UNITARY PERFECT NUMBERS

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1. Introduction. Let $\sigma^*(N)$ denote the sum of the unitary divisors of N, that is,

$$\sigma^*(N) = \sum_{\substack{d \mid N \\ (d,\, N/d) = 1}} d .$$

It is easily seen that $\sigma^*(N)$ is multiplicative. In fact $\sigma^*(1) = 1$ $\sigma^*(N) = (1 + p_1^{\alpha_1}) \dots (1 + p_r^{\alpha_r})$ if $N > 1$ has the prime decomposition $N = p_1^{\alpha_1} \dots p_r^{\alpha_r}$. Let us define a positive integer to be unitary perfect whenever $\sigma^*(N) = 2N$. The first four such numbers are 6, 60, 90 and 87,360. In a recent abstract [1] published by one of us, the last of these numbers was overlooked. No other unitary perfect numbers are known to the authors.

It would appear from some of the results to follow that the next unitary perfect number, if it exists, must indeed be quite large. It might seem reasonable to conjecture that there is no unitary perfect number larger than 87,360. When the problem of the determination of all unitary perfect numbers was mentioned to P. Erdos, he expressed the opinion that it might be a difficult one, comparable to the problem of odd perfect numbers. We present here a partial solution.

2. Notation. In all that follows, unless otherwise specified, n represents an odd integer larger than 1, and N an even integer given by $N = 2^m n$ with m a positive integer.

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We further assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ with $p_1 < p_2 < \ldots < p_r$ and the p_i odd primes, and the α_i positive integers. We also write $n = n_1 n_2 n_3$ with the n_i relatively prime in pairs, every prime divisor of n_1 is congruent to 1 modulo 4, every prime divisor of n_2 is congruent to 3 modulo 4 and occurs with an even exponent, and the prime divisors of n_3 are each congruent to 3 modulo 4 but occur with odd exponent. For any fixed n, let a, b, and c denote the number of distinct primes in n_1, n_2 and n_3 respectively. For given non-negative integers a, b, and c not all zero, the class of all odd numbers $n = n_1 n_2 n_3$ associated with a, b, and c will be denoted by K(a, b, c).

It is trivial to observe that if $n = n_1 n_2 n_3$ and $n' = n'_1 n'_2 n'_3$ are both members of K(a, b, c), then

(2.1) $$\sigma^*(n)/n \geq \sigma^*(n')/n'$$

whenever $n'_1 \geq n_1$, $n'_2 \geq n_2$, and $n'_3 \geq n_3$.

Define

(2.2) $$B(a, b, c) = \max \{ \sigma^*(x)/x \},$$

$$x \in K(a, b, c), \quad x \text{ not square-free if } b = c = 0.$$

The reason for x not square-free if b = c = 0 will be apparent after Lemma 2 of section 3. Finally notice

(2.3) $B(a, b, c) \geq B(a', b', c')$ if $a \geq a'$, $b \geq b'$, and $c \geq c'$.

3. Some Lemmas and Theorems. We proceed to prove several results for unitary perfect numbers. For a prime p, we write $p^t \| x$ to mean, as usual, that $p^t | x$ and $p^{t+1} \nmid x$.

THEOREM 1. There are no odd unitary perfect numbers.

Proof. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ is odd and unitary perfect, then $2 \| (1 + p_1^{\alpha_1})(1 + p_2^{\alpha_2}) \ldots (1 + p_r^{\alpha_r})$. Hence r = 1 and $1 + p_1^{\alpha_1} = 2p_1^{\alpha_1}$, which is not possible.

LEMMA 1. If $N = 2^m n$ is unitary perfect, then:

(3.1) $p_r \mid (2^m + 1)$ if $\alpha_1 = \alpha_2 = \ldots \alpha_{r-1} = 1$;

(3.2) $a + b + 2c \leq m + 1$ and equality holds when $c = 0$;

(3.3) $B(a, b, c) \geq 2^{m+1} / (2^m + 1)$ for at least one set of values of a, b, and c satisfying (3.2).

Proof. The proof of (3.1) is trivial. To show (3.2), for any prime $p \equiv 1 \pmod 4$ and all positive integers k, we have $2 \parallel (1 + p^k)$. This also holds when $p \equiv 3 \pmod 4$ provided k is even. But if $p \equiv 3 \pmod 4$ and k is odd, we have $4 \mid (1 + p^k)$ and indeed $2^t \parallel (1 + p^k)$ if and only if $2^t \parallel (1 + p)$. Applying these remarks to the relation

$$(3.4) \qquad 2^{m+1} n = (1 + 2^m)(1 + p_1^{\alpha_1}) \ldots (1 + p_r^{\alpha_r}),$$

which holds if $N = 2^m n$ is unitary perfect, provides us with (3.3).

