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Maximal perfect spaces

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Let (X, T) be a topological space (we assume T_1 throughout) where every point is a limit point. The purpose of this note is to present an internal construction of a maximal perfect topology on (X, T). The existence of a maximal connected Hausdorff space has not been demonstrated. However, this construction of a maximal perfect topology is useful in constructing connected Hausdorff spaces which cannot be embedded in a maximal connected Hausdorff space.

Let (X, T) be a topological space in which every point is a limit point, then (X, T) is said to be perfect. (Throughout this note, all spaces are assumed to be T_1.) A topological space (X, T) is maximal perfect if (X, T) is perfect and for every topology $T' \supsetneq T$, (X, T') is not perfect. It follows from an application of Zorn's Lemma that, if (X, T) is a perfect topological space, there exists a topology $T' \supset T$ such that (X, T') is maximal perfect. The main aim of this note is to present an internal construction of a maximal perfect topology for any given perfect topological space. The advantage of this construction arises when one wishes to know which sets are open in a maximal perfect topology. This maximal perfect topology is constructed by using N-sets (see Definition 1) and a particular filter, F, of dense subsets of (X, T). It is also shown that, if $\sigma \supset T \vee F$ and (X, σ) is perfect, then, if $G \in T \vee F$, $G = F \cap M$, where $F \in F$ and M is an N-set.

The problem of the existence of a maximal perfect topology arose

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naturally from the consideration of the existence of a maximal connected Hausdorff space. A connected Hausdorff space (X, T) is maximal connected if for every topology $T' \supsetneq T$, (X, T') is not connected (see [2]). The author used the type of particular construction of a maximal perfect topology, as presented in this note, to construct an example of a countable connected Hausdorff space (X, T) which is not first countable at any point and such T is not contained in any maximal connected topology on X [1].

If (X, T) is a topological space, then we will say a set $K \subset X$ is T-dense in X if X is contained in the closure of K in the T topology.

Also, K is T-open if $K \in T$, and K^c will be used to denote the complement of K.

For the remainder of this note, we will assume that (X, T) is an arbitrary but fixed perfect topological space. Let

$$H = \{F \subset X \mid F^c \text{ is nowhere dense in } X\} .$$

Then, H is a filter of dense subsets of X. It follows from Zorn's Lemma that there exists a maximal filter $\mathcal{F}$, containing H, such that for every $F \in \mathcal{F}$, F is a dense subset of X. Throughout, we will use $\mathcal{F}$ to denote this maximal filter on X.

DEFINITION 1. A set $G \subset X$ is an N-set if for every $x \in G$ and every open set $U \in T$ and containing x, $U \cap G$ contains a T-open set.

It follows, immediately, from the definition, that if $U \in T$, then U is an N-set. However, there may exist N-sets which are not T-open; for example, if G is the subset of the real line defined as follows:

$$G = \{0\} \cup \left\{\left(\frac{1}{2n}, \frac{1}{2n-1}\right) \mid n = 1, 2, \dots\right\} .$$

DEFINITION 2. A collection of N-sets, $\mathcal{G}$, is an N-family if $\mathcal{G}$ is closed under finite intersection.

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two N-families, we will say that $\mathcal{G}_1 \leq \mathcal{G}_2$ if for every $G \in \mathcal{G}_1$, $G \in \mathcal{G}_2$. It follows from Zorn's Lemma that given any

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N-family G, there exists a maximal N-family M, such that $G \leq M$. In what follows, M will be used to denote a maximal N-family on X.

If A and B are two families of subsets of X, by $A \vee B$ is meant the topology which is generated by the subbase

$$P = \{C \mid C \in A \text{ or } C \in B\} .$$

LEMMA 1. *Let* $T' = F \vee M$; *then* $T' \supset T$ *and* (X, T') *is perfect.*

Proof. Since $T \subset M$, it follows that $T \subset T'$. Suppose (X, T') is not perfect. This implies that there exists some $x \in X$ such that $\{x\} \in T \vee F$. Therefore, $\{x\} = F \cap G$, where $F \in F$ and $G \in M$. However, since G contains a non-empty T-open set U and $F \cap U$ is T-dense in U, it follows that $F \cap G \neq \{x\}$; a contradiction. So, (X, T') is perfect.

