \end{aligned}$$

The details of the proof of this are similar to the proof of Lemma 4.2 and are omitted. Let $Y \in \mathcal{N}_G \cap (\mathfrak{m}_s' + \mathfrak{m}_{[(di+1)/2]})$. Then, as remarked in the proof of Lemma 4.10,

$$Y = Y' + Y^\perp \quad \text{for some } Y' \in \mathfrak{m}_s' \text{ and } Y^\perp \in \mathfrak{m}'^{\perp}_{[(di+1)/2]}.$$

A straightforward calculation shows that

$$c(-Y')c(Y) \in c(Y^\perp - [Y^\perp, Y'] - 2Y'Y^\perp Y' + [Y'Y^\perp Y', Y'])P_{di+1} \subset c(\mathfrak{m}'^{\perp}_{[(di+1)/2]})P_{di+1} \subset \tilde{J}_i.$$

Combining this with above remarks concerning the definition of κ_α and (5.9), results in

$$\begin{aligned} \frac{\chi_{\kappa_\alpha}\big(c(Y)\big)}{\chi_{\kappa_\alpha}(1)} &= \frac{\chi_{\kappa_\alpha}\big(c(Y')c(-Y')c(Y)\big)}{\chi_{\kappa_\alpha}(1)} \\ &= \begin{cases} \Omega_\alpha\big(c(Y')\big), & \text{if } Y^\perp \in \mathfrak{m}_{[di/2]+1}, \text{ that is, } c(Y) \in TP_s'J_i \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Now we evaluate $\Omega_\alpha\big(c(Y')\big)$. Let $\tilde{\mathfrak{g}}$ denote the Lie algebra of 3×3 matrices with entries in E. Let $\tilde{\mathfrak{g}}'$ be the centralizer of α in $\tilde{\mathfrak{g}}$. For $\ell \in \mathbf{Z}$,

$$\tilde{\mathfrak{m}}_\ell = (\tilde{\mathfrak{m}}_\ell \cap \tilde{\mathfrak{g}}') + (\tilde{\mathfrak{m}}_\ell \cap \tilde{\mathfrak{g}}'^\perp)$$

It follows from $Y^\perp Y'^2 + Y'Y^\perp Y' + Y'^2 \in \tilde{\mathfrak{g}}'^\perp$, $Y' \in \mathfrak{m}_r$, and $Y^3 = (Y' + Y^\perp)^3 = 0$, that

$$Y'^3 \in (Y'Y^{\perp 2} + Y^\perp Y' Y^\perp + Y'^2 Y^\perp) + \tilde{\mathfrak{f}}_{3[di/2]+3} \subset \tilde{\mathfrak{m}}_{di+r} \subset \tilde{\mathfrak{f}}_{i+1}.$$

Apply (5.7) and note that $\langle \alpha, Y^\perp \rangle = 0$ to conclude that $\Omega_\alpha\big(c(Y')\big) = \psi_F(\langle \alpha, -2Y \rangle)$.

Suppose that $s = r - 1$. Since $c(Y)$ is unipotent and $c(Y) \in c(Y')P_r'P_{[(di+1)/2]}$, the image of $c(Y')$ in $P_s'/P_r' \simeq \mathbf{G}'(\mathbf{F}_q)$ is unipotent. Thus, if $c(Y') \notin P_r'$, no P_s-conjugate of $c(Y')$ is in TP_r. Combining this with earlier remarks about the character of κ', we conclude that

$$\chi_{\kappa'}\big(c(Y')\big) = 0 \quad \text{if } Y' \in \mathfrak{m}_s' - \mathfrak{m}_r'.$$

Recall that $\chi_{\kappa'}|P_r'$ is given by (5.10).

We can now conclude that for $Y \in \mathcal{N}_G \cap (\mathfrak{m}_s' + \mathfrak{m}_{[(di+1)/2]})$,

$$\frac{\chi_\kappa\big(c(Y)\big)}{\chi_\kappa(1)} = \begin{cases} \psi_F(\langle \alpha + \beta, -2Y \rangle), & \text{if } Y \in \mathfrak{m}_r' + \mathfrak{m}_{[di/2]+1} \\ 0 & \text{otherwise.} \end{cases}$$

The above formula gives values of f_π on $TP_s'P_{[(di+1)/2]} \cap c(\mathcal{N}_G) = c\big((\mathfrak{m}_s' + \mathfrak{m}_{[(di+1)/2]}) \cap \mathcal{N}_G\big)$. From this and Lemma 4.10, it is now clear that Proposition 5.1(1) holds with $K_\pi = P$ and $X_\pi = \alpha + \beta$. ∎

LEMMA 5.11. *Suppose $\pi \in {}^0\mathcal{E}(G)$ contains the representation Ω_α of K_i, α as in (4.8), and the corresponding representation π' of G' has inducing data as in (b) or (d). Then Proposition 5.1(1) holds with $X_\pi = \alpha + \beta$, β as in (4.8i) or (ii), $j = 0$, and $K_\pi = K$ or L, respectively.*

PROOF. Let $\mathfrak{m}_\ell = \mathfrak{k}_\ell$, resp. $\mathfrak{l}_\ell$, $\ell \in \mathbf{Z}$, $P = K$, resp. L, $d = 1$, resp. 2, in case (b), resp. (d).

Let J_i and $\tilde{J}_i$ be as in the proof of Lemma 5.8. Extend Ω_α to from P_{di} to $P'J_i$. Then produce a representation κ_α of $P'\tilde{J}_i$ whose restriction to P'_1J_i is a multiple of Ω_α.

Observe that

$$P'/P'_1 \simeq \begin{cases} \mathbf{H}_{qs}(\mathbf{F}_q) \times \mathbf{U}(1)(\mathbf{F}_q), & \text{in case (b)} \\ \mathbf{H}_{an}(\mathbf{F}_q) \times \mathbf{U}(1)(\mathbf{F}_q), & \text{in case (d).} \end{cases}$$

P'/P'_1 has no cuspidal unipotent representations and it can be shown, by an argument similar to that for Lemmas 5.2 and 5.3, that if the image of $Y' \in \mathfrak{m}'_0$ in $\mathfrak{m}'_0/\mathfrak{m}'_1$ is nilpotent, then

$$\frac{\chi_{\kappa'}\big(c(Y')\big)}{\chi_{\kappa'}(1)} = \int_{P'} \psi_E\Big(\mathrm{tr}\big(\beta \,\mathrm{Ad}\, k^{-1}(-2Y')\big)\Big)\, dk,$$

for some β as in (4.9i) of (4.9ii) with $j = 0$ in cases (b) and (d), respectively. (The measure on P' is assumed to be normalized so that P' has volume one.)

$\pi = \mathrm{Ind}_{P'\tilde{J}_i}^G \kappa$, where

$$\kappa|P' = \kappa_\alpha \otimes \kappa' \quad \text{and} \quad \kappa|\tilde{J}_i = \kappa_\alpha \otimes 1_{\dim \kappa'}.$$

If $J_i = \tilde{J}_i$, then an argument as in the proof of Lemma 5.8 yields

$$\frac{\chi_\kappa\big(c(Y)\big)}{\chi_\kappa(1)} = \psi_F(\langle \alpha, -2Y\rangle) \int_{P'} \psi_F\Big(\big\langle \beta, -2\,\mathrm{Ad}\, k^{-1}(Y)\big\rangle\Big)\, dk, \quad Y \in \mathcal{N}_G \cap (\mathfrak{m}'_0 + \mathfrak{m}_{[(di+1)/2]}).$$

If $J_i \neq \tilde{J}_i$, then the main idea of the proof is along the same general lines as for Lemma 5.8, except that it is much longer, as the calculation of χ_{κ_α} is more involved. We omit the details. Proofs of analogous results for $\mathrm{GL}_n(F)$ and classical groups appear in Lemma 3.20 of [Mu2] and Lemma 9.2 of [Mu4]. The value of $\chi_{\kappa_\alpha}\big(c(Y)\big)$ is given by:

$$\frac{\chi_{\kappa_\alpha}\big(c(Y)\big)}{\chi_{\kappa_\alpha}(1)} = \int_{P_{[(di+1)/2]}} \psi_F\Big(\big\langle \alpha, \mathrm{Ad}\, h^{-1}(-2Y)\big\rangle\Big)\, dh, \quad Y \in \mathcal{N}_G \cap (\mathfrak{m}'_0 + \mathfrak{m}_{[(di+1)/2]}).$$

Combining this with the formula for $\chi_{\kappa'}$ results in

$$\frac{\chi_\kappa\big(c(Y)\big)}{\chi_\kappa(1)} = \int_{P'P_{[(di+1)/2]}} \psi_F\Big(\big\langle \alpha + \beta, \mathrm{Ad}\, h^{-1}(-2Y)\big\rangle\Big)\, dh.$$

The desired result now follows after an application of Lemma 4.10(1) or (2) in the case $j = 0$. ∎

We now consider those π which contain a nondegenerate representation Ω_α of L_{2i}, where α is as in (4.11).

https://doi.org/10.4153/CJM-1995-032-x Published online by Cambridge University Press

LEMMA 5.12. *Let $i \geq 1$ and assume α is as in (4.11). If $\pi \in {}^0\mathcal{E}(G)$ contains the nondegenerate representation Ω_α of L_{2i}, then Proposition 5.1(1) holds with $X_\pi = \alpha + \beta$ and $K_\pi = L$, where β is one of the following:*

(1) $\beta = 0$

(2) $\beta = \begin{pmatrix} a_1\sqrt{\varepsilon} & 0 & \varpi^{-1}b_1\sqrt{\varepsilon} \\ 0 & c\sqrt{\varepsilon} & 0 \\ \varpi b_1\sqrt{\varepsilon} & 0 & a_1\sqrt{\varepsilon} \end{pmatrix}$, *$a_1, b_1, c \in F$,*

$|a_1|, |b_1|, |c| \leq |(a_1 - c)^2 - b_1^2|^{1/2} = q^{j+1}$

(3) $\beta = \begin{pmatrix} 0 & \lambda c & 0 \\ \varpi\bar{\lambda}c & 0 & -\bar{\lambda}c \\ 0 & -\varpi\lambda c & 0 \end{pmatrix}$, *$\lambda \in E$ as in Section 3, $c \in F$ $|c| = q^{j+1}$.*

PROOF. The proof is similar to the proof of Lemma 5.8, so we omit the details. Along with α as in (4.11), Jabon and Moy also consider elements α of the form

$$\begin{pmatrix} a\sqrt{\varepsilon} & 0 & \varpi^{-1}b\sqrt{\varepsilon} \\ 0 & (a+b)\sqrt{\varepsilon} & 0 \\ \varpi b\sqrt{\varepsilon} & 0 & a\sqrt{\varepsilon} \end{pmatrix}.$$

Since such elements are conjugate by L to matrices of the form (4.11), we need only consider α as in (4.11).

