1

Elements of Operator Theory

In this chapter, we collect some material from the theory of self-adjoint operators. While the main focus of this book is on the specific cases of Laplace and Schrödinger operators, already the definition of these operators and the formulation of the questions we ask requires language from operator theory. At the same time, this theory provides the foundations of the spectral analysis of the operators of interest.

One of our main goals in this chapter is to prove the variational principle, which will be the essential tool in our proofs of spectral inequalities. It transforms the problem of counting eigenvalues into an optimization problem for quadratic forms. In this principle, and therefore in all our presentation, quadratic forms will play a prominent role, for the most part even more than the underlying operator.

Operator theory also provides a framework for perturbation theory, which considers operators given, for instance, as a sum of an 'unperturbed' operator, which is in some sense well understood, and a perturbation, which is in some sense small. This point of view will be particularly relevant when dealing with Schrödinger operators.

Let us now give a brief overview of the content of this chapter. We begin by recalling some basic facts from Hilbert space theory and, in particular, the notion of the spectrum of an operator and that of self-adjointness of an operator. Next, in §1.1.6, we state without proof the spectral theorem for self-adjoint operators and discuss the resulting functional calculus. As a consequence of the spectral theorem, we prove Weyl's theorem (Theorem 1.14) on the stability of the essential spectrum.

The second section is devoted to quadratic forms. In §1.2.1 we present the fundamental connection between lower semibounded self-adjoint operators and lower semibounded closed quadratic forms. In our applications to Laplace and Schrödinger operators, we use this connection to define the relevant operators.

In §§1.2.3 and 1.2.5 we formulate the variational principle for eigenvalues and their sums, which are naturally formulated in the language of quadratic forms and, as we already mentioned, are fundamental for the developments in the following chapters.

In §1.2.6 we discuss Riesz means, which will play an important role in the study of Lieb–Thirring inequalities. As an application of the variational principle, we prove various continuity results that are frequently used in applications.

Finally, in §§1.2.7 and 1.2.8 we discuss perturbations of operators in terms of quadratic forms and the Birman–Schwinger principle. The latter translates the problem of counting negative eigenvalues of unbounded operators to a related problem for compact operators.

While we try to be rather self-contained concerning the material on the theory of unbounded operators, it is probably advantageous if the reader has had some previous exposure to a standard course on functional analysis and measure theory; see, for instance, Brezis (2011), Friedman (1970), Folland (1999), Lax (2002), and Rudin (1991). For detailed expositions of operator theory and spectral analysis, we refer to the references in §1.3.

1.1 Hilbert spaces, self-adjoint operators and the spectral theorem

In this section we briefly recall the theory of self-adjoint operators in Hilbert spaces and we use this opportunity to fix our notation.

1.1.1 Bounded operators

Let $\mathcal{V}$ be a complex vector space together with an inner product $(\cdot,\cdot)$; that is, a complex-valued function on $\mathcal{V} \times \mathcal{V}$ satisfying

$$
\begin{aligned}
&(f,g) = \overline{(g,f)} && \text{for all } f,g \in \mathcal{V}\,,\\
&(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 (f_1,g) + \alpha_2 (f_2,g) && \text{for all } \alpha_1,\alpha_2 \in \mathbb{C},\ f_1,f_2,g \in \mathcal{V}\,,\\
&(f,f) > 0 && \text{for all } 0 \neq f \in \mathcal{V}\,.
\end{aligned}
$$

Here and in all the following, our sesquilinear forms are linear in the first and anti-linear in the second argument.

If $(\cdot,\cdot)$ is an inner product on $\mathcal{V}$, then

$$\|f\| := \sqrt{(f,f)}$$

defines a norm on $\mathcal{V}$. The vector space $\mathcal{V}$, together with its inner product, is

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called a *Hilbert space* if it is complete with respect to the corresponding norm. It is called *separable* if it has a countable orthonormal basis.

From now on, let $\mathcal{H}$ be a separable Hilbert space.

We say that a sequence $(f_n) \subset \mathcal{H}$ *converges* to $f \in \mathcal{H}$ if $\|f_n - f\| \to 0$ as $n \to \infty$. We say that $(f_n) \subset \mathcal{H}$ *converges weakly* to $f \in \mathcal{H}$ if, for every $g \in \mathcal{H}$, $(f_n, g) \to (f, g)$ as $n \to \infty$. If we want to emphasize the difference between convergence (in norm) and weak convergence, we also call the former *strong convergence*.

The following facts about weak convergence are standard results from functional analysis. First, if $(f_n) \subset \mathcal{H}$ is weakly convergent, then $(\|f_n\|)$ is bounded. Second, the closed unit ball in $\mathcal{H}$ is weakly sequentially compact; that is, if $(\|f_n\|)$ is bounded, then there is a subsequence (f_{n_m}) that converges weakly to some $f \in \mathcal{H}$.

Another standard result from functional analysis is the Riesz representation theorem. It states that, if ℓ is a bounded, linear functional on $\mathcal{H}$, then there is a unique $g \in \mathcal{H}$ such that $\ell(f) = (f, g)$ for all $f \in \mathcal{H}$. Moreover, $\|g\| = \|\ell\|$.

Next, we discuss operators on $\mathcal{H}$. A continuous, linear map $T : \mathcal{H} \to \mathcal{H}$ is called a *bounded* (linear) operator. This name comes from the fact that a linear map T is continuous if and only if

$$\|T\| := \sup_{\|f\|=1} \|Tf\| < \infty \,.$$

The set of bounded operators is complete with respect to the above norm.

By the Riesz representation theorem, for any bounded operator T on $\mathcal{H}$ one can define a unique bounded operator T^* on $\mathcal{H}$, called the *adjoint of* T, such that

$$(T^* f, g) = (f, Tg) \qquad \text{for all } f, g \in \mathcal{H} \,.$$

This implies, in particular, that

$$\|T^*\| = \|T\|$$

and that, if $f_n \to f$ weakly in $\mathcal{H}$, then $Tf_n \to Tf$ weakly in $\mathcal{H}$.

An operator K on a Hilbert space $\mathcal{H}$ is called *compact* if the image of the closed unit ball in $\mathcal{H}$ is relatively compact; that is, if any sequence (f_n) with $\|f_n\| \leq 1$ has a subsequence (f_{n_m}) such that (Kf_{n_m}) converges. Clearly, compact operators are bounded and the product of a compact operator with a bounded operator is compact.

The following lemma characterizes compactness in terms of weak convergence.

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Lemma 1.1 *A bounded operator K is compact if and only if it transforms every weakly convergent sequence into a strongly convergent sequence.*

Proof First, assume that K is compact and that $f_n \to 0$ weakly. (We may assume, without loss of generality, that the weak limit is zero.) Then, as recalled above, $Kf_n \to 0$ weakly. Moreover, as also mentioned before, $\sup \|f_n\| < \infty$. We choose a subsequence (f_{n_m}) such that $\lim_{m\to\infty} \|Kf_{n_m}\| = \limsup_{n\to\infty} \|Kf_n\| =: a$. Since (f_n) is bounded, the compactness of K implies that (Kf_{n_m}) is relatively compact and, therefore, there is a $g \in \mathcal{H}$ and a further subsequence $(f_{n_{m_l}})$ such that $Kf_{n_{m_l}} \to g$ strongly in $\mathcal{H}$ as $l \to \infty$. Since $Kf_n \to 0$ weakly, we have $g = 0$ and therefore $\|Kf_{n_{m_l}}\| \to 0$ as $l \to \infty$. Thus $a = 0$ which means that $Kf_n \to 0$ strongly as $n \to \infty$.

To prove the converse implication, let (f_n) be a sequence with $\|f_n\| \leq 1$. Then, by the weak sequential compactness of the unit ball, there is a subsequence (f_{n_m}) that converges weakly. By assumption, (Kf_{n_m}) converges strongly. This shows that the image of the closed unit ball is relatively compact, as claimed. □

Lemma 1.2 *Let K be a bounded operator. Then K is compact if and only if K^* is compact if and only if K^*K is compact.*

Proof The product of a bounded and a compact operator is compact, so if K is compact, then so is K^*K. Conversely, assume that K^*K is compact. To show that K is compact, let (f_n) be a sequence converging weakly to zero. By Lemma 1.1, (K^*Kf_n) converges strongly to zero, so, since (f_n) is bounded, $\|Kf_n\|^2 = (K^*Kf_n, f_n) \leq \|K^*Kf_n\|\|f_n\| \to 0$. By Lemma 1.1 again, this means that K is compact.

Now assume that K^* is compact. Then K^*K is compact as a product of a compact and a bounded operator, and therefore, by what we have just shown, K is compact. Applying this to $K = (K^*)^*$, we see that compactness of K implies that of K^*. □

Let us conclude this subsection by discussing modes of convergence for operators. Let (T_n) be a sequence of bounded operators on $\mathcal{H}$ and let T be a bounded operator on $\mathcal{H}$. We say that T is the *weak limit* of (T_n) and write

$$T = \text{w-lim}_{n\to\infty} T_n \text{ if } \lim_{n\to\infty} (T_n f, g) = (Tf, g) \qquad \text{for all } f, g \in \mathcal{H}\,.$$

We say that T is the *strong limit* of (T_n) and write

$$T = \text{s-lim}_{n\to\infty} T_n \text{ if } \lim_{n\to\infty} \|(T_n - T)f\| = 0 \qquad \text{for all } f \in \mathcal{H}\,.$$

Finally, we say that T is the *norm limit* of (T_n) and write $T = \lim_{n\to\infty} T_n$ if

$$\lim_{n\to\infty} \|T_n - T\| = 0\,.$$

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Lemma 1.3 *Let (K_n) be a sequence of compact operators and let K be a bounded operator such that $\lim_{n\to\infty} K_n = K$. Then K is compact.*

Proof Let (f_n) be a sequence that converges weakly to zero. According to Lemma 1.1, we need to prove that (Kf_n) tends strongly to zero. For any n, m we have the bound

$$\|Kf_n\| \leq \|K_m f_n\| + \|K_m - K\| \|f_n\| \, .$$

Since K_m is compact, we have $K_m f_n \to 0$ strongly as $n \to \infty$. Thus,

$$\limsup_{n\to\infty} \|Kf_n\| \leq \|K_m - K\| \sup_n \|f_n\| \, .$$

The supremum on the right side is finite, as recalled before. Letting $m \to \infty$, we obtain $\limsup_{n\to\infty} \|Kf_n\| = 0$, which proves the claimed convergence. □

Lemma 1.4 *Let K be a compact operator, let T and S be bounded operators and let $(T_n), (S_n)$ be sequences of bounded operators with $T = \text{s-}\lim_{n\to\infty} T_n$ and $S = \text{s-}\lim_{n\to\infty} S_n$. Then $\lim_{n\to\infty} T_n K S_n^* = TKS^*$.*

Proof *Step 1.* We first prove the assertion in the case where $S_n = S$ for all n and we abbreviate $L := KS^*$. Assume the stated convergence would not hold. Then there are $\varepsilon > 0$ and, passing to a subsequence if necessary, $f_n \in \mathcal{H}$ such that $\|f_n\| = 1$ and $\|(T_n - T)Lf_n\| \geq \varepsilon$. By weak compactness, passing to another subsequence if necessary, we may assume that $f_n \to f$ weakly for some f. Since L is compact as the product of a bounded and a compact operator, we infer from Lemma 1.1 that $Lf_n \to Lf$ strongly. Writing

$$\begin{aligned}
\|(T_n - T)Lf_n\| &\leq \|(T_n - T)Lf\| + \|(T_n - T)L(f_n - f)\| \\
&\leq \|(T_n - T)Lf\| + (\|T_n\| + \|T\|)\|Lf_n - Lf\| \, ,
\end{aligned}$$

we see that the first term on the right side tends to zero by strong convergence of T_n and the second term tends to zero since $\|T_n\|$ remains bounded by the uniform boundedness principle (see, for instance, Friedman, 1970, Theorem 4.5.1). Thus, $\|(T_n - T)Lf_n\| \to 0$, a contradiction.

Step 2. We now prove the assertion in the case where $T_n = T$ for all n and we abbreviate $M := TK$. Since M is compact as the product of a bounded and a compact operator, we infer from Lemma 1.2 that M^* is compact. Therefore, Step 1 implies that $S_n M^* \to SM^*$ in norm. Since the norm of the adjoint equals the norm of the operator itself, this implies that $MS_n^* \to MS^*$ in operator norm, which proves the assertion in this case.

Step 3. To prove the assertion in the general case, we write

$$T_n K S_n^* - TKS^* = (T_n - T)KS_n^* + TK(S_n^* - S^*) \, .$$

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The first term on the right side tends to zero in operator norm by Step 1 (applied with S replaced by the identity) and the fact that $\|S_n^*\| = \|S_n\|$ is uniformly bounded, and the second term tends to zero in operator norm by Step 2. This concludes the proof of the proposition. □

1.1.2 Unbounded operators

We now extend the notion of a bounded linear operator to that of a not necessarily bounded operator, often called an *unbounded* operator. In the following, an operator in $\mathcal{H}$ is, for us, a linear map T from its domain $\operatorname{dom} T$, a subspace of $\mathcal{H}$, to $\mathcal{H}$. (We emphasize that in this book subspaces are *not* necessarily assumed to be closed.) In particular, two operators T and S coincide, by definition, if $\operatorname{dom} S = \operatorname{dom} T$ and $Tu = Su$ for all $u \in \operatorname{dom} S = \operatorname{dom} T$.