Henceforth, T' will be used to denote the topology $F \vee M$. Let us assume there exists a topology T^* on X, such that (X, T^*) is perfect and $T^* \supsetneq T'$.

LEMMA 2. *If* $P \in T^*$, $P \subset U$, *where* $U \in T$, *and* P *is* T-*dense in* U, *then* $P \in T \vee F$.

Proof. Suppose $P \subset U$ and P is T-dense in $U \in T$ and $P \notin T \vee F$. If, for each $F \in F$, $P \cap F$ is T-dense in U, then $F \cap (P \cup U^c)$ is dense in X for all $F \in F$. So, since F is maximal, this would imply that $P \cup U^c \in F$. However, if $P \cup U^c \in F$, then $P = (P \cup U^c) \cap U$ and $P \in T \vee F$. If, on the other hand, there exists some $F \in F$ such that $F \cap P$ is not T-dense in U, then there exists some T-open set $V \subset U$ such that $V \cap (F \cap P) = \emptyset$. Let $x \in V$ such that $x \in P$ and $x \notin F$. Put $F_1 = F \cup \{x\}$. Then $F_1 \in F$ and is, therefore, open in $T \vee F$. This would imply that $F_1 \cap (P \cap V) = \{x\}$ and since $F_1 \cap P \in T^*$, this implies $\{x\}$ is open in T^*. However, since T^* is perfect, this is impossible. Therefore, for each $F \in F$, $F \cap P$ is T-dense in U, and so $P \in T \vee F$.

COROLLARY. *If* $P \in T^*$, $P \subset U$, *where* $U \in T$ *and* P *is* T-*dense in* U, *then* $P \in T'$.

DEFINITION 3. Let $P \in T^*$. P has property β if for every $x \in P$ there exists a T^*-open set $U_x \subset P$ and containing x such that

$U_x = \left(\bigcup_{\alpha \in A} K_\alpha \right) \cup L$, where

(a) for each $\alpha \in A$, $K_\alpha \subset U_\alpha \in T$ and K_α is T-dense in U_α ,

(b) no subset of L is T-dense in any T-open set, and

(c) if $x \in L$, then $x \notin K_\alpha$, for any α .

LEMMA 3. *Let P be a set with property β such that $P \in T^*$. Let $x \in P$ and $U_x = \left(\bigcup_{\alpha \in A} K_\alpha \right) \cup L$ be as in Definition 3. Then, for every $y \in L \cap U_x$ and for every T'-open set U containing x , $U \cap U_\alpha \neq \emptyset$, for some $\alpha \in A$.*

Proof. Suppose there exists some $x \in P$ and some $y \in L \cap U_x$ and a T'-open set U containing y such that $U \cap U_\alpha = \emptyset$, for all $\alpha \in A$. We may assume without loss of generality that $U = F \cap (I \cap M)$, where $F \in \mathcal{F}$, $I \in T$ and $M \in \mathcal{M}$. Since M is an N-set, $M \cap I$ contains a non-empty T-open set. Let Q be any non-empty T-open set contained in $I \cap M$. Then, $Q \cap U_\alpha = \emptyset$, for all $\alpha \in A$; for, if there exists some $\alpha \in A$, such that $Q \cap U_\alpha \neq \emptyset$, and since $Q \subset (I \cap M)$, this implies that $U \cap U_\alpha \neq \emptyset$. Also $(I \cap M) \cap K_\alpha = \emptyset$, for all $\alpha \in A$. If there exists an $x \in (I \cap M) \cap K_\alpha$, for some $\alpha \in A$, then, since $K_\alpha \subset U_\alpha$ and $I \cap M$ is an N-set, if U' is an T-open set containing x , $U' \cap (I \cap M) \cap U_\alpha$ would contain a T-open set. This would contradict the fact that $U_\alpha \cap Q = \emptyset$, for each open set $Q \subset (I \cap M)$. Therefore, $(I \cap M) \cap K_\alpha = \emptyset$, for all $\alpha \in A$.