The centralizer $G'' = \mathbf{H}_{an}(F)$ of α in G is compact. The element β represents a representation of G'' which is trivial on $L_{2i} \cap G''$. If this representation is trivial on $L_1 \cap G''$, then $\beta = 0$. Otherwise, β is given by Proposition 3.30 of [J].

Compactness of G'' can be used to show that for $Y \in \mathcal{N}_G$, $c(Y)$ belongs to the inducing subgroup if and only if $Y \in \mathfrak{l}_i$ (similar to Lemma 4.12). Furthermore it can be shown that

$$\int_L f_\pi\big(k^{-1}c(Y)k\big)\,dk = \begin{cases} \mathcal{J}(\alpha+\beta, Y; L) = \mathcal{J}(\alpha, Y; L), & \text{if } Y \in \mathfrak{l}_{2i}, \\ 0 & \text{if } Y \in \mathfrak{l}_i - \mathfrak{l}_{2i}. \end{cases}$$

To finish, apply Lemma 4.13. ∎

REMARK. In Lemma 5.12, we could have taken β equal to zero for all of the representations π considered. However, we chose a β which reflected the inducing data for π. This is useful for expressing coefficients in the local character expansion in terms of Shalika germs (*cf.* Corollary 6.6).

To conclude the proof of Proposition 5.1, we have the following lemma.

LEMMA 5.13. *Suppose $\pi \in {}^0\mathcal{E}(G)$ does not contain a nondegenerate representation. Choose a one-dimensional representation χ of G such that $\pi \otimes \chi$ contains a nondegenerate representation.*

(1) If $\pi \notin {}^0\mathcal{E}_u(G)$, then Proposition 5.1(1) holds with $X_\pi = X_{\pi\otimes\chi}$ and $K_\pi = K_{\pi\otimes\chi}$.

(2) If $\pi \in {}^0\mathcal{E}_u(G)$, Proposition 5.1(2) holds.

PROOF. Since χ is trivial on the unipotent subset of G, it is easy to check that $f_\pi(x) = f_{\pi\otimes\chi}(x)$ for x unipotent. ∎

6. **Main results.** Given f in $C_c^\infty(\mathfrak{g})$, the space of locally constant, compactly supported, complex-valued functions on $\mathfrak{g}$, let $\hat{f}$ in $C_c^\infty(\mathfrak{g})$ be the Fourier transform of f defined relative to the character ψ_F and the bilinear form $\langle\cdot,\cdot\rangle$. That is,

$$\hat{f}(X) = \int_{\mathfrak{g}} \psi_E\big(\mathrm{tr}(XY)\big) f(Y)\, dY,$$

where dY is a self-dual (with respect to $\hat{}$) Haar measure on $\mathfrak{g}$. Given Y in $\mathfrak{g}$, $O(Y)$ denotes the Ad G-orbit of Y. Let $\mu_{O(Y)}$ be the distribution given by integration over the orbit $O(Y)$. The Fourier transform $\hat{\mu}_{O(Y)}$ of $\mu_{O(Y)}$ is defined by $\hat{\mu}_{O(Y)}(f) = \mu_{O(Y)}(\hat{f})$, f in $C_c^\infty(\mathfrak{g})$. Recall ([HC2]) that $\hat{\mu}_{O(Y)}$ can be realized as a locally integrable function on $\mathfrak{g}$ which is locally constant on $\mathfrak{g}_{\mathrm{reg}}$. We use the same notation $\hat{\mu}_{O(Y)}$ for this function.

Let Y be a semisimple element in $\mathfrak{g}$. Choose a Cartan subgroup T such that Y belongs to the Lie algebra of T. Suppose that the stabilizer G_Y of Y in G is compact modulo the split component A of T. This is always the case if Y is regular, since $G_Y = T$. Choose an open compact subgroup K_c of G, and normalize Haar measure on K_c so that the volume of K_c equals one. Then the integral

$$\Phi(X:Y) = \int_{A\backslash G} \int_{K_c} \psi_E\Big(\mathrm{tr}\big(Y\,\mathrm{Ad}(kx)^{-1}(X)\big)\Big)\, dk\, dx$$

converges ([HC2], Lemma 18). Furthermore, if dx is normalized so as to correspond to $\mu_{O(Y)}$, then ([HC2], Lemma 19)

$$\hat{\mu}_{O(Y)}(X) = \Phi(X:Y) \tag{6.1}$$

Harish-Chandra stated the result for Y regular, but it generalizes to the situation above. Note that the centre of G is compact, so the split component of an elliptic Cartan subgroup of G is trivial.

LEMMA 6.2. *Suppose $\pi \in {}^0\mathcal{E}(G)$. Define f_π and K_π as in Section 5. Assume that Haar measure dh on K_π is normalized so that the volume of K_π is one.*

(1) Suppose $\pi \notin {}^0\mathcal{E}_u(G)$. Let X_π be defined as in Section 5. Let $X \in \mathfrak{g}_{\mathrm{reg}}$ and $y \in G_{\mathrm{reg}}$. Then

$$\Phi(X:X_\pi) = \int_G \int_{K_c} \Big[\int_{K_\pi} \psi_E\Big(\mathrm{tr}\big(X_\pi\,\mathrm{Ad}(kxh)^{-1}(X)\big)\Big)\, dh\Big]\, dk\, dx$$

(2) Let $X_{u,j}$, $j = 1, 2$ be defined as in Section 5. Then the conclusion of (1) holds with X_π replaced by $X_{u,j}$, $j = 1$ or 2.

(3) $\Theta_\pi(y) = \frac{d(\pi)}{f_\pi(1)} \int_G \int_{K_c} \Big[\int_{K_\pi} f_\pi\big((kxh)^{-1} ykxh\big)\, dh\Big]\, dk\, dx$

REMARK. The proof of Lemma 6.2 is the same as for Lemma 4.2 of [Mu2]. (3) follows from Harish-Chandra's character formula ([HC1], p. 60):

$$\Theta_\pi(y) = \frac{d(\pi)}{f_\pi(1)} \int_G \int_{K_c} f_\pi\big((kx)^{-1} ykx\big)\, dk\, dx.$$

Θ_π does not depend on a choice of measure on G. In order for (3) to hold, the formal degree $d(\pi)$ of π must be taken relative to the measure dx on G.

LEMMA 6.3. *If $x \in G$ and $X \in \mathfrak{k}_\ell$ for some $\ell \geq 1$, then* $\mathrm{Ad}\,x(X) \in \mathfrak{k}_\ell + \mathcal{N}_G$.

PROOF. This lemma is due to Howe in the case of the general linear group. For G, the proof works the same way. Given $x \in G$, $x = k_1 a k_2$ for some $k_1, k_2 \in K$, and some diagonal matrix a having diagonal entries ϖ^r, 1, and ϖ^{-r}, where r is a non-negative integer. Conjugating the element $\mathrm{Ad}\,k_2(X)$ of $\mathfrak{k}_\ell$ by a is easily seen to produce an element which is a sum of an upper triangular element in $\mathcal{N}_G$ and an element in $\mathfrak{k}_\ell$. It is now immediate that $\mathrm{Ad}\,x(X) \in \mathcal{N}_G + \mathfrak{k}_\ell$, as $\mathrm{Ad}\,K$ leaves both $\mathcal{N}_G$ and $\mathfrak{k}_\ell$ invariant. ■

For each $\pi \in {}^0\mathcal{E}(G)$, let $V_\pi = \mathfrak{k}_\ell$, where $\ell = \ell(\pi)$ is defined as follows:

(i) If π contains a nondegenerate representation, and $K_\pi = K$ or I, resp. L, choose ℓ so that $X_\pi \in \mathfrak{k}_\ell^*$, resp. $X_\pi \in \mathfrak{l}_{2\ell-2}^*$.

(ii) If $\pi \otimes \chi$ contains a nondegenerate representation, where χ is a non-trivial one-dimensional representation of G, let $\ell = \ell(\pi) = \max\{\ell(\pi\otimes\chi), m\}$, where where m is chosen so that χ is trivial on K_m.

THEOREM 6.4. *Let $\pi \in {}^0\mathcal{E}(G)$. Suppose $X \in V_\pi \cap \mathfrak{g}_{\mathrm{reg}}$.*

(1) If $\pi \notin {}^0\mathcal{E}_u(G)$, then $\Theta_\pi\big(c(-X/2)\big) = d(\pi)\hat{\mu}_{O(X_\pi)}(X)$.