From (3.4) we have

$$2^{m+1} (2^m + 1) = \sigma^*(n) / n$$

from which (3.3) follows by applying the definition of B(a, b, c).

Remark. Result (3.2) can be sharpened as follows:

$$(3.6) \qquad a + b + \sum_i i\, C_i = m + 1,$$

where C_i is the number of prime divisors p of n of the form $p \equiv 3 \pmod 4$ where exponents in the prime factorization are odd, and i is given by $2^i \parallel (1 + p)$.

LEMMA 2. If $N = 2^m n$ is unitary perfect, and $3 \nmid N$, then:

(3.7) m is an even integer;

(3.8) if $p^\alpha \parallel n$, then $p^\alpha \equiv 1 \pmod 6$;

(3.9) there is a prime p such that $p \mid n$, $p \equiv 5 \pmod 6$, and p occurs with an even exponent in n;

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(3.10) n has an even number of distinct primes.

Proof. The first two, (3.7) and (3.8), are quite trivial. To obtain (3.9), notice that with m even, $1 + 2^m \equiv 5 \pmod 6$. Hence there is a prime $p | n$ such that $p \equiv 5 \pmod 6$. From (3.8) it follows that p must occur to an even exponent. From (3.4), (3.7), and (3.8)

$$2^{m+1} n \equiv 2 \equiv 2^{r+1} \pmod 3)$$

and therefore r must be even which proves (3.10).

Notice that (3.9) justifies the remark after (2.2). The authors have not been able to find any unitary perfect numbers not divisible by 3 nor have they been able to prove that there are none.

THEOREM 2. Let $N = 2^m p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ be unitary perfect.

(1) If $r = 1$, then $N = 6$.

(2) If $m = 1$, then $N = 6$ or 90.

(3) If $m = 2$, then $N = 60$.

(4) If $r = 2$, then $N = 60$ or 90.

(5) It is not possible for $m = 3, 4, 5$, or 7.

(6) It is not possible for $r = 3$ or 5.

(7) If $m = 6$, then $N = 87,360$.

(8) If $r = 4$, then $N = 87,360$.

Proof. The basic tools are Lemmas 1 and 2. Let us illustrate the procedure for a few selected cases.

Suppose that $r = 1$, then by Lemma 2 the one odd prime must be 3. Thus $2^{m+1} 3^\alpha = (1 + 2^m)(1 + 3^\alpha)$ and therefore $1 + 3^\alpha = 2^{m+1}$ and $1 + 2^m = 3^\alpha$ which quickly forces $m = \alpha = 1$.

Next, if $m = 1$ then 3 must once more divide N. Hence not both b and c can be zero. We have two cases: (1) $a = b = 0$ and $c = 1$ or (2) $a = b = 1$ and $c = 0$. In the first case $r = 1$ and so $N = 6$. In case (2), $4 \times 3^\alpha \times p^\beta = 3(3^\alpha + 1)(p^\beta + 1)$ with α even. Thus $3 | (p^\beta + 1)$

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which implies that $p \equiv 2 \pmod 3$ and β is odd. Since it is known that $p \equiv 1 \pmod 4$, it follows that $p \equiv 5 \pmod{12}$. If $p^\beta \geq 17$, then we apply (2.1) to obtain

$$4 = 3(3^\alpha + 1)(p^\beta + 1)/3^\alpha p^\beta \leq 3 \times 10 \times 18/9 \times 17$$

which is a false statement. Therefore $p^\beta = 5$ and $4 \times 3^\alpha \times 5 = 3(3^\alpha + 1) 6$ which forces $\alpha = 2$.

The situation for $m = 6$ is somewhat more complicated. Since $2^6 + 1 = 5 \times 13$, it is apparent that $a \geq 2$. We have the following cases:

(1) $c = 0$ and $a + b = 7$; (2) $c = 1$ and $a + b \leq 5$; (3) $c = 2$ and $a + b \leq 3$.

By direct computation $B(a, 7-a, 0) < 2^7/(2^6 + 1)$ for $2 \leq a \leq 7$. Thus there are no unitary perfect numbers with $a + b = 7$ and $c = 0$. In case (2), $B(a, b, 1)$ is also less than $2^7/(2^6 + 1)$ which excludes case (2) from consideration. The computations are simplified because

$$B(2, 3, 1) \geq B(2, 2, 1) \geq B(2, 1, 1) \geq B(2, 0, 1)$$

and

$$B(3, 2, 1) \geq B(3, 1, 1) \geq B(3, 0, 1) \text{ by (2.3).}$$

In case (3), $B(2, 1, 2) < 2^7/(2^6 + 1)$.