The operator T is called *closed* if $\operatorname{dom} T$ is complete with respect to the norm $(\|Tu\|^2 + \|u\|^2)^{1/2}$. Clearly, every bounded operator defined on a closed subspace of $\mathcal{H}$ is closed. If the kernel, $\ker T$, of a closed operator T is trivial, then its inverse T^{-1}, defined on its range, $\operatorname{ran} T$, is also closed.

The operator T is called *densely defined* if $\operatorname{dom} T$ is dense in $\mathcal{H}$. For such a T we now define the *adjoint* T^* as follows. Its domain is

$$\begin{aligned}\operatorname{dom} T^* := \{v \in \mathcal{H} \colon &\text{ there is a } g \in \mathcal{H} \text{ such that} \\ &\text{for all } u \in \operatorname{dom} T \text{ one has } (g,u) = (v,Tu)\}\,.\end{aligned}$$

Since T is densely defined, for $v \in \operatorname{dom} T^*$ the corresponding g is unique and we define $T^* v := g$. For bounded T, this coincides with the definition given above.

One can show that T^* is always closed. If T is closed, then T^* is densely defined and $T^{**} := (T^*)^* = T$. Moreover, one has

$$\ker T^* = (\operatorname{ran} T)^\perp\,, \tag{1.1}$$

which, in turn, implies that

$$(\ker T^*)^\perp = \overline{\operatorname{ran} T}\,. \tag{1.2}$$

If T is densely defined, has trivial kernel and dense range, then, by (1.1), T^* has trivial kernel and its inverse is given by

$$(T^*)^{-1} = \left(T^{-1}\right)^*\,. \tag{1.3}$$

For a closed operator T, the *resolvent set* $\rho(T)$ is defined by

$$\begin{aligned}\rho(T) := \{z \in \mathbb{C} \colon T - z &\text{ is a bijection from } \operatorname{dom} T \text{ onto } \mathcal{H} \\ &\text{with a bounded inverse}\}\end{aligned}$$

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and the operator $(T-z)^{-1}$ is called the *resolvent* of T at $z \in \rho(T)$. Using the closed graph theorem, we could deduce the boundedness of the inverse from the fact that the range of $T-z$ is equal to $\mathcal{H}$, although we will not use this fact explicitly in this book. We note too that here and in what follows, in the notation $T-z$ we identify the number z with z times the identity operator on $\mathcal{H}$.

The *spectrum* $\sigma(T)$ is defined by

$$\sigma(T) := \mathbb{C} \setminus \rho(T)$$

and the *point spectrum* is defined by

$$\sigma_p(T) := \{z \in \sigma(T) : \ker(T-z) \neq \{0\}\} .$$

A number $z \in \sigma_p(T)$ is called an *eigenvalue* of T and any $0 \neq u \in \ker(T-z)$ is called a corresponding *eigenvector*. Moreover, $\dim\ker(T-z)$ is called the *(geometric) multiplicity* of the eigenvalue z of T.

We now discuss the orthogonal sum of operators. Let $\mathcal{N}$ be a countable (possibly finite) index set and assume that, for each $n \in \mathcal{N}$, $\mathcal{H}_n$ is a separable Hilbert space with a norm $\|\cdot\|_{\mathcal{H}_n}$ and an inner product $(\cdot,\cdot)_{\mathcal{H}_n}$. We recall that the Hilbert space

$$\mathcal{H} := \bigoplus_{n\in\mathcal{N}} \mathcal{H}_n$$

is the space of all elements $f = (f_n)_{n\in\mathcal{N}}$ in the Cartesian product $\prod_{n\in\mathcal{N}} \mathcal{H}_n$ (that is, $f_n \in \mathcal{H}_n$ for all $n \in \mathcal{N}$) such that

$$\|f\|_{\mathcal{H}} := \Big(\sum_{n\in\mathcal{N}} \|f_n\|^2_{\mathcal{H}_n}\Big)^{1/2} < \infty .$$

This is a separable Hilbert space with inner product

$$(f,g)_{\mathcal{H}} := \sum_{n\in\mathcal{N}} (f_n, g_n)_{\mathcal{H}_n} .$$

For each $n \in \mathcal{N}$, let T_n be an operator in $\mathcal{H}_n$. We define an operator, denoted by $\bigoplus_{n\in\mathcal{N}} T_n$, in $\mathcal{H}$ with domain

$$\operatorname{dom}\Big(\bigoplus_{n\in\mathcal{N}} T_n\Big) := \Big\{u \in \mathcal{H} : u_n \in \operatorname{dom} T_n \text{ for all } n \in \mathcal{N}\,,\ \sum_{n\in\mathcal{N}} \|T_n u_n\|^2_{\mathcal{H}_n} < \infty\Big\}$$

by

$$\Big(\bigoplus_{n\in\mathcal{N}} T_n u\Big)_m := T_m u_m \quad \text{for all } m \in \mathcal{N}\,,\ u \in \operatorname{dom}\Big(\bigoplus_{n\in\mathcal{N}} T_n\Big).$$

If all the T_n are bounded, then

$$\Big\|\bigoplus_{n\in\mathcal{N}} T_n\Big\| = \sup_{n\in\mathcal{N}} \|T_n\| , \tag{1.4}$$

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where the right side may or may not be finite. In particular, $\bigoplus_{n\in\mathcal{N}} T_n$ is bounded if and only if the T_n are uniformly bounded. If all the T_n are closed, then $\bigoplus_{n\in\mathcal{N}} T_n$ is closed. If all the T_n are densely defined, then $\bigoplus_{n\in\mathcal{N}} T_n$ is densely defined, and in this case one finds for the adjoint

$$\Big(\bigoplus_{n\in\mathcal{N}} T_n\Big)^* = \bigoplus_{n\in\mathcal{N}} T_n^* .$$

The relation between the spectrum of $\bigoplus_{n\in\mathcal{N}} T_n$ and the spectra of the T_n is as follows.

Lemma 1.5 *Assume that the T_n are closed. Then*

$$\rho\Big(\bigoplus_{n\in\mathcal{N}} T_n\Big) = \Big\{z \in \bigcap_{n\in\mathcal{N}} \rho(T_n) :\ \sup_{n\in\mathcal{N}} \big\|(T_n - z)^{-1}\big\| < \infty\Big\}. \tag{1.5}$$

Moreover,

$$\sigma_p\Big(\bigoplus_{n\in\mathcal{N}} T_n\Big) = \bigcup_{n\in\mathcal{N}} \sigma_p(T_n)$$

and, for each $z \in \mathbb{C}$,

$$\dim\ker\Big(\bigoplus_{n\in\mathcal{N}} T_n - z\Big) = \sum_{n\in\mathcal{N}} \dim\ker(T_n - z) .$$

Proof Writing $T := \bigoplus_{n\in\mathcal{N}} T_n$, we first note that

$$\ker(T - z) = \bigoplus_{n\in\mathcal{N}} \ker(T_n - z) .$$

This immediately implies the assertion about the point spectrum of T and the multiplicity of eigenvalues. To prove the assertion about the resolvent set, let ρ denote the set on the right side of (1.5). If $z \in \rho$, then the operator

$$R(z) := \bigoplus_{n\in\mathcal{N}} (T_n - z)^{-1}$$

is well defined and bounded by (1.4). Moreover, one easily checks that

$$(T - z)R(z) = 1 \quad \text{and} \quad R(z)(T - z) \text{ is the identity on } \operatorname{dom} T.$$

Thus, $z \in \rho(T)$.

Conversely, if $z \in \rho(T)$, then $(T - z)^{-1}$ is well defined and bounded. This means that $z \in \rho(T_n)$ for each n and that $(T - z)^{-1} = R(z)$. Again by (1.4), boundedness of $R(z)$ implies $\sup_{n\in\mathcal{N}} \big\|(T_n - z)^{-1}\big\| < \infty$. Thus $z \in \rho$. □

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1.1.3 Self-adjoint operators and their spectra

A densely defined operator A is called *symmetric* if $\operatorname{dom} A \subset \operatorname{dom} A^*$ and $A^* u = Au$ for every $u \in \operatorname{dom} A$. Equivalently, A is symmetric if and only if

$$(Au, v) = (u, Av) \quad \text{for all } u, v \in \operatorname{dom} A\,.$$

This, in turn, is equivalent to

$$(Au, u) \in \mathbb{R} \quad \text{for all } u \in \operatorname{dom} A\,.$$

Note that eigenvalues of a symmetric operator are necessarily real since $Au = zu$ implies

$$z\|u\|^2 = (Au, u) = (u, Au) = \overline{z}\|u\|^2\,.$$

Moreover, if u and v are eigenvectors of a symmetric operator A corresponding to distinct eigenvalues z and ζ, then $(u, v) = 0$. Indeed,

$$z(u, v) = (Au, v) = (u, Av) = \overline{\zeta}(u, v) = \zeta(u, v)\,.$$

An operator A is called *self-adjoint* if it is densely defined, symmetric and $\operatorname{dom} A = \operatorname{dom} A^*$. Clearly, any self-adjoint operator is closed.

A symmetric operator A that is bounded (and defined on all of $\mathcal{H}$) is self-adjoint.

Lemma 1.6 *Let A be self-adjoint. Then $\sigma(A) \subset \mathbb{R}$. Moreover, $z \in \rho(A)$ if and only if there is an $\varepsilon > 0$ such that*

$$\|(A - z)u\| \geq \varepsilon\|u\| \quad \text{for all } u \in \operatorname{dom} A\,. \tag{1.6}$$

Note that the second part of this lemma says that $z \in \sigma(A)$ if and only if there is a sequence $(u_n) \subset \operatorname{dom} A$ with $\|u_n\| = 1$ for all n and $\|(A - z)u_n\| \to 0$ as $n \to \infty$.

Proof Indeed, for any symmetric operator A,

$$\|(A - z)u\|^2 = \|(A - \operatorname{Re} z)u\|^2 + (\operatorname{Im} z)^2\|u\|^2 \geq (\operatorname{Im} z)^2\|u\|^2\,. \tag{1.7}$$

Hence, if $\operatorname{Im} z \neq 0$ the operator $A - z$ is injective and its inverse, defined on $\operatorname{ran}(A - z)$, is bounded. If, in addition, A is closed, the inequality above implies that $\operatorname{ran}(A - z)$ is closed. Hence, if A is self-adjoint, by (1.2) for $\operatorname{Im} z \neq 0$ we get

$$\operatorname{ran}(A - z) = \overline{\operatorname{ran}(A - z)} = (\ker(A^* - \overline{z}))^\perp = (\ker(A - \overline{z}))^\perp = \mathcal{H}\,,$$

and therefore $z \in \rho(A)$. Thus, we have shown $\sigma(A) \subset \mathbb{R}$, as claimed.

The same argument implies that, for all $z \in \mathbb{R}$ for which (1.6) holds, one has $z \in \rho(A)$.

Conversely, if $z \in \sigma_p(A)$, then (1.6) clearly fails for eigenvectors u. If $z \in \sigma(A)\backslash\sigma_p(A) \subset \mathbb{R}$, then $A-z$ is invertible. We show that its inverse is unbounded, which contradicts (1.6). Indeed, as above, we have $\overline{\operatorname{ran}(A-z)} = (\ker(A-z))^\perp = \mathcal{H}$. Since A is closed, the inverse of $A-z$ is closed as well. For a bounded inverse of $A-z$ this would mean that its domain $\operatorname{ran}(A-z)$ is closed in $\mathcal{H}$ and $\operatorname{ran}(A-z) = \mathcal{H}$. But then $z \in \rho(A)$, which contradicts $z \in \sigma(A)$. We conclude that the inverse of $A-z$ is unbounded. □

1.1.4 The spectrum of a multiplication operator

Let X be a set, $\mathcal{A}$ a sigma-algebra on X and μ a non-negative measure on $(X,\mathcal{A})$. The measure space $(X,\mathcal{A},\mu)$ is called *separable* if there is a countable subset $\mathcal{B} \subset \mathcal{A}$ such that for any $E \in \mathcal{A}$ with $\mu(E) < \infty$ and any $\varepsilon > 0$ there is a $B \in \mathcal{B}$ with $\mu(B\Delta E) \le \varepsilon$. In that case, the Hilbert space $L^2(X,\mathcal{A},\mu)$ is separable (in the sense of having a countable orthonormal basis).

In this subsection we assume that $(X,\mathcal{A},\mu)$ is separable and sigma-finite.