(2) If $\pi \in {}^0\mathcal{E}_u(G)$, then

$$\Theta_\pi\big(c(-X/2)\big) = \{(q-1)(q+1)^2\hat{\mu}_{O(X_{u,1})}(X) - (q-1)(q^2-q+1)\hat{\mu}_{O(X_{u,2})}(X)\}/3.$$

REMARK. $c(-X/2)$ may be replaced by $\exp X$ if X is sufficiently close to zero (see the proof of Corollary 6.6).

PROOF. Assume that π contains a nondegenerate representation. Suppose that $\pi \notin {}^0\mathcal{E}_u(G)$. Then (6.1) and Lemma 6.2 imply that it suffices to show

$$\mathcal{J}\big(X_\pi, \mathrm{Ad}\,x^{-1}(X); K_\pi\big) = f_\pi(1)^{-1}\int_{K_\pi} f_\pi\Big(h^{-1}c\big(-\mathrm{Ad}\,x^{-1}(X)/2\big)h\Big)\,dh \tag{6.5}$$

for any $x \in G$. Fix $x \in G$. By Lemma 6.3, we can write $\mathrm{Ad}\,x^{-1}(X) = Y+Z$, with $Y \in \mathcal{N}_G$ and $Z \in V_\pi$. If $K_\pi = K$ or I and $h \in K_\pi$, $\mathrm{Ad}\,h^{-1}(Z) \in V_\pi$. If $K_\pi = L$ and $h \in L$, since $\mathfrak{k}_\ell \subset \mathfrak{l}_{2\ell-2}$, $\mathrm{Ad}\,h^{-1}(Z) \in \mathfrak{l}_{2\ell-1}$. Therefore,

$$\psi_E\Big(\mathrm{tr}\big(X_\pi\,\mathrm{Ad}\,h^{-1}(Y+Z)\big)\Big) = \psi_E\Big(\mathrm{tr}\big(X_\pi\,\mathrm{Ad}\,h^{-1}(Y)\big)\Big), \quad h \in K_\pi.$$

This implies that the left side of (6.5) equals $\mathcal{J}(X_\pi, Y; K_\pi)$.

Let $h \in K_\pi$. Set $Y_1 = \mathrm{Ad}\,h^{-1}(Y)$ and $Z_1 = \mathrm{Ad}\,h^{-1}(Z)$. Suppose $K_\pi = K$ or I. It can be shown that if $Y_1 \notin \mathfrak{k}_0$, then neither $c(-Y_1/2)$ or $c\big(-(Y_1+Z_1)/2\big)$ is in K, so $f_\pi\big(c(-Y_1/2)\big) = f_\pi\Big(c\big(-(Y_1+Z_1)/2\big)\Big) = 0$. If $Y_1 \in \mathfrak{k}_0$, then it is easy to see that $c(-Y_1/2) \in c(-(Y_1+Z_1)/2)K_\ell$. Also K_ℓ is in the support of f_π. Thus $f_\pi\big(c(-Y_1/2)\big) = f_\pi\Big(c\big(-(Y_1+Z_1)/2\big)\Big)$.

If $K_\pi = L$, argue as above, replacing $\mathfrak{k}_0$ by $\mathfrak{l}_0$, and K_ℓ by $L_{2\ell-2}$, to conclude that $f_\pi\Big(c\big(-(Y_1+Z_1)/2\big)\Big) = f_\pi\big(c(-Y_1/2)\big)$.

We have shown that the right side of (6.5) equals $f_\pi(1)^{-1}\int_{K_\pi} f_\pi\big(h^{-1}c(-Y/2)h\big)\,dh$. Since $Y\in\mathcal{N}_G$, by Proposition 5.1(1), this integral equals $\mathcal{J}(X_\pi, Y; K_\pi)$, which, as seen above, equals the left side of (6.5). This completes the proof of (1) in the case where π contains a nondegenerate representation.

The proof of (2) is omitted, as it is the same as the proof of (1), except that Proposition 5.1(2) is used.

Suppose $\pi\otimes\chi$, χ a one-dimensional representation of G, contains a nondegenerate representation. To prove the theorem for π, use $\Theta_\pi(x)=\chi^{-1}(x)\Theta_{\pi\otimes\chi}(x)$, $x\in G_{\mathrm{reg}}$. Note that χ is trivial on V_π. ∎

Given $O\in(\mathcal{N}_G)$, let $\Gamma_O\colon\mathfrak{g}_{\mathrm{reg}}\longrightarrow\mathbf{R}$ be the Shalika germ ([HC2]) associated to O. Then, if f is in $C_c^\infty(\mathfrak{g})$ and X is regular and sufficiently close to zero,

$$\mu_{O(X)}(f)=\sum_{O\in(\mathcal{N}_G)}\Gamma_O(X)\mu_O(f).$$

Let $c_O(\pi)$, $O\in(\mathcal{N}_G)$, be the coefficient of $\hat\mu_O$ in the local character expansion of π at the identity (Section 1).

COROLLARY 6.6. *Let $O\in(\mathcal{N}_G)$ and $\pi\in{}^0\mathcal{E}(G)$.*

(1) If $\pi\notin{}^0\mathcal{E}_u(G)$ and $X_\pi\in\mathfrak{g}_{\mathrm{reg}}$, then $c_O(\pi)=d(\pi)\Gamma_O(X_\pi)$.

(2) If $\pi\notin{}^0\mathcal{E}_u(G)$ and $X_\pi\notin\mathfrak{g}_{\mathrm{reg}}$, then $c_O(\pi)=d(\pi)\Gamma_O(X_\pi+Z)$ for any $Z\in\mathfrak{g}$ which commutes with X_π, is sufficiently close to zero, and is such that $X_\pi+Z\in\mathfrak{g}_{\mathrm{reg}}$.

(3) If $\pi\in{}^0\mathcal{E}_u(G)$, then

$$c_O(\pi)=\{(q-1)(q+1)^2\Gamma_O(X_{u,1})-(q-1)(q^2-q+1)\Gamma_O(X_{u,2})\}/3.$$

REMARKS. (a) In case (2), some twist of of π by a one-dimensional representation of G is as in Lemma 5.12, and $X_\pi=\alpha$, where α is given by (4.11). More generally, if $X_\pi=\alpha+\beta$ with α given by (4.11), it follows from the proof of (2) that $c_O(\pi)$ is independent of β.

(b) In Section 7, we will determine whether the coefficient $c_{\mathrm{reg}}(\pi)$ corresponding to the regular nilpotent orbit is nonzero. Also, in Section 8, for certain $\pi\in{}^0\mathcal{E}(G)$, the coefficients $c_O(\pi)$ will be computed for all $O\in(\mathcal{N}_G)$.

PROOF (COROLLARY 6.6). Harish-Chandra ([HC2], Lemma 21) showed that:

$$\Phi(X_1:X_2)=\sum_{O\in(\mathcal{N}_G)}\Gamma_O(X_2)\hat\mu_O(X_1)$$

for $X_1, X_2\in\mathfrak{g}_{\mathrm{reg}}$ contained in certain subsets of $\mathfrak{g}$. Assume π is as in (1). Arguing as in the proof of Theorem 4.4 of [Mu2], we see that there exists an open neighbourhood W_π of zero in $\mathfrak{g}$ such that the above relation holds with $X_2=X_\pi$, as long as $X_1=X\in W_\pi\cap\mathfrak{g}_{\mathrm{reg}}$. Applying Theorem 6.4(1) results in

$$\Theta_\pi\big(c(-X/2)\big)=d(\pi)\sum_{O\in(\mathcal{N}_G)}\Gamma_O(X_\pi)\hat\mu_O(X),\quad X\in\mathfrak{g}_{\mathrm{reg}}\cap W_\pi\cap V_\pi.$$

There exists an open neighbourhood $W'_\pi \subset V_\pi$ such that if $X \in W'_\pi \cap \mathfrak{g}_{\mathrm{reg}}$ and $x \in G$, $\mathrm{Ad}\,x^{-1}(X) = Y + Z$, with $Y \in \mathcal{N}_G$ and $Z \in W'_\pi$, exp is defined on W'_π, and $f_\pi\big(h^{-1}\exp(Y+Z)h\big) = f_\pi\big(h^{-1}(\exp Y)h\big)$ for every $h \in K_\pi$. To see this argue as in the proof of Theorem 4.3 of [Mu2]. In fact, the proof is much the same as the proof in Theorem 6.4 that $f_\pi\Big(h^{-1}c\big(-(Y+Z)/2\big)h\Big) = f_\pi\big(h^{-1}c(-Y/2)h\big)$, except that it is necessary to work on a smaller neighbourhood of zero on account of the exponential map. Since $Y \in \mathcal{N}_G$, in particular $Y^3 = 0$, $c(-Y/2) = \exp Y$. So we have

$$\begin{aligned} f_\pi\big(h^{-1}\exp(Y+Z)h\big) &= f_\pi\big(h^{-1}(\exp Y)h\big) \\ &= f_\pi\big(h^{-1}c(-Y/2)h\big) = f_\pi\Big(h^{-1}c\big(-(Y+Z)/2\big)h\Big). \end{aligned}$$

It now follows from Lemma 6.2(3) that $\Theta_\pi(\exp X) = \Theta_\pi\big(c(-X/2)\big)$ for $X \in W'_\pi \cap \mathfrak{g}_{\mathrm{reg}}$.