$y \in U \cap U_x$ and T^* is perfect, so there exists some $p \in U \cap U_x$ and $p \neq y$. Since $U \cap U_\alpha = \emptyset$, for all α and $K_\alpha \subset U_\alpha$, $p \notin K_\alpha$, for any $\alpha \in A$. Therefore, $p \in L$. Let V be any T-open set containing p . Put $H = V \cap (I \cap M)$. Since $p \in U$ and $U = F \cap (I \cap M)$, it follows that

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$p \in H$. Also, since $(I \cap M) \cap K_\alpha = \emptyset$, for all $\alpha \in A$, $L \supset H \cap P$. From the definition of L , it follows that $H \cap P$ is nowhere dense in (X, T) . So, $(H \cap P)^c \in F$ and it is clear from the definition of F that $(H \cup P)^c \cup \{p\}$ is also an element of F . But, this says that $|(H \cap P)^c \cup \{p\}| \cap (H \cap P) = \{p\}$. This implies that $\{p\} \in T^*$, which contradicts the assumption that T^* is perfect. Therefore, for each $y \in U_x$ and for each T'-open set U containing y , there exists some $\alpha \in A$, such that $U \cap U_x \neq \emptyset$.

LEMMA 4. *If* $P \in T^*$ *and* P *has property* β *, then* $P \in T'$.

Proof. Let $x \in P$ and let $U_x \subset P$ be a T^*-open set such that $U_x = \left\{ \bigcup_{\alpha \in A} K_\alpha \right\} \cup L$, where the K_α's and L satisfy conditions (a) through (c) of Definition 3. We will show that for every $y \in U_x$, there exists a T' open set U , containing y , such that $U \subset U_x$.

For each $y \in L$, put $U(y) = \left\{ \bigcup_{\alpha \in A} U_\alpha \right\} \cup \{y\}$. It follows from Lemma 3, that for each $y \in L$, $U\{y\}$ is an N-set since for every $p \in U(y)$ and every T-open set U containing p , $U \cap U_\alpha \neq \emptyset$ for some α .

We will show that $U(y) \in M$. Suppose not. Then, since M is a maximal N-family, there must exist some $M \in M$, such that $M \cap U(y)$ is not an N-set. This implies there exists some T-open set V containing y such that $V \cap (M \cap U(y))$ does not contain a T-open set. It follows from the definition of an N-set, and the fact that $\{U(y)-\{y\}\}$ is T-open that $\{U(y)-\{y\}\} \cap M \cap V = \emptyset$. Therefore, $V \cap (M \cap U_x) \subset L$ since $U_x = \left\{ \bigcup_{\alpha \in A} U_\alpha \right\} \cup L$. Put $H = V \cap (M \cap U_x)$, then $H \in T^*$. Since L is nowhere dense in (X, T) , H must also be nowhere dense in (X, T) .

Therefore, $H^c \in F$, since F contains each set whose complement is nowhere dense in (X, T) . This implies that $(H^c \cap \{y\}) \cap H = \{y\} \in T^*$, which contradicts the assumption that T^* is perfect. Therefore, $U(y) \cap M$ must be an element of M , for all $M \in M$, and, so, $U(y) \in M$.

This implies that $U(y) \in T'$.

Since $\bigcup_{\alpha \in A} K_\alpha$ is T-dense in $\bigcup_{\alpha \in A} U_\alpha$, it follows from Lemma 2 that $\bigcup_{\alpha \in A} K_\alpha = F \cap \left(\bigcup_{\alpha \in A} U_\alpha\right)$, for some $F \in F$. Therefore, there exists an $F_1 \in F$ (namely, $F_1 = F \cup \{y\}$), such that $F_1 \cap U(y) = \{y\} \cup \left(\bigcup_{\alpha \in A} K_\alpha\right)$. Since both F_1 and $U(y)$ are open in T' , it follows that $U = \{y\} \cup \left(\bigcup_{\alpha \in A} K_\alpha\right)$ is open in T' and clearly, $U \subset U_x$. Therefore, for every $y \in U_x$, there exists a T'-open set U containing y and contained in U_x . This implies that U_x and, consequently P , is an element of T' .

THEOREM 1. *(X, T') is maximal perfect.*

Proof. Let T^* be any perfect topology on X such that $T^* \supset T'$. Let $x \in X$ and let U_x be any T^*-open set containing x . For each $y \in U_x$, let K_α be a subset of U_x , containing y , such that K_α is T-dense in some $U_\alpha \in T$ and $K_\alpha \subset U_\alpha$, if such a K_α exists. Otherwise, let $L = \{y \in U_x \mid \text{no such } K_\alpha \text{ exists}\}$. Clearly $U_x = \left\{\bigcup_{\alpha \ A} K_\alpha\right\} \cup L$, and, therefore, by Lemma 4, $U_x \in T'$ and T' is maximal perfect.