Let φ be an $\mathcal{A}$-measurable, complex-valued and μ-a.e. finite function on X, and consider the multiplication operator T_φ in $L^2(X)$ defined by

$$T_\varphi u := \varphi u\,, \qquad \operatorname{dom} T_\varphi := \left\{u \in L^2(X)\colon\ \varphi u \in L^2(X)\right\}.$$

The completeness of $L^2(X,(1+|\varphi|^2)\mu)$ implies that T_φ is a closed operator. Moreover, T_φ is densely defined because, for any $u \in L^2(X)$, the functions $\chi_{\{|\varphi|\le n\}}u \in \operatorname{dom} T_\varphi$ converge to u as $n \to \infty$ by dominated convergence, using the μ-a.e. finiteness of φ. The adjoint of T_φ is given by

$$T_\varphi^* = T_{\overline{\varphi}}\,.$$

In particular, T_φ is self-adjoint if and only if φ is real-valued μ-a.e. The operator T_φ is bounded if and only if φ is μ-essentially bounded, and in this case

$$\|T_\varphi\| = \|\varphi\|_{L^\infty(X,\mathcal{A},\mu)}\,.$$

Let us characterize the spectrum of the operator T_φ.

Theorem 1.7 *Let φ be an $\mathcal{A}$-measurable, complex-valued and μ-a.e. finite function on X. Then*

$$\sigma(T_\varphi) = \left\{z \in \mathbb{C}\colon\ \mu(\varphi^{-1}(\{\zeta \in \mathbb{C}\colon\ |\zeta - z| \le \varepsilon\})) > 0 \textit{ for all } \varepsilon > 0\right\} \tag{1.8}$$

and

$$\sigma_p(T_\varphi) = \left\{z \in \mathbb{C}\colon\ \mu(\varphi^{-1}(\{z\})) > 0\right\}, \tag{1.9}$$

where $\varphi^{-1}(E) := \{x \in X\colon \varphi(x) \in E\}$ denotes the pre-image of E under φ.

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Proof We begin with proving (1.9). If $\varphi u = zu$ μ-a.e. on X for some $u \not\equiv 0$, then $\varphi = z$ μ-a.e. on the set $\{x \in X : u(x) \neq 0\}$, which has positive measure. Conversely, if $Y \subset \varphi^{-1}(\{z\})$ is measurable with $0 < \mu(Y) < \infty$ (such a set exists by sigma-finiteness), then $0 \not\equiv \chi_Y \in \operatorname{dom} T_\varphi$ and $T_\varphi \chi_Y = z\chi_Y$.

We turn to (1.8). By what we have just shown, if $z \in \sigma_p(T_\varphi)$, then z belongs to the set on the right side of (1.8). If $z \notin \sigma_p(T_\varphi)$, then the operator $T_\varphi - z$ is invertible on $\operatorname{ran} T_\varphi$ and the inverse is given by T_{ψ_z} with

$$\psi_z(x) := \frac{1}{\varphi(x) - z} \quad \text{for all } x \in X\,.$$

As we have noticed before, this operator is bounded if and only if ψ_z is μ-essentially bounded; that is, if and only if there is an $\varepsilon > 0$ such that for μ-a.e. $x \in X$ one has $|\varphi(x) - z| \geq \varepsilon$. This means that z does not belong to the right side in (1.8). □

1.1.5 Functional calculus

A bounded operator Π on a Hilbert space $\mathcal{H}$ is called an *orthogonal projection* if $\Pi = \Pi^* = \Pi^2$. A *projection-valued measure* is a map $P : \omega \mapsto P_\omega$ on the Borel sigma-algebra on $\mathbb{R}$ taking values in the set of orthogonal projections such that

(a) If $(\omega_n)_{n\in\mathcal{N}}$, $\mathcal{N} \subset \mathbb{N}$, is a finite or countable family of disjoint Borel sets, then

$$P_{\bigcup_n \omega_n} = \operatorname*{s-lim}_{N\to\infty} \sum_{n\leq N} P_{\omega_n}\,.$$

(b) $P_{\mathbb{R}} = 1$.

One can deduce from these properties that $P_\emptyset = 0$ and that

$$P_{\omega_1} P_{\omega_2} = P_{\omega_1 \cap \omega_2} \qquad \text{for all Borel sets } \omega_1, \omega_2\,. \tag{1.10}$$

Notions for scalar measures have natural analogues for projection-valued measures. The support of P is

$$\operatorname{supp} P := \left\{\lambda \in \mathbb{R} : P_{(\lambda-\varepsilon,\lambda+\varepsilon)} \neq 0 \text{ for all } \varepsilon > 0\right\}. \tag{1.11}$$

A property hold P-a.e. if it holds outside of a Borel set ω with $P_\omega = 0$. The P-essential supremum of a real-valued measurable function φ on $\mathbb{R}$ is

$$P\text{-}\sup_\lambda \varphi(\lambda) := \inf\left\{a \in \mathbb{R} : P_{\{\varphi > a\}} = 0\right\}$$

and a measurable function φ is P-bounded if $P\text{-}\sup_\lambda |\varphi(\lambda)| < \infty$.

Given a projection-valued measure P and $f, g \in \mathcal{H}$, then $\omega \mapsto (P_\omega f, g)$ is

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a complex Borel measure on $\mathbb{R}$ and we denote integration with respect to this measure by $d(P_\lambda f, g)$. If $f = g$, the measure $\omega \mapsto (P_\omega f, f)$ is non-negative and, since $P_\mathbb{R} = 1$, we have

$$\int_\mathbb{R} d(P_\lambda f, f) = \|f\|^2 .$$

The next result provides us with the existence and fundamental properties of a *functional calculus*.

Theorem 1.8 *Let P be a projection-valued measure.*

(a) *For every measurable, P-a.e. finite function φ on $\mathbb{R}$ there is a unique operator J_φ in $\mathcal{H}$ satisfying*

$$\operatorname{dom} J_\varphi = \left\{ u \in \mathcal{H} \colon \int_\mathbb{R} |\varphi(\lambda)|^2 \, d(P_\lambda u, u) < \infty \right\}$$

and

$$(J_\varphi u, g) = \int_\mathbb{R} \varphi(\lambda) \, d(P_\lambda u, g) \qquad \text{for all } u \in \operatorname{dom} J_\varphi ,\ g \in \mathcal{H} . \tag{1.12}$$

The operator J_φ is closed and densely defined.

(b) *If φ, ψ are measurable, P-a.e. finite functions on $\mathbb{R}$, then*

$$J_\varphi^* = J_{\overline{\varphi}} , \tag{1.13}$$

$$J_1 = 1 , \tag{1.14}$$

$$\left\| J_\varphi u \right\|^2 = \int_\mathbb{R} |\varphi(\lambda)|^2 \, d(P_\lambda u, u) \quad \text{for all } u \in \operatorname{dom} J_\varphi , \tag{1.15}$$

$$\left\| J_\varphi \right\| = P\text{-}\sup_\lambda |\varphi(\lambda)| . \tag{1.16}$$

If $\alpha, \beta \in \mathbb{C}$ and if $|\varphi| + |\psi| \le C(1 + |\alpha\varphi + \beta\psi|)$ P-a.e. for some $C < \infty$, then

$$\alpha J_\varphi + \beta J_\psi = J_{\alpha\varphi + \beta\psi} , \tag{1.17}$$

and, if $|\varphi| + |\psi| \le C(1 + |\varphi\psi|)$ P-a.e. for some $C < \infty$, then

$$J_\varphi J_\psi = J_{\psi\varphi} . \tag{1.18}$$

(c) *If φ_n are measurable, P-bounded functions that converge pointwise P-a.e. to a P-a.e. finite function φ and satisfy $|\varphi_n| \le C(1 + |\varphi|)$ P-a.e. for all n, then $J_{\varphi_n} u \to J_\varphi u$ for all $u \in \operatorname{dom} J_\varphi$.*

In the following we will sometimes use the notation

$$J_\varphi =: \int_{\mathbb{R}} \varphi(\lambda)\, dP_\lambda \, .$$

We note that the conditions for (1.17) and (1.18) are satisfied, in particular, if φ and ψ are P-bounded. Hence, Theorem 1.8 shows that the map $\varphi \mapsto J_\varphi$ restricts to an isometric isomorphism from the C^*-Banach algebra of all measurable, P-bounded functions on $\mathbb{R}$ (with the norm P-$\sup_\lambda$, the constant function 1 as unit, and the involution $\varphi \mapsto \overline{\varphi}$) onto a commutative subalgebra of the C^*-Banach algebra of bounded operators on $\mathcal{H}$ (with the operator norm, the identity as unit, and the involution $T \mapsto T^*$).

For general, measurable, P-a.e. finite functions φ and ψ the identities (1.17) and (1.18) hold pointwise on the domains of the operator expressions on the left side and, moreover, these domains are dense in the operator norm of the right sides.

Proof We only sketch the main steps of the construction and refer for details to, for instance, Birman and Solomjak (1987, §§5.3 and 5.4) or Teschl (2014, §3.1). We begin by defining the operator J_φ for simple functions $\varphi = \sum_{n=1}^N c_n \chi_{\omega_n}$ with $c_n \in \mathbb{C}$ and disjoint Borel $\omega_n \subset \mathbb{R}$ by

$$J_\varphi := \sum_n c_n P_{\omega_n} \, .$$

Using the properties of a projection-valued measure one easily verifies the assertions in parts (a) and (b) of the theorem. By the density of simple functions in the set of measurable, P-bounded functions with respect to the P-essential supremum, the operator $\varphi \mapsto J_\varphi$ can be uniquely extended to the latter class of functions φ. The assertions in parts (a) and (b) carry over to this extension. Moreover, the assertion in (c) in the case where the φ_n are uniformly P-bounded follows easily by dominated convergence.

For general measurable, P-a.e. finite functions φ one first verifies that the set $\operatorname{dom} J_\varphi$ in part (a) is, indeed, a dense subspace. On the other hand, using dominated convergence, one sees that, for $u \in \operatorname{dom} J_\varphi$ and for every sequence (φ_n) as in (c), the elements $J_{\varphi_n} u$ converge to a limit that is independent of the choice of the (φ_n). This defines a linear operator J_φ as in (a), except for the claimed closedness, which will be established momentarily. The assertion in (c) holds by construction and the properties in (b) follow by approximation from the corresponding properties in the bounded cases. Finally, (1.13) implies that $J_\varphi = (J_{\overline{\varphi}})^*$, so J_φ is closed as an adjoint operator. This completes our sketch of the proof of Theorem 1.8. $\square$

1.1.6 The spectral theorem

The content of the spectral theorem for self-adjoint operators is that any such operator can be obtained via the procedure, in the previous subsection, of integration against a spectral measure. This theorem generalizes the diagonalization of a Hermitian matrix and plays a fundamental role in the theory of self-adjoint operators.

Theorem 1.9 *Let A be a self-adjoint operator in $\mathcal{H}$. Then there is a unique projection-valued measure P on $\mathcal{H}$ such that*

$$A = \int_{\mathbb{R}} \lambda \, dP_\lambda \, .$$

Proof Various proofs can be found in the textbooks mentioned in §1.3. Here we briefly sketch the main lines of the argument in Teschl (2014, §3.1).

Let $f \in \mathcal{H}$. Since, by Lemma 1.6, $\mathbb{C}_+ \subset \rho(A)$, we can consider the function $\mathbb{C}_+ \ni z \mapsto ((A-z)^{-1} f, f)$, and, by (1.7), we obtain the bound

$$\left|((A-z)^{-1} f, f)\right| \leq \left\|(A-z)^{-1}\right\| \|f\|^2 \leq |\mathrm{Im}\, z|^{-1} \|f\|^2 \, .$$

By a Neumann series expansion we see that the function is analytic with respect to z. Moreover, using

$$(A-z)^{-1} - (A-\zeta)^{-1} = (z-\zeta)(A-\zeta)^{-1}(A-z)^{-1}, \qquad z, \zeta \in \rho(A), \quad (1.19)$$

with $\zeta = \overline{z}$ and, by (1.3), $((A-z)^{-1})^* = (A-\overline{z})^{-1}$, we find

$$\begin{aligned} \mathrm{Im}\left((A-z)^{-1} f, f\right) &= (2i)^{-1}\left(((A-z)^{-1} f, f) - \overline{((A-z)^{-1} f, f)}\right) \\ &= (2i)^{-1}\left(\left((A-z)^{-1} - (A-\overline{z})^{-1}\right) f, f\right) \\ &= (\mathrm{Im}\, z)\left((A-\overline{z})^{-1}(A-z)^{-1} f, f\right) \\ &= (\mathrm{Im}\, z)\left\|(A-z)^{-1} f\right\|^2 . \end{aligned} \qquad (1.20)$$

Thus, by the Herglotz representation theorem (see, e.g., Teschl, 2014, Theorem 3.20), there is a non-negative Borel measure μ_f on $\mathbb{R}$ with $\mu_f(\mathbb{R}) \leq \|f\|^2$ such that

$$\left((A-z)^{-1} f, f\right) = \int_{\mathbb{R}} \frac{d\mu_f(\lambda)}{\lambda - z} \quad \text{for all } z \in \mathbb{C}_+ \, .$$

For $f, g \in \mathcal{H}$ we define a complex Borel measure

$$\mu_{f,g} := \frac{1}{4}\left(\mu_{f+g} - \mu_{f-g} + i\mu_{f+ig} - i\mu_{f-ig}\right)$$

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and obtain

$$\big((A-z)^{-1}f,g\big)=\int_{\mathbb{R}}\frac{d\mu_{f,g}(\lambda)}{\lambda-z}\,. \tag{1.21}$$

The right side is the Stieltjes transform of $\mu_{f,g}$. By a uniqueness theorem for this transform (see, e.g., Teschl, 2014, Theorem 3.21), the measure $\mu_{f,g}$ depends linearly on f and anti-linearly on g. For any Borel $\omega\subset\mathbb{R}$, $f\mapsto\mu_f(\omega)$ is a non-negative quadratic form that is bounded by $\mu_f(\omega)\le\mu_f(\mathbb{R})\le\|f\|^2$. Thus, by the Riesz representation theorem, there is a self-adjoint operator P_ω in $\mathcal{H}$ with $\|P_\omega\|\le 1$ such that

$$(P_\omega f,g)=\int_\omega d\mu_{f,g}(\lambda)\qquad\text{for all } f,g\in\mathcal{H}\,.$$

We will show that P is a projection-valued measure. The defining property (a) of such a measure follows, with a weak limit instead of a strong limit, from the sigma-additivity of the measures $\mu_{f,g}$. Once we have shown that P is projection-valued, the weak limit can be replaced by a strong limit. To prove the defining property (b) of such a measure, let $f\in\ker P_{\mathbb{R}}$. Then $0=(P_{\mathbb{R}}f,f)=\mu_f(\mathbb{R})$ and therefore $((A-z)^{-1}f,f)=0$ for all $z\in\mathbb{C}_+$. By (1.20), this implies $f=0$. Thus, $\ker P_{\mathbb{R}}=\{0\}$ and, once we have shown that $P_{\mathbb{R}}$ is an orthogonal projection, we deduce that $P_{\mathbb{R}}=1$, as claimed.