To finish the proof, compare the above expression for $\Theta_\pi\big(c(-X/2)\big)$ with the local character expansion of π around the identity:

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O}\in(\mathcal{N}_G)} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X),$$

$X \in \mathfrak{g}_{\mathrm{reg}}$ near zero. Note ([HC2]) that the functions $\hat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in (\mathcal{N}_G)$ are linearly independent on any open neighbourhood of zero intersected with $\mathfrak{g}_{\mathrm{reg}}$.

Suppose π is as in (2). Then $X_\pi = \alpha$, where α is given by (4.11). Suppose $[\alpha, Z] = 0$, $\alpha + Z \in \mathfrak{g}_{\mathrm{reg}}$, and $Z \in \mathfrak{l}_{-2i-1}$. Then, by Lemma 6.2(1) and Lemma 4.13,

$$\hat{\mu}_{\mathcal{O}(\alpha)} = \hat{\mu}_{\mathcal{O}(\alpha+Z)}.$$

Combining this with Proposition 5.1(1), we get

$$\Theta_\pi\big(c(-X/2)\big) = d(\pi)\hat{\mu}_{\mathcal{O}(\alpha+Z)}(X)$$

for $X \in \mathfrak{g}_{\mathrm{reg}}$ close to zero. Now proceed as for (1).

The proof of (3) is like that of (1), except that Theorem 6.4(2) is used. ∎

7. **Whittaker models.** In this section we determine which of the representations in ${}^0\mathcal{E}(G)$ have Whittaker models (Theorem 7.13 and Corollary 7.16). We begin with a few remarks about nondegenerate characters and Whittaker models. Let U be the unipotent radical of the upper triangular Borel subgroup of G. An element u of U has the form

$$u = \begin{pmatrix} 1 & B & b\sqrt{\varepsilon} - B\bar{B}/2 \\ 0 & 1 & -\bar{B} \\ 0 & 0 & 1 \end{pmatrix}, \quad B \in E,\ b \in F.$$

Given $\tau \in E$, define a character χ_τ of U by:

$$\chi_\tau(u) = \psi_E(\tau B), \quad u \in U.$$

Any linear character of U is trivial on the commutator subgroup of U and therefore is equal to χ_τ for some τ in E. As follows from the definition ([Sh, p. 191]), χ_τ is nondegenerate (or generic) if and only if τ is nonzero. This use of of the term nondegenerate is not the same as Moy's nondegenerate representations in [Mo].

A smooth admissible representation π of G has a χ_τ-Whittaker model, or is χ_τ-generic, if there exists a linear functional λ on the representation space $\mathcal{V}$ of π satisfying

$$\lambda(\pi(u)v) = \chi_\tau(u)\lambda(v), \quad u \in U,\ v \in \mathcal{V}. \tag{7.1}$$

LEMMA 7.2. *Let $\pi \in {}^0\mathcal{E}(G)$. The dimension of the space of linear functionals on the representation space of π satisfying (7.1) is either zero or one.*

PROOF. The result is stated and proved in [Sh] for irreducible unitary admissible representations of $\mathrm{GL}_n(F)$. However, as remarked in the introduction of [Sh], the result holds for quasi-split groups. Note that since the centre of G is trivial, every $\pi \in {}^0\mathcal{E}(G)$ is unitary. ∎

In the case of a general reductive group, a representation may have a Whittaker model with respect to one nondegenerate character, but not with respect to another (nonconjugate) nondegenerate character. However, this will not happen for G because all nondegenerate characters of U are conjugate by elements of the diagonal Cartan subgroup T_d of G. In fact, if $x \in T_d$ has diagonal entries τ^{-1}, 1, and $\bar\tau$, then $\chi_\tau(xux^{-1}) = \chi_1(u)$. Thus we say that π has a Whittaker model if π has a χ_τ-Whittaker model for some (hence all) $\tau \in E^\times$. Otherwise we say that π does not have a Whittaker model.

There is one regular nilpotent orbit $\mathcal{O}_{\mathrm{reg}}$ in $\mathfrak{g}$ ([R2]). The notations Γ_{reg}, μ_{reg}, and $c_{\mathrm{reg}}(\pi)$ will be used in place of $\Gamma_{\mathcal{O}}$, $\mu_{\mathcal{O}}$ and $c_{\mathcal{O}}(\pi)$ if $\mathcal{O} = \mathcal{O}_{\mathrm{reg}}$. Recall that $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in (\mathcal{N}_G)$, is the coefficient of $\hat\mu_{\mathcal{O}}$ in the local character expansion of π at the identity.

LEMMA 7.3. *Suppose $\pi \in {}^0\mathcal{E}(G) - {}^0\mathcal{E}_u(G)$ and $X_\pi \in \mathfrak{g}_{\mathrm{reg}}$. Then π has a Whittaker model if and only if $\Gamma_{\mathrm{reg}}(X_\pi) \neq 0$.*

PROOF. By Corollary I.17 of [MW], $c_{\mathrm{reg}}(\pi) \neq 0$ if and only if π has a Whittaker model. The lemma now follows from Corollary 6.6(1). ∎

We now proceed to determine whether $\Gamma_{\mathrm{reg}}(X)$ is nonzero for various X in $\mathfrak{g}_{\mathrm{reg}}$.

LEMMA 7.4. *Let $X \in \mathfrak{g}_{\mathrm{reg}}$. If $X \in \mathcal{T}_{\mathrm{unr}}$, $\mathcal{T}_{\mathrm{ram},\zeta}$, or $\mathcal{T}_{\theta,2} \cap (\mathfrak{l}_{2j+1} - \mathfrak{l}_{2j+2})$, $j \in \mathbf{Z}$, then $\Gamma_{\mathrm{reg}}(X) \neq 0$.*

PROOF. Since $\Gamma_{\mathrm{reg}}(X+Z) = \Gamma_{\mathrm{reg}}(X)$ for Z in the centre of $\mathfrak{g}$ ([HC2]), we can assume that $\operatorname{tr} X = 0$. Define $i \in \mathbf{Z}$ by $X \in \mathfrak{k}_i - \mathfrak{k}_{i+1}$.

Given $X \in \mathcal{T}_{\mathrm{unr}} \cap \mathfrak{g}_{\mathrm{reg}}$, there exist $B, C \in \mathfrak{p}_E^i$ and $b, c \in \mathfrak{p}_f^i$ such that

$$X = \begin{pmatrix} 0 & B & b\sqrt{\varepsilon} \\ C & 0 & -\bar B \\ c\sqrt{\varepsilon} & -\bar C & 0 \end{pmatrix}.$$

The image X_q of $\varpi^{-i}X$ in $\mathfrak{k}_0/\mathfrak{k}_1 \simeq \mathfrak{g}(\mathbf{F}_q)$ is regular and is contained in a degree 3 unramified extension of $\mathbf{F}_{q^2}$. If $B, C \in \mathfrak{p}_E^{i+1}$, then zero is an eigenvalue of X_q, which is impossible. After conjugating by the matrix J (J appears in the definition of G in Section 2) if necessary, we can assume that $B \in \mathfrak{p}_E^i - \mathfrak{p}_E^{i+1}$. After conjugating X by the diagonal matrix in G having diagonal entries ϖ^{-1}, 1, and ϖ, we obtain the matrix

$$\begin{pmatrix} 0 & \varpi^{-1}B & \varpi^{-2}b\sqrt{\varepsilon} \\ \varpi C & 0 & -\varpi^{-1}\bar{B} \\ \varpi^2 c\sqrt{\varepsilon} & -\varpi\bar{C} & 0 \end{pmatrix}$$

which lies in the set $\mathcal{S} = Y + \mathfrak{k}_{i+1}$, where

$$Y = \begin{pmatrix} 0 & \varpi^{-1}B & \varpi^{-2}b\sqrt{\varepsilon} \\ 0 & 0 & \varpi^{-1}\bar{B} \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $f \in C_c^\infty(\mathfrak{g})$ be the characteristic function of $\mathcal{S}$. Then $\mu_{O(X)}(f) \neq 0$. Furthermore, as f is invariant under translation by $\mathfrak{k}_{i+1}$, an unpublished result of Hales ([H]) implies that the Shalika germ expansion of f is valid on $\mathfrak{k}_i \cap \mathfrak{g}_{\mathrm{reg}}$. If $Z \in \mathcal{N}_G \cap \mathcal{S}$ then, since $Y^2 \notin \mathfrak{k}_{2i+1}$, it follows that $Z^2 \neq 0$. This implies that $O(Z)$ is the regular nilpotent orbit. Thus the germ expansion of $\mu_{O(X)}(f)$ is:

$$0 \neq \mu_{O(X)}(f) = \mu_{\mathrm{reg}}(f)\Gamma_{\mathrm{reg}}(X).$$

Therefore $\Gamma_{\mathrm{reg}}(X) \neq 0$.