In the preceding, we constructed a maximal perfect topology T^* on a perfect topological space (X, T) by adding to T a particular maximal filter F of dense subsets of (X, T) and a maximal N-family M . We will now show that if σ is any perfect topology on X , larger than $T \vee F$, then σ can be generated by T, F and a family of N-sets. On the other hand, if γ is a perfect topology, larger than $T \vee M$, we will give an example to show that γ may not be generated by T, M and a family of dense subsets of (X, T) .

Let (X, T) be a perfect topological space and let F be a maximal family of dense subset of (X, T) , such that, if H is a nowhere dense subset of (X, T) , then $H^c \in F$.

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THEOREM 2. *Let* σ *be any perfect topology on* X *such that* $\sigma \subset T \vee F$. *If* $G \in \sigma$ *and* $G \notin T \vee F$ *, then* $G = M \cap F$ *, where* M *is an* N*-set and* $F \in F$.

Before presenting the proof of Theorem 2, we present a lemma which will be helpful in completing the proof of this theorem. The symbols used in Lemma 5 have the same meaning as in Theorem 2.

LEMMA 5. *If* V *is any* T*-open subset of* X *and* $G \cap V \neq \emptyset$ *, then* $G \cap V$ *is* T*-dense in some* T*-open subset of* V .

Proof. Suppose $G \cap V$ is not T-dense in any T-open subset of V . Then, $G \cap V$ is nowhere dense in (X, T) . This implies that $(G \cap V)^c \in F$. Let $y \in G \cap V$. Then $(G \cap V)^c \cup \{y\} \in F$ and since $\{y\} = [(G \cap V)^c \cup \{y\}] \cap (G \cap V)$, this implies that $\{y\} \in \sigma$, and contradicts the fact that (X, σ) is perfect. Therefore, $G \cap V$ is T-dense in some T-open subset of V .

Proof of Theorem 2. Put $G = \left\{\bigcup_{\alpha \in A} K_\alpha\right\} \cup L$, where for each $\alpha \in A$, $K_\alpha \subset U_\alpha$, U_α is T-open and K_α is T-dense in U_α ; and no subset of L is T-dense in any T-open set. Put $M = \left\{\bigcup_{\alpha \in A} U_\alpha\right\} \cup L$. We will show that M is an N-set. Let $x \in M$. If $x \in U_\alpha$ for some α , then clearly, for every T-open set U containing x , $U \cap U_\alpha$ is a T-open set. If $x \in L$ and U is a T-open set containing x , then it follows from Lemma 5, that $U \cap U_\alpha \neq \emptyset$, for some α , so, again, $U \cap U_\alpha$ is a T-open set and is contained in M . Therefore, M is an N-set.

Finally, we will show that for some $F \in F$, $F \cap M = G$. It follows from Lemma 2 that there exists some $F_1 \in F$, such that $F_1 \cap \left\{\bigcup_\alpha U_\alpha\right\} = G \cap \left\{\bigcup_{\alpha \in A} U_\alpha\right\}$. Put $F = F_1 \cup \{x \mid x \in L\}$. Then $F \cap M = G$ and the theorem is established.

We now give an example of a maximal N-family and a perfect topology γ on X , such that γ is larger than $T \vee M$ but γ cannot be generated by T, M and a family of dense subsets of (X, T) .

EXAMPLE. Let X be the closed unit interval and let T be the usual topology on X. Let K be the Cantor subset of X and let $C_n = (a_n, b_n)$, for $n = 1, 2, \ldots$, be the open intervals which are removed to form the Cantor set. Put $R_n = (0, a_n]$, and $G_n = [b_n, 1)$, for $n = 1, 2, \ldots$. Then for each n, R_n and G_n is an N-set. Let M be a maximal N-family which contains R_n and G_n, for all n. Put $\gamma = (T \vee M) \vee K$. Then, γ is perfect and K is a γ-open set. It is easily seen that $K \neq M \cap F$, where F is T-dense in X and $M \in M$.

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