Let us show that $P_\omega^2=P_\omega$. For all $z,\zeta\in\mathbb{C}_+$ with $z\ne\zeta$, by (1.19),

$$\begin{aligned}\int_{\mathbb{R}}\frac{d\mu_{f,g}(\lambda)}{(\lambda-z)(\lambda-\zeta)}&=\frac{1}{z-\zeta}\left(\int_{\mathbb{R}}\frac{d\mu_{f,g}(\lambda)}{\lambda-z}-\int_{\mathbb{R}}\frac{d\mu_{f,g}(\lambda)}{\lambda-\zeta}\right)\\&=\frac{1}{z-\zeta}\Big(\big((A-z)^{-1}-(A-\zeta)^{-1}\big)f,g\Big)=\Big((A-\zeta)^{-1}f,(A-\overline{z})^{-1}g\Big)\\&=\big((A-z)^{-1}(A-\zeta)^{-1}f,g\big)=\int_{\mathbb{R}}\frac{d\mu_{f,(A-\overline{z})^{-1}g}(\lambda)}{\lambda-\zeta}\,.\end{aligned}$$

By a uniqueness theorem for the Stieltjes transform, this implies that

$$\frac{d\mu_{f,g}(\lambda)}{\lambda-z}=d\mu_{f,(A-\overline{z})^{-1}g}(\lambda)\,.$$

Using this formula, we find for any Borel $\omega\subset\mathbb{R}$,

$$\begin{aligned}\int_\omega\frac{d\mu_{f,g}(\lambda)}{\lambda-z}&=\int_\omega d\mu_{f,(A-\overline{z})^{-1}g}(\lambda)=\Big(P_\omega f,(A-\overline{z})^{-1}g\Big)\\&=\big((A-z)^{-1}P_\omega f,g\big)=\int_{\mathbb{R}}\frac{d\mu_{P_\omega f,g}(\lambda)}{\lambda-z}\,.\end{aligned}$$

Again by the uniqueness theorem for the Stieltjes transform, this implies that

$$\chi_\omega(\lambda)d\mu_{f,g}(\lambda)=d\mu_{P_\omega f,g}(\lambda)\,.$$

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Multiplying this identity by $\chi_\omega(\lambda)$ and integrating, we find

$$(P_\omega f, g) = \int_\omega d\mu_{f,g}(\lambda) = \int_\omega d\mu_{P_\omega f,g}(\lambda) = (P_\omega^2 f, g)\,.$$

Thus, $P_\omega = P_\omega^2$. We have shown that P is a projection-valued measure.

In terms of this measure, (1.21) means that $(A-z)^{-1} = \int_{\mathbb{R}} (\lambda - z)^{-1} dP_\lambda$ for $z \in \mathbb{C}_+$. By the functional calculus (Theorem 1.8), this implies $A = \int_{\mathbb{R}} \lambda\, dP_\lambda$, as claimed. Uniqueness of the projection-valued measure follows from (1.21) and the uniqueness of the Stieltjes transform. □

In the situation of Theorem 1.9 the projection-valued measure is also called the *spectral measure* of A. Its relation to the spectrum is clarified in Corollary 1.10 below.

Given the spectral measure of a self-adjoint operator A one can apply the construction in the previous subsection and define functions of a self-adjoint operator. More precisely, if A is a self-adjoint operator, P its spectral measure, and if φ is a measurable, P-a.e. finite function on $\mathbb{R}$, then, according to Theorem 1.8, there is an operator

$$\varphi(A) := \int_{\mathbb{R}} \varphi(\lambda)\, dP_\lambda$$

with domain

$$\operatorname{dom} \varphi(A) = \left\{ u \in \mathcal{H} \colon \int_{\mathbb{R}} |\varphi(\lambda)|^2 d(P_\lambda u, u) < \infty \right\} \tag{1.22}$$

satisfying

$$(\varphi(A)u, g) = \int_{\mathbb{R}} \varphi(\lambda)\, d(P_\lambda u, g) \qquad \text{for all } u \in \operatorname{dom}\varphi(A),\ g \in \mathcal{H}\,. \tag{1.23}$$

Note, in particular, that by (1.18), the operator $\varphi(A)$ for $\varphi(\lambda) = \lambda^n$, $n \in \mathbb{N}$, coincides with the n-fold product of A defined in the sense of operator products. Moreover, if $z \in \mathbb{C} \setminus \sigma_p(A)$, then $(A-z)^{-1}$, defined by the functional calculus, coincides with the resolvent defined as a possibly unbounded operator. The spectral projections P_ω of Borel sets ω reappear through the characteristic functions χ_ω, namely, $P_\omega = \chi_\omega(A)$.

Recall that the support of a projection-valued measure was defined in (1.11).

Corollary 1.10 *We have $\sigma(A) = \operatorname{supp} P$ and, for all $z \in \rho(A)$,*

$$\left\|(A-z)^{-1}\right\| = \operatorname{dist}(z, \sigma(A))^{-1}\,. \tag{1.24}$$

Moreover $\sigma_p(A) = \{\lambda \in \mathbb{R} \colon P_{\{\lambda\}} \neq 0\}$ and $\ker(A-\lambda) = \operatorname{ran} P_{\{\lambda\}}$, for any $\lambda \in \sigma_p(A)$.

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Proof The proof is based on the fact that, for $u \in \operatorname{dom} A$, $z \in \mathbb{C}$, and Borel $\omega \subset \mathbb{R}$, we have $P_\omega u \in \operatorname{dom} A$ and

$$\|(A-z)P_\omega u\|^2 = \int_\omega |\mu - z|^2 d(P_\mu u, u)\,. \tag{1.25}$$

Indeed, by (1.15) and (1.18),

$$\begin{aligned}\|(A-z)P_\omega u\|^2 &= \|(A-z)\chi_\omega(A)u\|^2 = \|((\cdot - z)\chi_\omega)(A)u\|^2 \\ &= \int_\omega |\mu - z|^2\, d(P_\mu u, u)\,.\end{aligned}$$

First, assume that $z \in \mathbb{C} \setminus \operatorname{supp} P$. Then, by (1.25) and the fact that the support of the measure $\omega \mapsto (P_\omega u, u)$ is contained in the support of the projection-valued measure P, for all $u \in \operatorname{dom} A$,

$$\begin{aligned}\|(A-z)u\|^2 &= \int_{\mathbb{R}} |\mu - z|^2 d(P_\mu u, u) = \int_{\operatorname{supp} P} |\mu - z|^2 d(P_\mu u, u) \\ &\geq \operatorname{dist}(z, \operatorname{supp} P)^2 \int_{\operatorname{supp} P} d(P_\mu u, u) = \operatorname{dist}(z, \operatorname{supp} P)^2 \|u\|^2\,.\end{aligned}$$

Thus, by Lemma 1.6, $z \in \rho(A)$ and

$$\|(A-z)^{-1}\| \leq \operatorname{dist}(z, \operatorname{supp} P)^{-1}\,.$$

This proves, in particular, that $\sigma(A) \subset \operatorname{supp} P$.

Conversely, if $\lambda \in \operatorname{supp} P$, there is a sequence (u_n) with $0 \neq u_n \in \operatorname{ran} P_{[\lambda - 1/n, \lambda + 1/n]}$ for all n. Then, by (1.25), for any $z \in \mathbb{C}$,

$$\begin{aligned}\|(A-z)u_n\|^2 &= \int_{[\lambda-1/n,\lambda+1/n]} |\mu - z|^2 d(P_\mu u_n, u_n) \\ &\leq \left((|\operatorname{Re} z - \lambda| + n^{-1})^2 + (\operatorname{Im} z)^2 \right) \int_{[\lambda-1/n,\lambda+1/n]} d(P_\mu u_n, u_n) \\ &= \left((|\operatorname{Re} z - \lambda| + n^{-1})^2 + (\operatorname{Im} z)^2 \right) \|\chi_{[\lambda-1/n,\lambda+1/n]}(A) u_n\|^2 \\ &= \left((|\operatorname{Re} z - \lambda| + n^{-1})^2 + (\operatorname{Im} z)^2 \right) \|u_n\|^2\,.\end{aligned}$$

In particular, with the choice $z = \lambda$ we see that the bound (1.6) is violated and therefore $\lambda \in \sigma(A)$. Thus, $\operatorname{supp} P \subset \sigma(A)$, which completes the proof of the first assertion.

Applying the above bound for $z \in \rho(A)$, we deduce that

$$\|(A-z)^{-1}\|^2 \geq \left((|\operatorname{Re} z - \lambda| + n^{-1})^2 + (\operatorname{Im} z)^2 \right)^{-1}\,.$$

Since n is arbitrary, we find

$$\|(A-z)^{-1}\|^2 \geq \left((\operatorname{Re} z - \lambda)^2 + (\operatorname{Im} z)^2 \right)^{-1}$$

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and, taking the supremum over all $\lambda \in \operatorname{supp} P$, we obtain the bound

$$\|(A-z)^{-1}\|^2 \geq \operatorname{dist}(z, \operatorname{supp} P)^{-2}.$$

This proves the formula for the norm of the resolvent.

Finally, by (1.25) with $\omega = \mathbb{R}$, $u \in \ker(A-z)$ if and only $d(P_\mu u, u)$ is a point measure of mass $\|u\|^2$ at $\mu = z$. The latter is equivalent to $u = P_{\{z\}} u$. This completes the proof. □

For a symmetric operator A we set

$$m_A := \inf_{0 \neq u \in \operatorname{dom} A} \frac{(Au, u)}{\|u\|^2}.$$

An operator A is called *lower semibounded* if it is symmetric and $m_A > -\infty$.

An operator A is called *non-negative* if it is symmetric and $m_A \geq 0$.

Corollary 1.11 *Let A be a self-adjoint operator. Then*

$$m_A = \inf \sigma(A).$$

Assuming that $m_A > -\infty$, we have that m_A is an eigenvalue of A if and only if the infimum $\inf_{0 \neq u \in \operatorname{dom} A}(Au, u)/\|u\|^2$ is a minimum, and the eigenvectors are precisely those vectors for which the infimum is attained.