The other cases are similar. ∎

In [S], Shelstad derived a formula for $\Gamma^G_{\mathrm{reg}}(x)$, for x in G_{reg}, where Γ^G_{reg} denotes the Shalika germ corresponding to the regular unipotent conjugacy class in G. A simple argument shows that if $X \in \mathfrak{g}_{\mathrm{reg}}$ is close enough to zero and $x = \exp X$, then $\Gamma_{\mathrm{reg}}(X)$ is a positive multiple of $\Gamma^G_{\mathrm{reg}}(x)$. Lemma 7.3 can be rephrased in the following way:

LEMMA 7.5. *Suppose $\pi \in {}^0\mathcal{E}(G) - {}^0\mathcal{E}_u(G)$ and $X_\pi \in \mathfrak{g}_{\mathrm{reg}}$. Then π has a Whittaker model if and only if $\Gamma^G_{\mathrm{reg}}\big(\exp(\varpi^{2m}X_\pi)\big) \neq 0$ for m sufficiently large.*

PROOF. The lemma is an immediate consequence of Lemma 7.3, the above remarks and the homogeneity property of Γ_{reg} ([HC2]). ∎

Suppose $\mathbf{T}$ is a Cartan subgroup of $\mathbf{G}$. Let $T = \mathbf{T}(F)$. The diagonal Cartan subgroup of $\mathbf{G}$ will be denoted by $\mathbf{T}_d$. Suppose $x = \exp X \in T, X \in \mathfrak{g}_{\mathrm{reg}}$ is near the identity. If α is a root of $\mathbf{T}$ in $\mathbf{G}$, define

$$\alpha(x)^{1/2} = \exp\big(\alpha(X)/2\big).$$

Let $\{a_\alpha\}$ be a-data for the action of $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ on the roots of $\mathbf{T}$, as defined in Section 2.2 of [LS]. Given α, let $\alpha^\vee$ be the corresponding co-root.

$$\sigma \mapsto \prod_{\substack{\alpha>0 \\ \sigma^{-1}\alpha<0}} \left[\frac{\alpha(x)^{1/2} - \alpha(x)^{-1/2}}{a_\alpha}\right]^{\alpha^\vee} \tag{7.6}$$

defines a 1-cocycle of Γ_F in $\mathbf{T}(F)$ ([S]) whose class in $H^1(\mathbf{T})$ will be denoted by $\mathrm{inv}(x)$.

Let $\mathrm{inv}(\mathbf{T})$ be the image in $H^1(\mathbf{T})$ of the class $\lambda(\mathbf{T}_{sc})$ defined in Section 2.3 of [LS].

THEOREM 7.7 ([S]). $\Gamma^G_{\mathrm{reg}}(x) \neq 0$ *if and only if* $\mathrm{inv}(x) = \mathrm{inv}(\mathbf{T})^{-1}$.

LEMMA 7.8. *For the given* $\mathbf{T}$ *and* h, $h\mathbf{T}(\bar{F})h^{-1} = \mathbf{T}_d(\bar{F})$. *Fix* $\tau \in E$ *such that* $2\tau\bar{\tau} = 1$.

(1) $\mathbf{T} = \mathbf{T}_{E,1}$

$$h = \begin{pmatrix} \tau & 0 & \tau \\ 0 & 1 & 0 \\ -\tau/2 & 0 & \tau/2 \end{pmatrix}$$

(2) $\mathbf{T} = \mathbf{T}_{E,2}$

$$h = \begin{pmatrix} \varpi\tau & 0 & \tau \\ 0 & 1 & 0 \\ -\varpi\tau/2 & 0 & \tau/2 \end{pmatrix}$$

(3) $\mathbf{T} = \mathbf{T}_{\theta,1}$, $\theta \in \{\varpi, \varepsilon\varpi\}$

$$h = \begin{pmatrix} \tau & 0 & \tau/\sqrt{\theta} \\ 0 & 1 & 0 \\ -\tau\sqrt{\theta} & 0 & \tau \end{pmatrix}$$

(4) $\mathbf{T} = \mathbf{T}_{\theta,2}$, $\theta \in \{\varpi, \varepsilon\varpi\}$. *Suppose* λ *is as in Section 3, that is* $\lambda\bar{\lambda} = \theta\varepsilon/2\varpi$.

$$h = \begin{pmatrix} \tau\sqrt{\varepsilon\varpi}/2\lambda & \tau & -\tau\sqrt{\varepsilon\varpi}/2\varpi\lambda \\ \sqrt{\varepsilon\varpi}/2\lambda & 0 & \sqrt{\varepsilon\varpi}/2\varpi\lambda \\ -\tau\sqrt{\varepsilon\varpi}/2\lambda & \tau & \tau\sqrt{\varepsilon\varpi}/2\varpi\lambda \end{pmatrix}$$

PROOF. In each case $\mathbf{T}(\bar{F})$ is equal to the invertible elements in the commuting algebra of $\mathcal{T}$ in the set of 3×3 matrices over $\bar{F}$. To check that $h\mathbf{T}(\bar{F})h^{-1} = \mathbf{T}_d(\bar{F})$ is straightforward. The details are omitted. ∎

Suppose $X \in \mathfrak{g}(\bar{F})$ is diagonal with diagonal entries λ_j, $1 \leq j \leq 3$. For $j = 1$ or 2, define $\alpha_j(X) = \lambda_j - \lambda_{j+1}$. Set $\alpha_3(X) = \alpha_1(X) + \alpha_2(X)$. $\{\pm\alpha_j \mid 1 \leq j \leq 3\}$ are the roots of $\mathbf{T}_d$ in $\mathbf{G}$. Let $\mathbf{B}$ be the upper triangular Borel subgroup of $\mathbf{G}$. Given this choice of Borel subgroup, α_j, $1 \leq j \leq 3$ are the positive roots. Define $X_{\alpha_j}, j = 1$ or 2, to be the matrix whose only nonzero entry is a one in the $j, j+1$ position. Let $X_{\alpha_3} = X_{\alpha_1} + X_{\alpha_2}$, and $X_{-\alpha_j} = {}^tX_{\alpha_j}$. This choice of F-splitting $(\mathbf{B}, \mathbf{T}_d, \{X_{\pm\alpha_j}\})$ of G will remain fixed throughout the section.

Given a Cartan subgroup $\mathbf{T}$ and an h in $\mathbf{G}(\bar{F})$ such that $h\mathbf{T}(\bar{F})h^{-1} = \mathbf{T}_d(\bar{F})$, for a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$, we take $h^{-1}\mathbf{B}h$. The roots of $\mathbf{T}$ in $\mathbf{G}$ will be identified, via h, with those of $\mathbf{T}_d$ in $\mathbf{G}$. h will be assumed to be as given in Lemma 7.8.

LEMMA 7.9. *The following table gives a-data for some Cartan subgroups in* $\mathbf{G}$. $a_{-\alpha_j}$ *is defined to be* $-a_{\alpha_j}$, $1 \leq j \leq 3$.

$\mathbf{T}$	a_{α_1}	a_{α_2}	a_{α_3}
$\mathbf{T}_{E,1}$	$\sqrt{\varepsilon}$	$-\sqrt{\varepsilon}$	$\sqrt{\varepsilon}^{-1}$
$\mathbf{T}_{E,2}$	$\sqrt{\varepsilon}$	$-\sqrt{\varepsilon}$	$\varpi\sqrt{\varepsilon}^{-1}$
$\mathbf{T}_{\theta,1}$	$\sqrt{\varepsilon}$	$-\sqrt{\varepsilon}$	$\sqrt{\varepsilon\varpi}^{-1}$
$\mathbf{T}_{\theta,2}$	$\sqrt{\varepsilon\varpi}$	$\sqrt{\varepsilon\varpi}$	$\sqrt{\varepsilon\varpi}^{-1}$

PROOF. In order that the given data be a-data, $a_{\sigma\alpha_j} = \sigma(a_{\alpha_j})$ must be satisfied for every $\sigma \in \Gamma_F$ ([LS]). This is straightforward. ∎

LEMMA 7.10. *Let* $\mathbf{T} \in \{\mathbf{T}_{E,1}, \mathbf{T}_{E,2}, \mathbf{T}_{\theta,1}, \mathbf{T}_{\theta,2}\}$, $\theta \in \{\varpi, \varepsilon\varpi\}$. *If* inv(**T**) *is defined relative to the F-splitting given above and the a-data in Lemma 7.9, then* inv(**T**) *is the trivial class in* $H^1(\mathbf{T})$.

PROOF. Fix **T**. Let Ω be the Weyl group of $\mathbf{T}_d$ in **G**. For each $\sigma \in \Gamma_F$, let $\sigma_T \in \Omega \cdot \Gamma_F$ be the action of σ on $\mathbf{T}_d$ which comes from transporting the action of σ on **T** to $\mathbf{T}_d$ via conjugation by h. Define $x(\sigma_T) \in \mathbf{T}_d$ by:

$$x(\sigma_T) = \prod_{\{j \mid \sigma^{-1}(\alpha_j) > 0\}} a_\alpha^{\alpha^\vee}$$

Conjugation by $h\sigma(h^{-1})$, h as in Lemma 7.8, defines an element $\omega(\sigma_T)$ of Ω. α_1 and α_2 are the simple roots of $\mathbf{T}_d$ in **G**. Using $X_{\pm\alpha_j}, j = 1, 2$, to define $n(\alpha_j)$ as in [LS], we obtain

$$n(\alpha_1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad n(\alpha_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

As in [LS], if $\omega(\sigma_T)$ is written in reduced form as a product of simple reflections corresponding to the α_j's, $j = 1, 2$, a representative $n\big(\omega(\sigma_T)\big)$ for $\omega(\sigma_T)$ is given by taking the corresponding product of $n(\alpha_j)$'s, $j = 1, 2$.

$$\sigma_T \longmapsto m(\sigma_T) = x(\sigma_T) n\big(\omega(\sigma_T)\big)$$

defines a 1-cocycle of $\{\sigma_T \mid \sigma \in \Gamma_F\}$ in $\mathbf{T}_d(\bar{F})$ ([LS]). $\lambda(\mathbf{T})$ is then given by:

$$\sigma \longmapsto h^{-1} m(\sigma_T)\sigma(h), \quad \sigma \in \Gamma_F.$$

With our choice of a-data and h, it turns out that $h^{-1}m(\sigma_T)\sigma(h) = 1$ for every $\sigma \in \Gamma_F$. Thus the 1-cocycle $\lambda(\mathbf{T})$, and hence its class inv(**T**), is trivial. We omit the details. ∎

LEMMA 7.11. *Let* $X_{E,j}$, $1 \leq j \leq 4$, *and* $X_{\theta,j}$, $j = 1, 2$, *be defined as in Section 3. Let* $m, n \in \mathbf{Z}$ *be such that* $n \geq m$.