Suppose that N is a unitary perfect number which occurs in $K(3, 0, 2)$. Set $N = 2^6 p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} p_5^{\alpha_5}$. Since 5 is an odd integer, $p_1 = 3$. Also 5 and 13 occur in N. The prime 7 cannot occur in N because $2^3 | 7 + 1$ and $3 + 2 + 3 > 7$ which violates (3.6). Assume $\alpha_1 > 1$; then $\alpha_1 \geq 3$, and using (2.1)

$$2^7 \leq 5 \times 13 \times 28 \times 6 \times 14 \times 18 \times 12/27 \times 5 \times 13 \times 17 \times 11,$$

which is false. Therefore $\alpha_1 = 1$. Now $p_2 = 5$ and the assumption that $\alpha_2 > 1$ leads to a similar contradiction as does the assumption that $p_3^{\alpha_3} > 11$. Hence

$$2N = 2^7 \times 3 \times 5 \times 11 \times 13 \times p_5^{\alpha_5} = 5 \times 13 \times 4 \times 6 \times 12 \times 14 \times (p_5^{\alpha_5} + 1).$$

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But $3 \| 2N$ and $3^2 \| \sigma^*(N)$. Thus there are no unitary perfect numbers in K(3, 0, 2). We are now left with a consideration of K(2, 0, 2). Suppose that $N = 2^6 p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4}$ is a unitary perfect number in K(2, 0, 2). Assume $3 \nmid N$; then with the aid of Lemma 2 and (2.1)

$$2^7 \le 5 \times 13 \times 26 \times 14 \times 8 \times 20 / 25 \times 13 \times 7 \times 19$$

which is false. Therefore $p_1 = 3$, $p_2 = 5$, and either p_3 or p_4 is 13. Now $\alpha_1 > 1$ leads to a contradiction as does $\alpha_2 > 1$. If $p_3^{\alpha_3} \ge 13$, then

$$2^7 \le 5 \times 13 \times 4 \times 6 \times 14 \times 20 / 3 \times 5 \times 13 \times 19$$

which is not correct. Therefore $p_3^{\alpha_3} = 7$ or 11, but $p_3^{\alpha_3} = 11$ not possible by (3.6). Thus

$$2^7 \times 3 \times 5 \times 7 \times 13^\beta = 5 \times 13\,(1 + 3)(1 + 5)(1 + 7)(1 + 13^\beta)$$

and

$$2 \times 7 \times 13^{\beta - 1} = 1 + 13^\beta$$

which implies that $\beta = 1$. Therefore $2^6 \times 3 \times 5 \times 7 \times 13$ is the only member of K(2, 0, 2) which can be unitary perfect. It is unitary perfect.

THEOREM 3. Let m be a fixed positive integer. There is at most a finite number of unitary perfect numbers N such that $2^m \| N$.

Proof. Suppose that there are infinitely many such unitary perfect numbers. There is an infinite subset of such unitary perfect numbers of the form $2^m C D_i$ with C a constant, $(C, D_i) = 1$, and D_i is composed of a fixed number of distinct primes. Furthermore each prime power of D_i is increasing. Thus $\lim \sigma^*(D_i)/D_i = 1$. If $C = 1$, then $\lim \sigma^*(D_i)/D_i = \lim 2^{m+1}/2^m + 1$ and $2^{m+1} = 2^m + 1$, which is not possible. If $C > 1$ then $2^{m+1} C = (1 + 2^m)\,\sigma^*(C)$ and $2^{m+1} \| \sigma^*(C)$. For any D_i, $2 \mid \sigma^*(D_i)$, and so

$2^{m+2} \mid \sigma^*(CD_i)$. But $2^{m+1} \| \sigma^*(CD_i)$. In any case there is at most a finite number of such unitary perfect numbers.

THEOREM 4. There is at most a finite number of unitary perfect numbers with a fixed number of primes.

Proof. Assume there are infinitely many such unitary perfect numbers. There is an infinite subset of the numbers of the form CD_i with C an odd constant, $(C, D_i) = 1$, and the prime powers of D_i increasing. As before $\lim \sigma^*(D_i)/D_i = 1$. Then if $C = 1$, $2 = \lim \sigma^*(D_i)/D_i = 1$, a contradiction. If $C > 0$, $2C = \lim \sigma^*(C)\,\sigma^*(D_i)/D_i = \sigma^*(C)$ and C is an odd unitary perfect number, violating Theorem 1.

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