Proof We assume that $m_A > -\infty$ and omit the minor modifications needed for $m_A = -\infty$. First note that, for $\lambda < m_A$ and $u \in \operatorname{dom} A$,

$$(m_A - \lambda)\|u\|^2 \leq (Au, u) - \lambda\|u\|^2 = ((A-\lambda)u, u) \leq \|(A-\lambda)u\|\|u\|,$$

and therefore, by Lemma 1.6, $\lambda \in \rho(A)$. Hence, $\sigma(A) \subset [m_A, +\infty)$ and, in particular, $m_A \leq \inf \sigma(A)$. Conversely, by the spectral theorem (Theorem 1.9) and Corollary 1.10,

$$(Au, u) = \int_{\mathbb{R}} \lambda d(P_\lambda u, u) = \int_{\sigma(A)} \lambda d(P_\lambda u, u) \geq \inf \sigma(A)\|u\|^2 \text{ for all } u \in \operatorname{dom} A.$$

Thus, $m_A \geq \inf \sigma(A)$ which proves the first assertion. Equality in the previous bound is attained if and only if $d(P_\lambda u, u)$ is a point measure of mass $\|u\|^2$ at $\lambda = m_A$. As in the proof of Corollary 1.10, this is equivalent to $u \in \ker(A - m_A)$. □

According to the *spectral mapping theorem*, the original spectral measure $P =: P(A)$ of A and the spectral measure $P(\varphi(A))$ of $\varphi(A)$, where φ is a measurable, real-valued function, are related by

$$P_\omega(\varphi(A)) = P_{\varphi^{-1}(\omega)}(A) \qquad \text{for all Borel sets } \omega \subset \mathbb{R}. \tag{1.26}$$

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Indeed, the right side defines a projection-valued measure $\tilde{P}$ and for all $u \in \mathcal{H}$, by a change of variables,

$$\int_{\mathbb{R}} \mu^2 \, d(\tilde{P}_\mu u, u) = \int_{\mathbb{R}} \varphi(\lambda)^2 \, d(P_\lambda u, u) \,.$$

Moreover, for all u for which this is finite one has, by the same change of variables,

$$\int_{\mathbb{R}} \mu \, d(\tilde{P}_\mu u, g) = \int_{\mathbb{R}} \varphi(\lambda) \, d(P_\lambda u, g) = (\varphi(A)u, g) \qquad \text{for all } g \in \mathcal{H} \,.$$

By the uniqueness assertion in Theorem 1.9, $\tilde{P}$ is the spectral measure of $\varphi(A)$, as claimed in (1.26). In particular, (1.26) implies that for all functions φ that are also continuous on $\operatorname{supp} P$ one has

$$\sigma(\varphi(A)) = \overline{\varphi(\sigma(A))}.$$

We now return to the study of orthogonal sums of operators. Let $\mathcal{N}$ be a countable (possibly finite) index set and, for each $n \in \mathcal{N}$, assume that $\mathcal{H}_n$ is a separable Hilbert space. Moreover, for each $n \in \mathcal{N}$, let A_n be a self-adjoint operator in $\mathcal{H}_n$. According to our discussion in §1.1.2, we know that $\bigoplus_{n\in\mathcal{N}} A_n$ is self-adjoint in $\bigoplus_{n\in\mathcal{N}} \mathcal{H}_n$. We denote by $P(A_n)$ the spectral measure of A_n and define a map $\bigoplus_{n\in\mathcal{N}} P(A_n)$ from Borel sets in $\mathbb{R}$ to operators on $\bigoplus_{n\in\mathcal{N}} \mathcal{H}_n$ by

$$\Big(\bigoplus_{n\in\mathcal{N}} P(A_n)\Big)_\omega := \bigoplus_{n\in\mathcal{N}} P_\omega(A_n) \qquad \text{for any Borel set } \omega \subset \mathbb{R} \,.$$

This is clearly a projection-valued measure.

Lemma 1.12 *One has*

$$\sigma\Big(\bigoplus_{n\in\mathcal{N}} A_n\Big) = \overline{\bigcup_{n\in\mathcal{N}} \sigma(A_n)}$$

and $\bigoplus_{n\in\mathcal{N}} P(A_n)$ *is the spectral measure of* $\bigoplus_{n\in\mathcal{N}} A_n$.

Proof According to (1.24), we have $\|(A_n - z)^{-1}\| = (\operatorname{dist}(z, \sigma(A_n)))^{-1}$, so by Lemma 1.5 the first assertion follows from the fact that

$$\sup_{n\in\mathcal{N}} \big(\operatorname{dist}(z, \sigma(A_n))\big)^{-1} = \Big(\inf_{n\in\mathcal{N}} \operatorname{dist}(z, \sigma(A_n))\Big)^{-1} = \left(\operatorname{dist}\left(z, \overline{\bigcup_{n\in\mathcal{N}} \sigma(A_n)}\right)\right)^{-1},$$

which is valid for all $z \in \bigcap_{n\in\mathcal{N}} \rho(A_n)$. For the second assertion we first check that $\bigoplus_{n\in\mathcal{N}} P(A_n)$ is a projection-valued measure. We then verify the defining relations in part (a) of Theorem 1.8. □

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1.1.7 The essential spectrum and Weyl's theorem

Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with spectral measure P. Let us define the *essential spectrum* $\sigma_{\text{ess}}(A)$ and the *discrete spectrum* $\sigma_{\text{disc}}(A)$ of A by

$$\begin{aligned}\sigma_{\text{ess}}(A) &:= \left\{\lambda \in \mathbb{R}\colon\ \dim\operatorname{ran} P_{(\lambda-\varepsilon,\lambda+\varepsilon)} = \infty \text{ for all } \varepsilon > 0\right\},\\ \sigma_{\text{disc}}(A) &:= \sigma(A) \setminus \sigma_{\text{ess}}(A)\\ &= \left\{\lambda \in \sigma(A)\colon\ \dim\operatorname{ran} P_{(\lambda-\varepsilon,\lambda+\varepsilon)} < \infty \text{ for some } \varepsilon > 0\right\}.\end{aligned}$$

It is easy to prove that $\lambda \in \sigma_{\text{disc}}(A)$ if and only if λ is an isolated point of $\sigma(A)$ (i.e., $\sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}$ for some $\varepsilon > 0$) and λ is an eigenvalue of finite multiplicity. Moreover, one can prove that $\lambda \in \sigma_{\text{ess}}(A)$ if and only if one or more of the following statements hold: $\operatorname{ran}(A - \lambda)$ is not closed; or λ is an accumulation point of eigenvalues; or λ is an eigenvalue of infinite multiplicity.

In practice, it is useful to have a criterion of whether a point belongs to the essential spectrum of A that is not expressed through the spectral measure of A, but rather through A itself. To state such a criterion, we shall say that a sequence $(u_n) \subset \mathcal{H}$ is a *singular sequence* for A at a point $\lambda \in \mathbb{R}$ if the following conditions are satisfied:

$$\inf_n \|u_n\| > 0\,, \tag{1.27}$$

$$u_n \to 0 \text{ weakly in } \mathcal{H}\,, \tag{1.28}$$

$$u_n \in \operatorname{dom} A\,, \tag{1.29}$$

$$(A - \lambda)u_n \to 0 \text{ strongly in } \mathcal{H}\,. \tag{1.30}$$

Lemma 1.13 *A point $\lambda \in \mathbb{R}$ belongs to $\sigma_{\text{ess}}(A)$ if and only if there is a singular sequence for A at λ.*

Proof First, let $\lambda \in \sigma_{\text{ess}}(A)$ and let (ε_n) be a decreasing sequence of positive numbers tending to zero. Then, by the definition of the essential spectrum, there is an orthonormal system (u_n) with $u_n \in \operatorname{ran} P_{(\lambda-\varepsilon_n,\lambda+\varepsilon_n)}$ for all n. Then (1.27), (1.28) and (1.29) are clearly satisfied and (1.30) follows from

$$\begin{aligned}\|(A-\lambda)u_n\|^2 &= \int_{(\lambda-\varepsilon_n,\lambda+\varepsilon_n)} (\mu-\lambda)^2 d(P_\mu u_n, u_n)\\ &\le \varepsilon_n^2 \int_{(\lambda-\varepsilon_n,\lambda+\varepsilon_n)} d(P_\mu u_n, u_n) = \varepsilon_n^2 \to 0\,.\end{aligned}$$

Conversely, assume that there is a singular sequence (u_n) for A at λ. We argue by contradiction assuming that $\lambda \notin \sigma_{\text{ess}}(A)$. Then $\dim\operatorname{ran} P_{(\lambda-\varepsilon,\lambda+\varepsilon)} < \infty$ for some $\varepsilon > 0$.

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Let $v_n := u_n - P_{(\lambda-\varepsilon,\lambda+\varepsilon)}u_n$. Then, by (1.25),

$$\varepsilon\|v_n\| \leq \|(A-\lambda)v_n\| = \|P_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon)}(A-\lambda)u_n\| \leq \|(A-\lambda)u_n\| .$$

Using (1.30), we deduce that $v_n \to 0$ in $\mathcal{H}$. Since $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}$ has finite rank, (1.28) implies that $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}u_n \to 0$ in $\mathcal{H}$. Thus $u_n = v_n + P_{(\lambda-\varepsilon,\lambda+\varepsilon)}u_n \to 0$ in $\mathcal{H}$. This contradicts (1.27) and completes the proof. □

The following theorem is due to Weyl and states the stability of the essential spectrum under certain perturbations. This is very useful in practice since it reduces the computation of the essential spectrum for general operators to that for certain model operators. It is an example of a perturbation-theoretic result.

Theorem 1.14 *Let A_1 and A_2 be self-adjoint operators and assume that, for some $z \in \rho(A_1) \cap \rho(A_2)$,*

$$(A_1 - z)^{-1} - (A_2 - z)^{-1} \qquad \textit{is compact}\,. \tag{1.31}$$

Then $\sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(A_2)$.

Note that assumption (1.31) holds for bounded A_1, A_2 with $A_1 - A_2$ compact, since

$$(A_1-z)^{-1}-(A_2-z)^{-1} = -(A_2-z)^{-1}(A_1-A_2)(A_1-z)^{-1}\,, \quad z \in \rho(A_1)\cap\rho(A_2)\,. \tag{1.32}$$

In our applications it is crucial, however, that it suffices that the compactness holds only for the resolvent difference, rather than the operator difference. A convenient way to verify this condition in terms of quadratic forms will be given in Theorem 1.51.

We note that if (1.31) holds for *some* $z \in \rho(A_1) \cap \rho(A_2)$, then it holds for *any* such z. Indeed, $D(\zeta) := (A_1 - \zeta)^{-1} - (A_2 - \zeta)^{-1}$ satisfies

$$D(z') = \big((A_1 - z)(A_1 - z')^{-1}\big)D(z)\big((A_2 - z)(A_2 - z')^{-1}\big).$$

Since the factors $(A_j - z)(A_j - z')^{-1}$ are bounded, compactness of $D(z)$ implies compactness of $D(z')$.

Proof Since the assertion is symmetric in the operators A_1 and A_2, it suffices to prove that $\sigma_{\mathrm{ess}}(A_1) \subset \sigma_{\mathrm{ess}}(A_2)$. Let $\lambda \in \sigma_{\mathrm{ess}}(A_1)$. Then, by Lemma 1.13, there is a singular sequence (u_n) for A_1 at λ. With z from (1.31), let

$$v_n := (A_2 - z)^{-1}(A_1 - z)u_n,$$

which is well defined because of (1.29). We would like to show that (v_n) is a singular sequence for A_2 at λ. Once this is done, the theorem follows again from Lemma 1.13.

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Obviously, (1.29) is satisfied for v_n and A_2.

Let us verify (1.27) and (1.28). With $K := (A_2 - z)^{-1} - (A_1 - z)^{-1}$, we have

$$v_n = K(A_1 - z)u_n + u_n = K(A_1 - \lambda)u_n + (\lambda - z)Ku_n + u_n\,.$$

Because of (1.28) and (1.30) for u_n and A_1, and Lemma 1.1, we have $v_n - u_n \to 0$ strongly in $\mathcal{H}$. Therefore, (1.27) and (1.28) for u_n imply (1.27) and (1.28) for v_n.

Finally, in order to verify (1.30), we compute

$$(A_2 - \lambda)v_n = (A_1 - z)u_n + (z - \lambda)v_n = (A_1 - \lambda)u_n + (z - \lambda)(v_n - u_n)\,.$$

By (1.30) for u_n and A_1, and by the fact that $v_n - u_n \to 0$ strongly in $\mathcal{H}$, we obtain (1.30) for v_n and A_2. This completes the proof. □

Here is a consequence of the spectral theorem for compact operators.

Lemma 1.15 *Let K be a bounded, self-adjoint operator in $\mathcal{H}$ with* $\dim \mathcal{H} = \infty$. *Then K is compact if and only if $\sigma_{\mathrm{ess}}(K) = \{0\}$. In this case, $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of K.*

Proof First, assume that K is compact. By (1.32), the assumption (1.31) in Weyl's theorem holds with $A_1 = K$ and $A_2 = 0$. Thus, the theorem implies that $\sigma_{\mathrm{ess}}(K) = \sigma_{\mathrm{ess}}(0) = \{0\}$. This means that the spectrum of K away from zero consists of isolated eigenvalues of finite multiplicities. Thus if $(\lambda_n^{\pm})$ denotes the positive and negative eigenvalues of K, *not* repeated according to multiplicities, then by Corollary 1.10, the spectral measure P of K has support

$$\{\lambda_n^+ : n\} \cup \{\lambda_n^- : n\} \cup \{0\}$$

and $\operatorname{ran} P_{\lambda_n^{\pm}}$ are the eigenspaces corresponding to non-zero eigenvalues. The defining properties of a projection-valued measure give that, in the sense of strong convergence,

$$P_{\{0\}} + \sum_n P_{\{\lambda_n^+\}} + \sum_n P_{\{\lambda_n^-\}} = 1\,.$$

This identity implies the completeness of an orthonormal basis obtained by combining orthonormal bases in the range of each of these projections.