(1) *Suppose* $a, b, c \in F$ *are such that* $|a+b-c| = |a-b-c| = q^{-m}$ *and* $|b| = q^{-n}$. *For m sufficiently large, the following are equivalent:*
   (a) inv($\exp X_{E,1}$) *is trivial*
   (b) $m + n$ *is even*
   (c) inv($\exp X_{E,2}$) *is non-trivial.*

   Furthermore, inv($\exp X_{E,3}$) *and* inv($\exp X_{E,4}$) *are non-trivial.*
(2) *Suppose* $X_{\theta,1}$ *is such that* $|b| = q^{-m}$ *and* $|a|, |c| \leq q^{-m-1}$. *If m is sufficiently large, then* inv($\exp X_{\theta,1}$) *is non-trivial.*
(3) *Suppose* $X_{\theta,1}$ *and* $X_{\theta,2}$ *are such that* $|a|, |b| \leq |a-c| = q^{-m}$. *Then, if m is sufficiently large,* inv($\exp X_{\theta,1}$) *is trivial and* inv($\exp X_{\theta,2}$) *is non-trivial.*

PROOF. Let $\mathbf{T} \in \{\mathbf{T}_{E,1}, \mathbf{T}_{E,2}, \mathbf{T}_{\theta,1}, \mathbf{T}_{\theta,2}\}$. Suppose $x = \exp X \in T \cap G_{\mathrm{reg}}$ is close to the identity and let η_x be the 1-cocycle defined by (7.6). The class inv(x) of η_x is trivial if and only if there exists $y \in \mathbf{T}(\bar{F})$ such that $\eta_x(\sigma) = y\sigma(y^{-1})$ for every $\sigma \in \Gamma_F$. By setting $t = hyh^{-1} \in \mathbf{T}_d(\bar{F})$ and taking σ_T as in the proof of Lemma 7.10, we see that this is equivalent to

$$(7.12) \qquad h\eta_x(\sigma)h^{-1} = th\sigma(h^{-1})\sigma(t^{-1})\sigma(h)h^{-1} = t\sigma_T(t^{-1}), \quad \sigma \in \Gamma_F.$$

If $\sigma \in \Gamma_F$ is such that $\sigma(\alpha_j) < 0$, $1 \le j \le 3$ set $\gamma_j = \gamma_j(x) = \left(\alpha_j(x)^{1/2} - \alpha_j(x)^{-1/2}\right)/a_{\alpha_j}$, $1 \le j \le 3$. Otherwise, $\sigma(\alpha_j) > 0$ for $1 \le j \le 3$, and we take $\gamma_j = 1$.

$$h\eta_x(\sigma)h^{-1} = \operatorname{diag}(\gamma_1\gamma_3, \gamma_1^{-1}\gamma_2, \gamma_2^{-1}\gamma_3^{-1}), \quad \sigma \in \Gamma_F,$$

where $\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ denotes the 3×3 diagonal matrix with diagonal entries λ_j, $1 \le j \le 3$. Throughout the proof, we assume that $t \in \mathbf{T}_d(\bar{F})$ is of the form $\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_j \in \bar{F}$.

(1) Since $\mathbf{T}_{E,1}$ and $\mathbf{T}_{E,2}$ split over E, it suffices to determine whether there exists a $t \in \mathbf{T}_d(E)$ satisfying (7.12). Let σ_ε denote the nontrivial element of $\operatorname{Gal}(E/F)$. If $t \in \mathbf{T}_d(E)$, a simple calculation shows that, for both $T = T_{E,1}$ and $T_{E,2}$,

$$t\sigma_{\varepsilon,T}(t^{-1}) = \operatorname{diag}(\lambda_1\bar{\lambda}_1, \lambda_2\bar{\lambda}_2, \lambda_3\bar{\lambda}_3).$$

To determine whether $\operatorname{inv}(\exp X_{i,4})$ is trivial, we must determine whether there exist $\lambda_j \in E^\times$, $1 \le j \le 3$ such that

$$\lambda_1\bar{\lambda}_1 = \gamma_1\gamma_3, \quad \lambda_2\bar{\lambda}_2 = \gamma_1^{-1}\gamma_2, \quad \lambda_3\bar{\lambda}_3 = \gamma_2^{-1}\gamma_3^{-1}.$$

Note that $\gamma_j(\exp X_{E,i}) \in F$, $1 \le j \le 3$, $1 \le i \le 4$. $\mathrm{N}_{E/F}(E^\times)$ consists of the set of elements in $F^\times$ of even valuation. In the case of $X_{E,1}$, for large m, $|\gamma_1^{-1}\gamma_2| = 1$ and $|\gamma_1\gamma_3| = |\gamma_2\gamma_3| = q^{-m-n}$ The equivalence of (a) and (b) is now clear. The case of $X_{E,2}$ is similar, except, due to the different a-data, $|\gamma_1\gamma_3| = |\gamma_2\gamma_3| = q^{-m-n-1}$, which implies the equivalence of (b) and (c). In the cases of $X_{E,3}$ and $X_{E,4}$, $|\gamma_2\gamma_3| = q^{-2m-1}$, and so $\gamma_2\gamma_3 \notin \mathrm{N}_{E/F}(E^\times)$.

The details for (2) and (3) are omitted. ∎

THEOREM 7.13. *Let $\pi \in {}^0\mathcal{E}(G) - {}^0\mathcal{E}_u(G)$. Let a, b, $c \in F$.*

(1) If $X_\pi \in \mathcal{T}_{\mathrm{unr}}$ or $\mathcal{T}_{\mathrm{ram},\zeta}$, then π has a Whittaker model.

(2) Suppose $|a+b-c| = |a-b+c| = q^{i+1}$ and $|b| = q^{j+1}$ for integers $i \ge j \ge 0$. If X_π is equal to $X_{E,1}$, resp. $X_{E,2}$, π has a Whittaker model if and only if $i+j$ is even, resp. odd. If $X_\pi = X_{E,3}$ or $X_{E,4}$, π does not have a Whittaker model.

(3) Suppose $|b| = q^{i+1}$ and $|a|$, $|c| \le q^{i-1}$, $i \ge 1$. If $X_\pi = X_{\theta,1}$, resp. $X_{\theta,2}$, then π does not, resp. does, have a Whittaker model.

(4) Suppose $|a|$, $|b| \le |a-c| = q^{i+1}$, $i \ge 1$. If $X_\pi = X_{\theta,1}$, resp. $X_{\theta,2}$, then π does, resp. does not, have a Whittaker model.

(5) If $X_\pi \notin \mathfrak{g}_{\mathrm{reg}}$, then π does not have a Whittaker model.

PROOF. Part (1) and the case $X_{\theta,2}$ in part (2) follow from Lemmas 7.3 and 7.4. For the other cases where $X_\pi \in \mathfrak{g}_{\mathrm{reg}}$, Lemma 7.5, Theorem 7.7, and Lemma 7.10 are combined with the appropriate part of Lemma 7.11. If $X_\pi \notin \mathfrak{g}_{\mathrm{reg}}$, then let Z be an element which commutes with X_π, is close to zero, and is such that $X_\pi + Z \in \mathfrak{g}_{\mathrm{reg}}$. We can choose Z so that $X_\pi + Z = X_{E,3}$ or $X_{E,4}$ as in (2), or else $X_\pi + Z = X_{\theta,2}$ as in (4). By Corollary I.17 of [MW] and Corollary 6.6(2), π does not have a Whittaker model, since $\Gamma_{\mathrm{reg}}(X_\pi + Z) = 0$. ■

REMARK 7.14. It can be seen from the definitions of the various X_π's in Section 5 that every $\pi \in {}^0\mathcal{E}(G) - {}^0\mathcal{E}_u(G)$ has an X_π which appears in Theorem 7.13. In fact,

(a) If π contains a nondegenerate representation of K or of L, then $X_\pi \in \mathcal{T}_{\mathrm{unr}}$ or $X_\pi \in \{X_{E,m} \mid 1 \leq m \leq 4\}, i = j = 0$.

(b) If π contains a nondegenerate representation Ω_α of K_i, $i \geq 1$, with $\varpi^{i+1}\alpha$ having regular elliptic image in $\mathfrak{g}(\mathbf{F}_q)$, then $X_\pi \in \mathcal{T}_{\mathrm{unr}} \cap \mathfrak{g}_{\mathrm{reg}}$ or $X_\pi = X_{E,1}$, $(i = j)$.

(c) If π contains a nondegenerate representation Ω_α of K_i, $i \geq 1$, with α as in (4.8), then $X_\pi \in \{X_{E,1}, X_{E,2}, X_{\theta,1}$ (as in (4))$\}$

(d) If π contains a nondegenerate representation Ω_α of L_{2i}, $i \geq 1$, α as in (4.11), then

$$X_\pi \in \{\alpha, X_{E,3}, X_{E,4}, X_{\theta,2} \text{ (as in (4))}\},$$

and π does not have a Whittaker model.

(e) If π is not as in one of (a)–(d), and π contains a nondegenerate representation Ω_α such that $X_\pi = \alpha \in \mathfrak{g}_{\mathrm{reg}}$ (see Lemma 5.6), then

$$X_\pi \in \{X_{E,m}, 2 \leq m \leq 4, (i = j), X_{\theta,r}, r = 1,2 \text{ (as in (3))}\}.$$

Recall that representatives for the Ad G-orbits within stable orbits of regular elements are given in Section 3. In part(1) of Theorem 7.13, the X_π's considered have the property that their their Ad G-orbits are stable orbits. In each of parts (2)–(4), the X_π's are representatives for the Ad G-orbits within a stable orbit which contains more than one Ad G-orbit. Suppose $\mathcal{S}$ is a finite subset of ${}^0\mathcal{E}(G) - {}^0\mathcal{E}_u(G)$ having the property that $\{X_\pi \mid \pi \in \mathcal{S}\}$ is a set of representatives for the Ad G-orbits within the stable orbit of a regular element (with each Ad G-orbit in the stable orbit represented once). Theorem 7.13 implies that exactly one of the representations in $\mathcal{S}$ has a Whittaker model, and that representation can be identified by the corresponding X_π. In fact, it can be seen from the inducing data for the representations in ${}^0\mathcal{E}(G)$, that, given elements as in Theorem 7.13(1)–(4), such sets $\mathcal{S}$ exist. However, since inequivalent representations may have the same X_π, they are not uniquely determined. Rogawski ([R2]) has defined a partition of the representations of G into sets called *L-packets*. We expect, although it is not proved here, that if an L-packet consists entirely of supercuspidal representations, then every π in the L-packet is in ${}^0\mathcal{E}(G) - {}^0\mathcal{E}_u(G)$, and X_π is regular. Furthermore, the set of X_π's corresponding to the representations in the L-packet should consist of representatives for one stable orbit.

In Section 8, the coefficients $c_\mathcal{O}(\pi)$, $\mathcal{O} \in (\mathcal{N}_G)$, will be computed for certain $\pi \in {}^0\mathcal{E}(G)$, including $\pi \in {}^0\mathcal{E}_u(G)$. As a consequence we will obtain the following result.

COROLLARY 7.15. *If $\pi \in {}^0\mathcal{E}_u(G)$, π does not have a Whittaker model.*

8. **Evaluation of coefficients.** We conclude the paper by computing the coefficients $c_O(\pi)$, $O \in (\mathcal{N}_G)$, for π belonging to the following families of representations:

$$\mathcal{F}_1 = \{\pi \in {}^0\mathcal{E}(G) \mid X_\pi \in \mathcal{T}_{\mathrm{unr}}\}$$
$$\mathcal{F}_2 = \{\pi \in {}^0\mathcal{E}(G) \mid X_\pi = X_{E,1} \text{ such that } |a - c \pm b| = |b|\}$$
$$\mathcal{F}_3 = {}^0\mathcal{E}_u(G)$$

LEMMA 8.1. *Assume that the measure of K in G is one.*
(1) For $\pi \in \mathcal{F}_1$ such that $X_\pi \in \mathfrak{k}_{-i-1} - \mathfrak{k}_{-i}$, $i \geq 1$, $d(\pi) = q^{3i}(q-1)(q+1)^2$.
(2) For $\pi \in \mathcal{F}_2$ such that $X_\pi \in \mathfrak{k}_{-i-1} - \mathfrak{k}_{-i}$, $d(\pi) = q^{3i}(q-1)(q^2-q+1)$.
(3) For $\pi \in \mathcal{F}_3$, $d(\pi) = q(q-1)$.

PROOF. Jabon ([J]) computed $d(\pi)$ for all $\pi \in {}^0\mathcal{E}(G)$. If $\pi \in \mathcal{F}_1$ or $\mathcal{F}_2$ and $i = 0$, then some twist of π by a one-dimensional representation of G contains a nondegenerate representation of K. If $i \geq 1$, then some twist of π contains the nondegenerate representation Ω_α of K_i, where $\alpha = X_\pi$. If $\pi \in \mathcal{F}_3$, then π is a twist of π_u. The formal degrees may be read off the table on p. 66 of [J]. ∎

Let O_{reg} be the regular nilpotent $\mathrm{Ad}\, G$-orbit. The two other nontrivial nilpotent orbits, O_1 and O_ϖ, are represented by ([R2] Section 3.9)

$$X_1 = \begin{pmatrix} 0 & 0 & \sqrt{\varepsilon} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_\varpi = \begin{pmatrix} 0 & 0 & \varpi\sqrt{\varepsilon} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\pi \in {}^0\mathcal{E}(G)$, let $c_{\mathrm{reg}}(\pi)$, $c_1(\pi)$, $c_\varpi(\pi)$ and $c_0(\pi)$, be the coefficients in the local character expansion, corresponding to O_{reg}, O_1, O_ϖ, and the trivial nilpotent orbit $O_0 = \{0\}$. The notation Γ_{reg}, Γ_1, Γ_ϖ and Γ_0 will be used for the Shalika germs associated to the nilpotent orbits.

To find the values of $c_O(\pi)$, we will compute $\Gamma_O(X_\pi)$ and then apply Corollary 6.6 and Lemma 8.1. The next lemma gives a normalization of measure on each $O \in (\mathcal{N}_G)$.

LEMMA 8.2. *Let dt be Haar measure on F normalized so that O_F has volume one. If $f \in C_c^\infty(\mathfrak{g})$, let $f_K \in C_c^\infty(\mathfrak{g})$ be defined by $f_K(X) = \int_K f(\mathrm{Ad}\, k^{-1}(X))\, dk$, where dk is normalized so that K has volume one. For each $O \in (\mathcal{N}_G)$, the distribution defined below is an* Ad *G-invariant measure on O.*
(1) $O = O_0$; $\mu_0(f) = f(0)$
(2) $O = O_1$; $\mu_1(f) = q^{-3}(q^3+1) \int_{\mathrm{N}_{E/F}(E^\times)} |t| f_K(tX_1)\, dt$
(3) $O = O_\varpi$; $\mu_\varpi(f) = q^{-5}(q^3+1) \int_{N_{E/F}(E^\times)} |t| f_K(tX_\varpi)\, dt$
(4) $O = O_{\mathrm{reg}}$; Let $\mathfrak{n}$ be the subalgebra of strictly upper triangular matrices in $\mathfrak{g}$. Assume Haar measure dX on $\mathfrak{n}$ is normalized so that $\mathfrak{n} \cap \mathfrak{k}_0$ has volume one.

$$\mu_{\mathrm{reg}}(f) = q^{-3}(q^3+1) \int_{\mathfrak{n}} f_K(X)\, dX$$

PROOF. In each case, Ranga Rao's formula for $\mu_O(f)$ ([RR]) is seen to be a positive multiple of the given formula. ∎

REMARK. By Corollary I.17 of [MW], there exists a positive constant a depending on normalizations of measures such that, if π is an irreducible admissible representation of G, $a^{-1}c_{\mathrm{reg}}(\pi)$ equals the dimension of the space of linear functionals satisfying (7.1). We claim that with the above normalizations of the Fourier transform and of μ_{reg}, $a = 1$. Choose a one-dimensional representation ν of a Borel subgroup B of G which is trivial on the unipotent radical of B and such that the representation $\mathrm{Ind}_B^G \nu$ is irreducible. Let Θ_ν denote the character of the representation $\mathrm{Ind}_B^G \nu$. then $\hat{\mu}_{\mathrm{reg}} = \Theta_\nu \circ \exp$ on some neighbourhood of zero. (This can be seen by the argument used for Lemma 5.1 of [Mu2].) By (9), p. 444 of [MW], $\mathrm{Ind}_B^G \nu$ has a Whittaker model and the corresponding space of linear functionals has dimension one. This implies that $a = 1$. It now follows from from Corollary 6.6(1) and Lemma 7.2 that if $\pi \in {}^0\mathcal{E}(G) - {}^0\mathcal{E}_u(G)$ has a Whittaker model and $X_\pi \in \mathfrak{g}_{\mathrm{reg}}$, then $\Gamma_{\mathrm{reg}}(X_\pi) = d(\pi)^{-1}$.

If t is a nonzero element of F and X is regular, $\Gamma_O(tX)$ can be expressed as a multiple of $\Gamma_{tO}(X)$. (Here tO denotes the nilpotent orbit obtained from O by multiplying the elements of O by t.)

LEMMA 8.3. *Let $X \in \mathfrak{g}_{\mathrm{reg}}$. Suppose $t \in F^\times$. If the valuation of t is even, then $\Gamma_O(tX) = |t|^{-\dim O/2}\Gamma_O(X)$, $O \in (\mathcal{N}_G)$. If the valuation of t is odd and if μ_1 and μ_ϖ are normalized as in Lemma 8.2,*

(1) $\Gamma_1(tX) = |t|^{-2}\Gamma_\varpi(X)$

(2) $\Gamma_\varpi(tX) = |t|^{-2}\Gamma_1(X)$

(3) $\Gamma_{\mathrm{reg}}(tX) = |t|^{-3}\Gamma_{\mathrm{reg}}(X)$

PROOF. The case of t a square in F is the standard homogeneity property of Shalika germs ([HC2]). Let f_0, f_1, ϕ_0 and ϕ_1 be the characteristic functions of $\mathfrak{k}_0$, $\mathfrak{k}_1$, $\mathfrak{i}_0$ and $\mathfrak{i}_3$, respectively.

For $t = \varepsilon$, a comparison of the Shalika germ expansions of f at $\varpi^{2j}X$ and $\varpi^{2j}\varepsilon X$, for $f = f_0$ and ϕ_0, for j sufficiently large, and an application of the standard homogeneity property yields the desired result.