Conversely, assume that $\sigma_{\mathrm{ess}}(K) = \{0\}$. To prove compactness of K, assume that $u_n \to 0$ weakly in $\mathcal{H}$. Let $\tau > 0$ and $P := \chi_{(-\tau,\tau)}(K)$. Since $\sigma_{\mathrm{ess}}(K) = \{0\}$, the operator $P^{\perp} := 1 - P$ has finite rank. Therefore, the weak convergence $u_n \to 0$ implies the strong convergence $P^{\perp}Ku_n \to 0$. On the other hand, by the spectral theorem, $\|PKu_n\| \le \tau\|u_n\|$. Since, by orthogonality,

$$\|Ku_n\|^2 = \|P^{\perp}Ku_n\|^2 + \|PKu_n\|^2,$$

we find that $\limsup_{n\to\infty} \|Ku_n\|^2 \le \tau^2 C$ with $C := \sup_n \|u_n\|^2$. Note that C is

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finite by uniform boundedness. Since $\tau > 0$ can be chosen arbitrarily small, we conclude that $Ku_n \to 0$ strongly in $\mathcal{H}$. Thus, by Lemma 1.1, K is compact. □

1.2 Semibounded operators and forms, and the variational principle

1.2.1 Semibounded operators and forms

A *sesquilinear form* $a[\cdot,\cdot]$ in $\mathcal{H}$ is a map from its domain $d[a] \times d[a]$ that is linear in its first and anti-linear in its second argument, where $d[a]$ is a subspace of $\mathcal{H}$. It is called *symmetric* if

$$a[u,v] = \overline{a[v,u]} \text{ for all } u,v \in d[a],$$

and it is called *densely defined* if $d[a]$ is dense in $\mathcal{H}$.

The *quadratic form* $a[\cdot]$ associated to a sesquilinear form $a[\cdot,\cdot]$ is defined by

$$a[u] := a[u,u] \text{ for all } u \in d[a].$$

The sesquilinear form $a[\cdot,\cdot]$ can be recovered from its quadratic form in view of the polarization identity

$$a[u,v] = \frac{1}{4}\left(a[u+v] - a[u-v] + ia[u+iv] - ia[u-iv]\right) \text{ for all } u,v \in d[a].$$

The quadratic form $a[\cdot]$ is real-valued if and only if $a[\cdot,\cdot]$ is symmetric.

A quadratic form $a = a[\cdot]$ with domain $d[a]$ is called *lower semibounded* in a Hilbert space $\mathcal{H}$ if it is real-valued and

$$m_a := \inf_{0 \neq u \in d[a]} \frac{a[u]}{\|u\|^2} > -\infty.$$

In this case, for each $m > -m_a$, the expression

$$a[u,v] + m(u,v)$$

defines an inner product on $d[a]$ and for different $m > -m_a$ the corresponding norms are equivalent.

A lower semibounded quadratic form a with domain $d[a]$ is called *closed* in a Hilbert space $\mathcal{H}$ if, for some (and hence any) $m > -m_a$, the set $d[a]$ is complete with respect to the norm

$$\left(a[u] + m\|u\|^2\right)^{1/2}.$$

A *form core* is a subspace $F \subset d[a]$ that is dense with respect to the norm $\left(a[u] + m\|u\|^2\right)^{1/2}$ in $d[a]$.

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Let us discuss the relation between lower semibounded quadratic forms and lower semibounded operators. The latter notion was introduced before Corollary 1.11 and we recall the notation

$$m_A := \inf_{0 \neq u \in \operatorname{dom} A} \frac{(Au, u)}{\|u\|^2} > -\infty .$$

Theorem 1.16 *Let A be a self-adjoint, lower semibounded operator. Then there is a unique lower semibounded, closed quadratic form a satisfying*

$$\operatorname{dom} A \subset d[a], \tag{1.33}$$

$$a[u, v] = (Au, v) \qquad \textit{for all } u \in \operatorname{dom} A, \ v \in d[a]. \tag{1.34}$$

The form a is densely defined, $\operatorname{dom} A$ *is a form core of a, and $m_a = m_A$. Moreover, for all $m \geq -m_A$, one has $d[a] = \operatorname{dom}(A+m)^{1/2}$ and*

$$a[u, v] = ((A+m)^{1/2}u, (A+m)^{1/2}v) - m(u, v) \qquad \textit{for all } u, v \in d[a], \tag{1.35}$$

and, if P is the spectral measure of A, then

$$a[u, v] = \int_{\mathbb{R}} \lambda \, d(P_\lambda u, v) \qquad \textit{for all } u, v \in d[a]. \tag{1.36}$$

In the following, for brevity, we refer to a described in Theorem 1.16 as the quadratic form *corresponding* to A.

We begin with a technical lemma.

Lemma 1.17 *Let A be a self-adjoint, lower semibounded operator and let a be a lower semibounded, closed quadratic form satisfying* (1.33) *and* (1.34). *Then a is densely defined,* $\operatorname{dom} A$ *is a form core of a, and $m_a = m_A$.*

Proof Since A is densely defined, (1.33) implies that a is densely defined. By (1.33) and (1.34) we have $a[u] = (Au, u)$ for all $u \in \operatorname{dom} A$. Thus, by enlarging the set over which the infimum is taken, we see that $m_A \geq m_a$.

Let us show that $\operatorname{dom} A$ is a form core of a. Indeed, if $v \in d[a]$ satisfies $a[u, v] + m(u, v) = 0$ for all $u \in \operatorname{dom} A$ and some fixed $m > -m_a$, then, by (1.34), $v \in (\operatorname{ran}(A+m))^\perp = \ker(A+m) = \{0\}$. Here we used (1.1), the assumption that A is self-adjoint, and the fact that, by Corollary 1.11, $m > -m_a \geq -m_A = -\inf \sigma(A)$.

Since the infimum defining m_a remains the same when restricted to a form core, we conclude from the above facts that $m_A = m_a$. □

Proof of Theorem 1.16 We fix $m \geq -m_A$ and use (1.35) to define a symmetric sesquilinear form a with domain $d[a] := \operatorname{dom}(A+m)^{1/2}$. Note that by Corollary 1.11 we have $\inf \sigma(A+m) \geq 0$, and therefore the square root $(A+m)^{1/2}$ is well defined as a self-adjoint, non-negative operator by the functional calculus

(Theorem 1.8). The quadratic form a is closed since the operator $(A+m)^{1/2}$ is closed.

Let us show (1.33) and (1.34). The former follows from (1.22) applied to $\varphi(\lambda)=\sqrt{\lambda+m}$ and $\varphi(\lambda)=\lambda$. Before proving (1.34), we note that (1.36) follows from (1.15) for $u=v$ and then by polarization for general u, v. The identity in (1.34) follows from (1.36) by the spectral theorem (Theorem 1.9).

The facts that a is densely defined, that $\operatorname{dom} A$ is a form core, and that $m_a = m_A$ all follow from Lemma 1.17.

To show uniqueness, and, in particular, independence of the parameter $m \geq -m_A$, let $\tilde{a}$ be a lower semibounded, closed quadratic form satisfying (1.33) and (1.34). Then, by these properties for a and $\tilde{a}$,

$$\tilde{a}[u,v] = (Au,v) = a[u,v] \qquad \text{for all } u,v \in \operatorname{dom} A\,. \tag{1.37}$$

By Lemma 1.17, $\operatorname{dom} A$ is a form core of both a and $\tilde{a}$. Therefore, (1.37) and the closedness of a and $\tilde{a}$ imply that $d[a]=d[\tilde{a}]$ and $\tilde{a}[u,v]=a[u,v]$ for all u,v from this set. Thus $a=\tilde{a}$. □

The usefulness of lower semibounded quadratic forms comes from the fact that the above construction can be reversed. That is, each densely defined, lower semibounded, and closed quadratic form gives rise to a unique self-adjoint operator. The precise statement is the following.

Theorem 1.18 *Let a be a densely defined, lower semibounded, and closed quadratic form. Then there is a unique self-adjoint operator A satisfying* (1.33) *and* (1.34). *Moreover, A is lower semibounded with $m_A = m_a$ and the domain of A is given by*

$$\begin{aligned}\operatorname{dom} A = \{u \in d[a] \colon &\text{ there exists } f \in \mathcal{H} \text{ such that for all } v \in d[a]\,,\\ &a[u,v] = (f,v)\}.\end{aligned}$$

In the following, we briefly refer to A described in Theorem 1.18 as the operator *corresponding* to a.

Proof We fix $m > -m_a$. We have for any $g \in \mathcal{H}$ and any $v \in d[a]$,

$$|(g,v)| \leq \|g\|\|v\| \leq (m_a+m)^{-1/2}\|g\| \left(a[v]+m\|v\|^2\right)^{1/2}. \tag{1.38}$$

This means that for fixed $g \in \mathcal{H}$, $v \mapsto (g,v)$ is a bounded, anti-linear functional on $d[a]$ endowed with the norm $\left(a[\cdot]+m\|\cdot\|^2\right)^{1/2}$. Therefore, by the Riesz representation theorem, there is a unique $u_g \in d[a]$ such that

$$(g,v) = a[u_g,v] + m(u_g,v) \quad \text{for all } v \in d[a]\,. \tag{1.39}$$

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The uniqueness of u_g implies that the map $g \mapsto u_g$ is linear. Moreover, taking $v = u_g$ in both (1.38) and (1.39), we obtain

$$a[u_g] + m\|u_g\|^2 \le (m_a + m)^{-1/2}\|g\| \left(a[u_g] + m\|u_g\|^2\right)^{1/2};$$

that is, $a[u_g] + m\|u_g\|^2 \le (m_a + m)^{-1}\|g\|^2$. This means that the map $g \mapsto u_g$ is bounded from $\mathcal{H}$ to $d[a]$. Since the embedding $d[a] \to \mathcal{H}$ is continuous, we can consider the mapping $g \mapsto Bg := u_g$ as a bounded linear operator on $\mathcal{H}$. Taking $v = u_g$ in (1.39) gives

$$(g, Bg) = a[u_g] + m\|u_g\|^2 \ge (m_a + m)\|u_g\|^2 \ge 0, \tag{1.40}$$

and therefore B is a self-adjoint operator on $\mathcal{H}$. Its kernel is trivial since, by (1.39), $u_g = 0$ implies $g = 0$. Moreover, since, by (1.2), $\overline{\operatorname{ran} B} = (\ker B)^\perp = \mathcal{H}$, its range is dense in $\mathcal{H}$.

These facts imply that $A := B^{-1} - m$ can be defined as a possibly unbounded operator with domain $\operatorname{dom} A := \operatorname{ran} B$. By (1.3), A is self-adjoint. By construction, we have $\operatorname{dom} A \subset d[a]$ and, for $u \in \operatorname{dom} A$ and $v \in d[a]$, we can apply (1.39) to $g := (A + m)u$ and get, using $Bg = u$,

$$\begin{aligned}(Au, v) &= (g, v) - m(u, v) = a[u_g, v] + m(u_g, v) - m(u, v)\\ &= a[Bg, v] + m(Bg, v) - m(u, v) = a[u, v].\end{aligned}$$

This means that a is associated to A in the sense of (1.33) and (1.34).

It follows from (1.33), (1.34), and the lower semiboundedness of a that A is semibounded, and then from Lemma 1.17 that $m_A = m_a$. The formula for $\operatorname{dom} A$ follows easily from this construction. Indeed, the element g in $\operatorname{dom} A = \{u_g : g \in \mathcal{H}\}$ and the element f in the formula in the theorem are related by $f = g - m\,u_g$.

Finally, to show uniqueness, let A_1 and A_2 be self-adjoint, lower semibounded operators satisfying (1.33) and (1.34), and let $v \in \operatorname{dom} A_1$. Then on the one hand, by (1.34) and the symmetry of a, we have for all $u \in d[a]$, $(u, A_1 v) = a[u, v]$ and, on the other hand, by Theorem 1.16, we have $d[a] = \operatorname{dom}(A_2+m)^{1/2}$ and for all u from this set, $\left((A_2 + m)^{1/2}u, (A_2 + m)^{1/2}v\right) - m(u, v) = a[u, v]$. Thus,

$$\left(u, (A_1 + m)v\right) = \left((A_2 + m)^{1/2}u, (A_2 + m)^{1/2}v\right) \quad \text{for all } u \in \operatorname{dom}(A_2 + m)^{1/2}.$$

By the definition of the adjoint operator, this means that $(A_2 + m)^{1/2}v \in \operatorname{dom}\left((A_2 + m)^{1/2}\right)^*$ and $\left((A_2 + m)^{1/2}\right)^*(A_2 + m)^{1/2}v = (A_1 + m)v$. Since A_2 is self-adjoint, using (1.18) we find $v \in \operatorname{dom} A_2$ and $(A_2 + m)v = (A_1 + m)v$; that is, $A_2 v = A_1 v$. Interchanging the roles of A_1 and A_2, we also infer that, if $v \in \operatorname{dom} A_2$, then $v \in \operatorname{dom} A_1$ and $A_1 v = A_2 v$. Thus, $A_1 = A_2$, as claimed. □

It is important in applications to describe spectral properties of A in terms of the quadratic form a. The following lemma concerns the bottom of the spectrum of A.

Lemma 1.19 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then*

$$\inf \sigma(A) = m_a \,.$$

Moreover, m_a is an eigenvalue of A if and only if the infimum

$$\inf_{0 \neq u \in d[a]} a[u]/\|u\|^2$$

is a minimum, and eigenvectors are precisely those vectors for which the infimum is attained.

Proof The equality follows immediately from $m_a = m_A$ and Corollary 1.11. Moreover, given the characterization of minimizers there, it remains to show that, if $u_0 \in d[a]$ is a minimizer for $\inf_{0 \neq u \in d[a]} a[u]/\|u\|^2$, then $u_0 \in \operatorname{dom} A$. This follows from (1.36), which implies $u_0 = P_{\{m_a\}} u_0$ and, therefore, $u_0 \in \operatorname{dom} A$. □

Next, we characterize the bottom of the essential spectrum.

Lemma 1.20 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then*

$$\inf \sigma_{\mathrm{ess}}(A) \\ = \inf \left\{ \liminf_{n \to \infty} a[u_n] \colon (u_n) \subset d[a] \,,\ \|u_n\| = 1 \,,\ u_n \to 0 \textit{ weakly in } \mathcal{H} \right\}$$

with the convention that $\inf \emptyset = +\infty$.

Proof Let us denote the left and right sides by Λ and $\tilde{\Lambda}$, respectively. For the proof of $\Lambda \geq \tilde{\Lambda}$ we may assume that $\Lambda < \infty$. Then, since $\sigma_{\mathrm{ess}}(A)$ is closed, $\Lambda \in \sigma_{\mathrm{ess}}(A)$ and, by Lemma 1.13, there is a sequence $(u_n) \subset \operatorname{dom} A \subset d[a]$ such that $\|u_n\| = 1$, $u_n \to 0$ weakly in $\mathcal{H}$, and $(A - \Lambda)u_n \to 0$ strongly in $\mathcal{H}$. Thus $a[u_n] - \Lambda = \big((A - \Lambda)u_n, u_n\big) \to 0$, which implies $\tilde{\Lambda} \leq \Lambda$.

For the proof of $\Lambda \leq \tilde{\Lambda}$, let $(u_n) \subset d[a]$ with $\|u_n\| = 1$ and $u_n \to 0$ weakly in $\mathcal{H}$. We will use the fact that for any $u \in d[a]$ and any Borel $\omega \subset \mathbb{R}$, one has $P_\omega u \in d[a]$ and

$$a[P_\omega u] = \int_\omega \mu \, d(P_\mu u, u) \,. \tag{1.41}$$

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Indeed, this follows from (1.15), (1.18) and Theorem 1.16. Let $\lambda \in \mathbb{R}$ with $\lambda < \Lambda$ and set $P := P_{(-\infty,\lambda)}$ and $P^{\perp} := 1 - P$. Then, by the above fact,

$$\begin{aligned} a[u_n] &= a[Pu_n] + \int_{[\lambda,\infty)} \mu \, d(P_\mu u_n, u_n) \geq a[Pu_n] + \lambda \|P^{\perp} u_n\|^2 \\ &= \big((PAP)u_n, u_n\big) + \lambda\big(1 - (Pu_n, u_n)\big). \end{aligned}$$

Since $\lambda < \Lambda$, the operators PAP and P have finite rank and, thus, by the assumed weak convergence, $\big((PAP)u_n, u_n\big) \to 0$ and $(Pu_n, u_n) \to 0$. This proves that $\liminf_{n\to\infty} a[u_n] \geq \lambda$. Since $\lambda < \Lambda$ is arbitrary, this proves $\tilde{\Lambda} \geq \Lambda$. □

We say that the operator A has *discrete spectrum* if the essential spectrum of A is empty. This is a slight abuse of terminology and means not only that the set $\sigma(A)$ is a discrete subset of $\mathbb{R}$, but also that the corresponding eigenvalue multiplicities are finite. In other words, the operator A has discrete spectrum if and only if $\dim \operatorname{ran} P_\omega < \infty$ for any compact interval $\omega \subset \mathbb{R}$. We say that A has a discrete spectrum in an interval $\omega \subset \mathbb{R}$ if $\sigma_{\mathrm{ess}}(A) \cap \omega = \emptyset$.

According to Lemma 1.20, A has discrete spectrum if and only if any sequence $(u_n) \subset d[a]$, with $\|u_n\| = 1$ and $u_n \to 0$ weakly in $\mathcal{H}$, satisfies $a[u_n] \to \infty$. The following corollary gives a necessary and sufficient condition in terms of the *embedding operator* $\mathcal{J} : d[a] \to \mathcal{H}$, which maps every $u \in d[a]$ to itself as an element in $\mathcal{H}$. This is a bounded operator when $d[a]$ is equipped with its norm $\sqrt{a[u] + m\|u\|^2}$ for some $m > -m_a$. By definition, this operator is compact if the closed unit ball in $d[a]$ is relatively compact in $\mathcal{H}$. As in Lemma 1.1, this compactness is equivalent to the assertion that every weakly convergent sequence in $d[a]$ converges strongly in $\mathcal{H}$.

Corollary 1.21 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then A has discrete spectrum if and only if the embedding from $d[a]$ to $\mathcal{H}$ is compact.*

Proof Assuming that A has discrete spectrum and that $(v_n) \subset d[a]$ with $v_n \to 0$ weakly in $d[a]$, we need to show that $v_n \to 0$ strongly in $\mathcal{H}$. If this was not the case, then, after passing to a subsequence, we may assume that $(\|v_n\|)$ converges to a positive constant. Then $u_n := v_n/\|v_n\| \to 0$ weakly in $d[a]$ and, since the embedding from $d[a]$ to $\mathcal{H}$ is continuous, also weakly in $\mathcal{H}$. By Lemma 1.20, we infer that $a[u_n] \to \infty$, which contradicts the boundedness of a weakly convergent sequence in $d[a]$.

Conversely, assuming that the embedding from $d[a]$ to $\mathcal{H}$ is compact and that $(u_n) \subset d[a]$ with $\|u_n\| = 1$ and $u_n \to 0$ weakly in $\mathcal{H}$, by Lemma 1.20, we need to show that $a[u_n] \to \infty$. If this was not the case, then, after passing to a subsequence, we may assume that $(a[u_n])$ converges. In particular, (u_n)

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is bounded in $d[a]$ and, by the compactness of the embedding, after passing to a subsequence, we may assume that (u_n) converges strongly in $\mathcal{H}$. Its limit coincides necessarily with its weak limit, which is zero, but this contradicts $\|u_n\| = 1$ for all n. □

As a final topic in this subsection, we discuss orthogonal sums of operators via quadratic forms. Let $\mathcal{N}$ be a countable (possibly finite) index set and, for each $n \in \mathcal{N}$, let $\mathcal{H}_n$ be a separable Hilbert space. For each $n \in \mathcal{N}$, let a_n be a densely defined, lower semibounded and closed quadratic form in $\mathcal{H}_n$ with lower bound m_{a_n} satisfying

$$M := \inf_{n\in\mathcal{N}} m_{a_n} > -\infty \,.$$

Then we define a quadratic form a in $\bigoplus_{n\in\mathcal{N}} \mathcal{H}_n$ with domain

$$d[a] := \Big\{u \in \bigoplus_{n\in\mathcal{N}} \mathcal{H}_n \colon u_n \in d[a_n] \text{ for all } n \in \mathcal{N}\,,\ \sum_{n\in\mathcal{N}} a[u_n] < \infty\Big\}$$

by

$$a[u] := \sum_{n\in\mathcal{N}} a[u_n] \qquad \text{for all } u \in d[a]\,.$$

Note that the sum converges absolutely since $M > -\infty$. It is easy to see that a is densely defined, lower semibounded with lower bound $m_a = M$, and closed. Therefore, by Theorem 1.18, it generates a unique self-adjoint and lower semibounded operator A in $\bigoplus_{n\in\mathcal{N}} \mathcal{H}_n$. Similarly, for each $n \in \mathcal{N}$, a_n generates a unique self-adjoint and lower semibounded operator A_n in $\mathcal{H}_n$. By verifying (1.33) and (1.34), we see that

$$A = \bigoplus_{n\in\mathcal{N}} A_n \,. \tag{1.42}$$

1.2.2 The operators T^*T and TT^*, and the polar decomposition

Let T be a densely defined, closed operator in a Hilbert space $\mathcal{H}$. Consider the quadratic form $a[u] := \|Tu\|^2$ with domain $d[a] := \operatorname{dom} T$. This form is densely defined, non-negative and closed. Hence, by Theorem 1.18, it induces a self-adjoint non-negative operator A with

$$\operatorname{dom} A = \big\{u \in \operatorname{dom} T\colon \text{there exists } f \in \mathcal{H} \text{ such that for all } v \in \operatorname{dom} T\,,\\ (Tu, Tv) = (f, v)\big\}\,.$$

The f here is unique and one has $f = Au$. This means $Tu \in \operatorname{dom} T^*$ and $Au = T^*Tu$ for all $u \in \operatorname{dom} A$, which means that $A = T^*T$ in the sense of composition of unbounded operators. We write $A = T^*T$ in the following.

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Since T^*T is self-adjoint and non-negative, its square root is defined by the spectral theorem. The operator

$$|T| = (T^*T)^{1/2}$$

is called the *absolute value* of T. It is the unique self-adjoint, non-negative operator on $\mathcal{H}$ with

$$\||T|f\| = \|Tf\| \quad \text{for any} \quad f \in \operatorname{dom} T = \operatorname{dom} |T|\,. \tag{1.43}$$

Indeed, since by its definition $|T|^2 = T^*T$, the corresponding quadratic forms $\||T|f\|^2$ and $\|Tf\|^2$ coincide, including equality of their respective domains. The uniqueness follows from the uniqueness of the positive square root of a non-negative operator and the uniqueness in Theorem 1.18.

Any complex number z has a polar representation $z = e^{i\varphi}r$ with $r = |z| \geq 0$ and $\varphi = \arg z \in [0, 2\pi)$. In this subsection we present an analogous representation of an operator in a Hilbert space.

The operator $|T|$ will correspond to the 'radial part' in the polar decomposition of T. The following theorem describes the 'angular part' of this decomposition.

Proposition 1.22 *Let T be a densely defined, closed operator. Then there is a unique bounded operator U such that $T = U|T|$ and*

$$\|Uf\| = \|f\| \quad \textit{for all} \quad f \in (\ker T)^\perp\,, \tag{1.44}$$

$$\ker U = \ker T\,, \tag{1.45}$$

$$\operatorname{ran} U = \overline{\operatorname{ran} T}\,. \tag{1.46}$$

Proof We define $U : \operatorname{ran} |T| \to \operatorname{ran} T$ by $U|T|f := Tf$. According to (1.43), the operator U is well defined, norm-preserving and maps onto $\operatorname{ran} T$. Thus, U extends to a unitary operator from $\overline{\operatorname{ran} |T|}$ to $\overline{\operatorname{ran} T}$ and, extending U by zero to $(\operatorname{ran} |T|)^\perp$, we obtain a bounded operator on $\mathcal{H}$ satisfying (1.46).

To prove (1.44) and (1.45), it remains to show that $\overline{\operatorname{ran} |T|} = (\ker T)^\perp$. Indeed, using the self-adjointness of $|T|$, we have $\overline{\operatorname{ran} |T|} = (\ker |T|)^\perp$. Again by (1.43), we have $\ker |T| = \ker T$. This proves the existence of U with the claimed properties.

Uniqueness of U comes from the fact that the extension of U from $\operatorname{ran} |T|$ to $\overline{\operatorname{ran} |T|}$ is unique. □

Because T is densely defined and closed, we have $T^{**} = T$, and the same construction as at the beginning of this subsection leads to the self-adjoint operator TT^* defined on $\{u \in \operatorname{dom} T^* : T^*u \in \operatorname{dom} T\}$ corresponding to the quadratic form $\|T^*u\|^2$ defined on $\operatorname{dom} T^*$. Our next result compares the operators T^*T and TT^* away from their respective kernels.

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Proposition 1.23 *Let T be a densely defined, closed operator. The operator T^*T restricted to $(\ker T^*T)^\perp$ is unitarily equivalent to the operator TT^* restricted to $(\ker TT^*)^\perp$.*

Proof Let $T = U|T|$ be the polar decomposition of T. First, we observe that $T^* = |T|U^*$ with $\operatorname{dom} T^* = \{f \in \mathcal{H} : U^* f \in \operatorname{dom} |T|\}$. Indeed, by the definition of the adjoint, we have

$$\begin{aligned}\operatorname{dom} T^* &= \{f \in \mathcal{H} : \text{there is a } g \in \mathcal{H} \text{ such that} \\ &\qquad\qquad \text{for all } h \in \operatorname{dom} |T| \text{ one has } (g, h) = (U^* f, |T|h)\} \\ &= \{f \in \mathcal{H} : U^* f \in \operatorname{dom} |T|^*\} .\end{aligned}$$

Since $|T|^* = |T|$, this means $\operatorname{dom} |T|U^* = \operatorname{dom} T^*$ and $T^* f = |T|U^* f$ for f in this set.