For $t = \varpi$, the proof involves a comparison of the Shalika germ expansion of f_0 at $\varpi^{2j+1}X$ with that of f_1 at $\varpi^{2j}X$, and similarly for ϕ_0 and ϕ_1. The details are omitted. ∎

THEOREM 8.4. *Assume μ_O, $O \in (\mathcal{N}_G)$, is normalized as in Lemma 8.2. The values of the various $c_O(\pi)$'s for $\pi \in \mathcal{F}_j$ are given in the j^{th} row of the table below, $1 \leq j \leq 3$. If $\pi \in \mathcal{F}_1$ or $\mathcal{F}_2$, let $i \geq 0$ be such that and $X_\pi \in \mathfrak{k}_{-i-1} - \mathfrak{k}_{-i}$.*

$c_0(\pi)$	$c_1(\pi)$	$c_\varpi(\pi)$	$c_{\mathrm{reg}}(\pi)$
$-\frac{(q+1)^2}{q^2+1}q^{3i}$	$(-1)^{i-1}q^i$	$(-1)^i q^i$	1
$-\frac{(q^2-q+1)}{q^2+1}q^{3i}$	$\frac{((-1)^i-3)}{2}q^i$	$\frac{((-1)^{i-1}-3)}{2}q^i$	1
$\frac{-q}{q^2+1}$	0	1	0

PROOF. Let $X \in \{\varpi X_{u,j} \mid j = 1,2\}$, where $X_{u,j}$ is defined as in Lemma 5.3. Let f_0 and ϕ_0 be the characteristic functions of $\mathfrak{k}_0$ and $\mathfrak{i}_0$, respectively. Note that $X \in \mathfrak{k}_0 - \mathfrak{k}_1$. By Proposition 7.1 of [Ko], for $x \in G$,

$$x^{-1}c(X)x \in K \Longleftrightarrow x \in K.$$

This is easily seen to be equivalent to

$$\operatorname{Ad} x^{-1}(X) \in \mathfrak{k}_0 \Longleftrightarrow x \in K.$$

Thus $\mu_{O(X)}(f_0)$ equals the volume of K in G, which equals one. If $k \in K$, because the image of $\operatorname{Ad} k^{-1}(X)$ in $\mathfrak{g}(\mathbf{F}_q)$ is regular and elliptic, it cannot lie in the Borel subalgebra of $\mathfrak{g}(\mathbf{F}_q)$. Thus $\operatorname{Ad} k^{-1}(X) \notin \mathfrak{i}_0$. Therefore $O(X)\cap\mathfrak{i}_0 = \operatorname{Ad} K(X)\cap\mathfrak{i}_0 = \emptyset$. An unpublished result of Hales ([H] implies that the Shalika germ expansions of f_0 and ϕ_0 hold on $\mathfrak{k}_0\cap\mathfrak{g}_{\mathrm{reg}}$ (because these functions are invariant under translation by $\mathfrak{i}_0$). Evaluation of $\mu_O(f_0)$ and $\mu_O(\phi_0)$, $O \in (\mathcal{N}_G)$, We find that $\mu_{O(X)}(f_0) = 1$ is equivalent to

(8.5)

$$1 = -(q-1)^{-1}(q^2+1)^{-1} + (q^2-q+1)(q^2+1)^{-1}\big(\Gamma_1(X) + q^{-2}\Gamma_\varpi(X)\big) + q^{-3}(q^3+1)\Gamma_{\mathrm{reg}}(X).$$

Also, $\mu_{O(X)}(\phi_0) = 0$ is equivalent to

(8.6)

$$0 = -(q-1)^{-1}(q^2+1)^{-1} + q^{-1}(q^2+1)^{-1}\Gamma_1(X) + q^{-2}(q^2-q+1)(q^2+1)^{-1}\Gamma_\varpi(X) + 2q^{-3}\Gamma_{\mathrm{reg}}(X).$$

Here we have used Rogawski's formula ([R1]) $\Gamma_0(X) = -d(St_G)^{-1}$. The formal degree of the Steinberg (or special) representation St_G is ((1.9) of [Mo])

$$d(St_G) = (q-1)(q^2+1)(q^3+1)^{-1}\mathrm{volume}_G(I)^{-1} = (q-1)(q^2+1).$$

By Lemma 5.3, if $\pi \in \mathcal{F}_j, j = 1,2$, is such that $X_\pi \in \mathfrak{k}_{-1} - \mathfrak{k}_0$, then $X_\pi = X_{u,j}$. By Theorem 7.13, the remark following Lemma 8.2, $\Gamma_{\mathrm{reg}}(X_{u,j}) = d(\pi)^{-1}$, and this value is given by Lemma 8.1. By Lemma 8.3(3), $\Gamma_{\mathrm{reg}}(X) = q^3\Gamma_{\mathrm{reg}}(\varpi^{-1}X) = q^3\Gamma_{\mathrm{reg}}(X_{u,j})$. Substituting the value of $\Gamma_{\mathrm{reg}}(X)$ into (8.5) and (8.6) we can solve for $\Gamma_1(X)$ and $\Gamma_\varpi(X)$.

For $\pi \in \mathcal{F}_j, j = 1, 2$, Lemma 8.3 can be applied to obtain $\Gamma_O(X_\pi)$ from $\Gamma_O(\varpi X_{u,j})$, $O \in (\mathcal{N}_G)$, and $d(\pi)$ is given in Lemma 8.1. By Corollary 6.6(1) the coefficients $c_O(\pi)$ are as given in the table. For $\pi \in \mathcal{F}_3$, apply Corollary 6.6(3). ∎

REMARKS. (a) The analogue of Theorem 8.4 was proved for $\mathbf{GSp}_4(F)$ in Theorem 8.3 of [Mu1], by different methods.

(b) The choice of additive character used in the Fourier transform has an effect on the $\hat{\mu}_O$'s. For example, suppose ψ_E is replaced by ψ' defined by $\psi'(x) = \psi_E(\varpi x), x \in E$. Then $\hat{\mu}_1$, resp. $\hat{\mu}_\varpi$, defined using ψ_E, becomes $q^{-2}\hat{\mu}'_\varpi$, resp. $q^{-2}\hat{\mu}'_1$, defined using ψ'. For the trivial and regular nilpotent orbits, changing the character has the effect of multiplying $\hat{\mu}_O$ by a positive constant.

(c) If $\pi \in \mathcal{F}_3$, since π does not have a (nondegenerate) Whittaker model, it follows from Corollary I.17 of [MW] that π admits a degenerate Whittaker model relative to the orbit O_ϖ and a one-parameter subgroup defined as in [MW].

References

[C] H. Carayol, *Représentations cuspidales du groupe linéaire*, Ann. Sci. École Norm. Sup. (4) **17**(1984), 191–225.

[DL] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. Math. **103**(1976), 103–161.

[E] V. Ennola, *On the characters of the finite unitary groups*, Ann. Acad. Sci. Fenn. **323**(1963), 1–35.

[H] T. Hales, *unpublished results*.

[HC1] Harish–Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Math. **162**, Springer-Verlag, Berlin, 1970.

[HC2] ———, *Admissible distributions on reductive p-adic groups, in Lie Theories and Their Applications*, Queen's Papers in Pure and Appl. Math. **48**(1978), 281–347.

[J] D. Jabon, *The supercuspidal representations of* $U(2,1)$ *and* GSp_4 *over a local field via Hecke algebra isomorphisms*, thesis, University of Chicago, 1989.

[K] D. Kazhdan, *Proof of Springer's hypothesis*, Israel J. Math. **28**(1977), 272–286.

[Ko] R. Kottwitz, *Stable Trace Formula: Elliptic Singular Terms*, Math. Ann. **275**(1986), 365–399.

[LS] R. P. Langlands and D. Shelstad, *On the Definition of Transfer Factors*, Math. Ann. **278**(1987), 219–271.

[L] G. Lusztig, *Irreducible representations of finite classical groups*, Invent. Math. **43**(1977), 125–175.

[MW] C. Mœglin and J. L. Waldspurger, *Modèles de Whittaker dégénérés pour des groupes p-adiques*, Math. Z. **196**(1987), 427–452.

[M1] L. Morris, *Tamely ramified supercuspidal representations of classical groups I. Filtrations*, Ann. Sci. École Norm. Sup. (4) **24**(1991), 705–738.

[M2] ———, *Tamely ramified supercuspidal representations of classical groups II. Representations*, Ann. Sci. École Norm. Sup. (4) **25**(1992), 233–274.

[Mo] A. Moy, *Representations of* $U(2,1)$ *over a p-adic field*, J. Reine Angew. Math. **372**(1986), 178–208.

[Mu1] F. Murnaghan, *1 Asymptotic behaviour of supercuspidal characters of p-adic* GSp_4, Compositio Math. **80**(1991), 15–54.

[Mu2] ———, *Local character expansions and Shalika germs for* GL_n, Math. Ann., to appear.

[Mu3] ———, *Characters of supercuspidal representations of* SL_n, Pacific J. Math., to appear.

[Mu4] ———, *Characters of supercuspidal representations of classical groups*, Ann. Sci. École Norm. Sup. to appear.

[RR] R. Ranga Rao, *Orbital integrals in reductive groups*, Ann. of Math. **96**(1972), 505–510.

[R1] J. Rogawski, *An application of the building to orbital integrals*, Compositio Math. **42**(1981), 417–423.

[R2] ———, *Automorphic Representations of Unitary Groups in Three Variables*, Ann. of Math. Stud. **123**, Princeton University Press 1990.

[Sh] J. A. Shalika, *Multiplicity one theorem for* GL_n, Ann. of Math. **100**(1974), 171–193.

[S] D. Shelstad, *A formula for regular unipotent germs*, Astérisque **171–172**(1989), 275–277.

Department of Mathematics
University of Toronto
Toronto, Ontario
M5S 1A1