Note that, because of $\operatorname{dom} |T| = \operatorname{dom} T$, we also have $\operatorname{dom} T^* = \operatorname{dom} TU^*$. By the formula for T^* and by (1.43), we have

$$\|T^* u\| = \||T|U^* u\| = \|TU^* u\| \quad \text{for all } u \in \operatorname{dom} T^* = \operatorname{dom} TU^* .$$

This means $TT^* = (TU^*)^* TU^*$. The same argument as before implies $(UT^*)^* = TU^*$. Since UT^* is closed, this gives $(TU^*)^* = UT^*$, and therefore

$$TT^* = UT^* TU^* . \tag{1.47}$$

According to Proposition 1.22, the operator U can be restricted to a unitary operator $V : (\ker T)^\perp \to \overline{\operatorname{ran} T}$. Consequently, we have a unitary operator $V^* : \overline{\operatorname{ran} T} \to (\ker T)^\perp$. Since $\ker TT^* = \ker T^* = (\operatorname{ran} T)^\perp$ and $\ker T^*T = \ker T$, we can restrict (1.47) to

$$TT^*|_{(\ker TT^*)^\perp} = V \left(T^*T|_{(\ker T^*T)^\perp}\right) V^* .$$

This provides the claimed unitary equivalence. □

Corollary 1.24 *Let T be a densely defined, closed operator and $\lambda \neq 0$. Then*

$$\dim \ker(T^*T - \lambda) = \dim \ker(TT^* - \lambda) .$$

The above proof shows that, if u is an eigenvector for T^*T corresponding to an eigenvalue $\lambda \neq 0$, then Tu is non-zero and an eigenvector for TT^* corresponding to the eigenvalue λ.

1.2.3 The variational principle

The goal of this subsection is to prove the variational principle. It translates the problem of counting eigenvalues below the bottom of the essential spectrum of a self-adjoint, lower semibounded operator into a problem for the corresponding quadratic form.

Let A be a self-adjoint, lower semibounded operator. Then the spectrum of A below $\inf \sigma_{\mathrm{ess}}(A) \in (-\infty, \infty]$ is discrete. If it is not empty, this portion of the spectrum consists of eigenvalues of finite multiplicities that may accumulate only at the value $\inf \sigma_{\mathrm{ess}}(A)$. Therefore, these eigenvalues can be enumerated in non-decreasing order

$$\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A) \le \cdots ,$$

where each eigenvalue is repeated according to its multiplicity.

Instead of the eigenvalues it is sometimes more convenient to consider their counting function. For a self-adjoint operator A with spectral measure P and for $\mu \in \mathbb{R}$, let

$$N(\mu, A) := \dim \operatorname{ran} P_{(-\infty,\mu)} ,$$

which can be a natural number or infinite. The function $\mu \mapsto N(\mu, A)$ is clearly non-decreasing. It is referred to as the *spectral counting function*. For a given $\mu \in \mathbb{R}$, one has $N(\mu, A) < \infty$ if and only if the spectrum of A in $(-\infty, \mu)$ consists of finitely many eigenvalues with finite multiplicities, and in this case, $N(\mu, A)$ is equal to the total multiplicity of these eigenvalues. Consequently, one has $\sigma_{\mathrm{ess}}(A) \neq \emptyset$ if and only if there is a $\mu \in \mathbb{R}$ with $N(\mu, A) = \infty$, and in this case one has $\inf \sigma_{\mathrm{ess}}(A) = \inf\{\mu : N(\mu, A) = \infty\}$. We write

$$N_A := \begin{cases} N(\inf \sigma_{\mathrm{ess}}(A), A) & \text{if } \sigma_{\mathrm{ess}}(A) \neq \emptyset , \\ \dim \mathcal{H} & \text{if } \sigma_{\mathrm{ess}}(A) = \emptyset . \end{cases}$$

We begin with a version of the variational principle that is sometimes called *Glazman's lemma*.

Theorem 1.25 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then for any $\mu \in \mathbb{R}$,*

$$N(\mu, A) = \sup \big\{ \dim F : F \subset d[a] \textit{ is a subspace such that } \\ \textit{for all } 0 \neq u \in F \textit{ one has } a[u] < \mu\|u\|^2 \big\} \tag{1.48}$$

and

$$N(\mu, A) + \dim \ker(A - \mu) \\ = \sup \big\{ \dim F : F \subset d[a] \textit{ is a subspace such that for all } u \in F \\ \textit{one has } a[u] \le \mu\|u\|^2 \big\}. \tag{1.49}$$

If $\mathcal{F}$ is a form core of a, then in (1.48) *it suffices to take the supremum only over $F \subset \mathcal{F}$.*

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In particular, if the right side in (1.48) is finite, then it coincides with the number of eigenvalues of A that are strictly less than μ (counting multiplicities).

Proof Let $F := \operatorname{ran} P_{(-\infty,\mu)}$ and note that this is contained in $d[a]$. Moreover, by Theorem 1.16 (see also (1.41)), for any $0 \neq u \in F$,

$$a[u] = \int_{(-\infty,\mu)} \lambda\, d(P_\lambda u, u) < \mu \int_{(-\infty,\mu)} d(P_\lambda u, u) = \mu \|u\|^2 ,$$

which proves $\leq$ in (1.48).

To prove the reverse inequality, we consider an arbitrary subspace $F \subset d[a]$ with $\dim F > N(\mu, A)$. Since $\dim \operatorname{ran} P_{(-\infty,\mu)} = N(\mu, A)$, there is a $0 \neq u_0 \in F \cap \left(\operatorname{ran} P_{(-\infty,\mu)}\right)^\perp$. Then, again by Theorem 1.16 (see also (1.41)), we have

$$a[u_0] = \int_{[\mu,\infty)} \lambda\, d(P_\lambda u_0, u_0) \geq \mu \int_{[\mu,\infty)} d(P_\lambda u_0, u_0) = \mu \|u_0\|^2 .$$

Therefore, this subspace F is not admissible in the right side of (1.48).

An analogous argument proves the second identity (1.49).

The proof when F is restricted to lie in a dense subspace $\mathcal{F}$ of $d[a]$ follows from approximating, in the form norm, a finite linear system from $\operatorname{ran} P_{(-\infty,\mu)}$ by elements from $\mathcal{F}$. This preserves the strict inequality $a[u] < \mu\|u\|^2$, $u \neq 0$, on the span of these elements and completes the proof. □

Here is another useful form of the variational principle, also called Glazman's lemma.

Theorem 1.26 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then for any $\mu \in \mathbb{R}$,*

$$N(\mu, A) = \inf \Big\{ \dim F :\ F \subset \mathcal{H} \textit{ is a subspace such that } \textit{for all } u \in F^\perp \cap d[a],\ a[u] \geq \mu \|u\|^2 \Big\} \tag{1.50}$$

and

$$N(\mu, A) + \dim\ker(A - \mu) = \inf \Big\{ \dim F :\ F \subset \mathcal{H} \textit{ is a subspace such that for all } 0 \neq u \in F^\perp \cap d[a],\ a[u] > \mu \|u\|^2 \Big\} . \tag{1.51}$$

Proof We first show that $N \geq \inf$, and then $N \leq \inf$ in (1.50). For $F := \operatorname{ran} P_{(-\infty,\mu)}$, we have $N(\mu, A) = \dim F$ and, by Theorem 1.16 (see also (1.41)), $a[u] \geq \mu\|u\|^2$ for all $u \in F^\perp \cap d[a]$. This proves $\geq$ in (1.50).

We now consider some arbitrary subspace $F \subset \mathcal{H}$ with $N(\mu, A) > \dim F$. Then there exists some $u_0 \neq 0$ with $u_0 \in \operatorname{ran} P_{(-\infty,\mu)} \subset d[a]$ and $u_0 \in F^\perp$. But

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for this u_0, the opposite inequality $a[u_0] < \mu\|u_0\|^2$ holds. Hence, this subspace is not admissible in the right side of (1.50), and we are done.

An analogous argument proves the second identity (1.51). $\square$

As well as the variational principles for the counting function in Theorems 1.25 and 1.26, there are closely related variational principles for individual eigenvalues that play a very important role in applications. While we will not use them in this book, it is worth stating them and sketching their proof.

The following theorem is sometimes referred to as the *Courant–Fischer–Weyl min–max principle*. We recall the definition of the $\lambda_n(A)$ and of N_A at the beginning of this subsection.

Theorem 1.27 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then, for all $n \in \mathbb{N}$,*

$$\sup_{u_1,\ldots,u_{n-1}\in\mathcal{H}} \inf_{0\neq u\in d[a]\cap\{u_1,\ldots,u_{n-1}\}^\perp} \frac{a[u]}{\|u\|^2} = \begin{cases} \lambda_n(A) & \text{if } n \leq N_A\,, \\ \inf \sigma_{\mathrm{ess}}(A) & \text{if } n > N_A\,. \end{cases}$$

If $n \leq N_A$, the supremum is attained if $u_1,\ldots,u_{n-1}$ are orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1(A),\ldots,\lambda_{n-1}(A)$ and, in this case, the infimum is attained if u is an eigenvector corresponding to $\lambda_n(A)$.

If $\dim\mathcal{H} < \infty$, then both sides of the equality in Theorem 1.27 are interpreted as $+\infty$ for $n > N_A = \dim\mathcal{H}$.

Proof For each $n \in \mathbb{N}$ we denote by μ_n the left side of the assertion and, fixing orthonormal eigenvectors e_m corresponding to the $\lambda_m(A)$, we define, for $n \leq N_A$,

$$\tilde{\mu}_n := \inf_{0\neq u\in d[a]\cap\{e_1,\ldots,e_{n-1}\}^\perp} \frac{a[u]}{\|u\|^2}\,.$$

We prove that

$$\mu_n = \tilde{\mu}_n = \lambda_n(A) \qquad \text{for all } n \leq N_A\,. \tag{1.52}$$

Clearly, $\mu_n \geq \tilde{\mu}_n$ for $n \leq N_A$. Let P be the spectral measure of A. Since $\{e_1,\ldots,e_{n-1}\}^\perp \subset \operatorname{ran} P_{[\lambda_n(A),\infty)}$, we have, as in the proof of Theorem 1.25, that $a[u] \geq \lambda_n(A)\|u\|^2$ for every $u \in d[a] \cap \{e_1,\ldots,e_{n-1}\}^\perp$. Note that equality holds if $u = e_n$. Thus, we have $\tilde{\mu}_n \geq \lambda_n(A)$. On the other hand, for any $u_1,\ldots,u_{n-1} \in \mathcal{H}$, there is a $0 \neq v \in \operatorname{span}\{e_1,\ldots,e_n\} \cap \{u_1,\ldots,u_{n-1}\}^\perp$. Since $v \in \operatorname{span}\{e_1,\ldots,e_n\}$, we have $v \in d[a]$ and, again as in the proof of Theorem 1.25, $a[v] \leq \lambda_n(A)\|v\|^2$. Hence,

$$\inf_{0\neq u\in d[a]\cap\{u_1,\ldots,u_{n-1}\}^\perp} \frac{a[u]}{\|u\|^2} \leq \frac{a[v]}{\|v\|^2} \leq \lambda_n(A)\,.$$

Thus, since $u_1, \ldots, u_{n-1}$ are arbitrary, we have $\mu_n \le \lambda_n(A)$. This completes the proof of (1.52), and thereby of all conclusions in the theorem for $n \le N_A$.

To prove the remaining conclusions, we may assume that $\dim \mathcal{H} = \infty$, $\kappa := \inf \sigma_{\mathrm{ess}}(A) < \infty$, and $N_A < \infty$. Then $\dim \operatorname{ran} P_{(-\infty,\kappa+\varepsilon)} = \infty$ for any $\varepsilon > 0$ and therefore, for any given $\varepsilon > 0$, we can extend the finite orthonormal system $(e_n)_{n=1}^{N_A}$ to an infinite orthonormal system $(e_n)_{n=1}^{\infty} \subset \operatorname{ran} P_{(-\infty,\kappa+\varepsilon)}$. Defining $\tilde{\mu}_n$ for $n > n_A$ by the same formula as before, we still have $\mu_n \ge \tilde{\mu}_n$. Moreover, by repeating the above reasoning, we get $\tilde{\mu}_n \ge \kappa$ and $\mu_n \le \kappa + \varepsilon$ for $n > n_A$. Since $\varepsilon > 0$ is arbitrary, this proves that $\mu_n = \kappa$, as claimed. This completes the proof. □

There is another version where the inf and the sup are interchanged.

Theorem 1.28 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then, for all $n \in \mathbb{N}$,*

$$\inf_{\substack{u_1,\ldots,u_n \in d[a] \\ \text{lin. independent}}} \ \sup_{0 \ne u \in \operatorname{span}\{u_1,\ldots,u_n\}} \frac{a[u]}{\|u\|^2} = \begin{cases} \lambda_n(A) & \text{if } n \le N_A\,, \\ \inf \sigma_{\mathrm{ess}}(A) & \text{if } n > N_A\,. \end{cases}$$

If $n \le N_A$, the infimum is attained if $u_1, \ldots, u_n$ are orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$ and, in this case, the supremum is attained if u is an eigenvector corresponding to $\lambda_n(A)$.

The proof of this theorem is similar to those of the others in this subsection and is omitted; see, e.g., Davies (1995, Theorems 4.5.1 and 4.5.2).

1.2.4 Applications of the variational principle

We begin by introducing the important notion of comparison of two operators. Let A and B be self-adjoint, lower semibounded operators and let a and b be the corresponding quadratic forms with form domains $d[a]$ and $d[b]$, respectively. We say that A is *greater or equal than* B, in symbols

$$A \ge B\,, \quad \text{or} \quad B \le A\,, \tag{1.53}$$

if the following two conditions are satisfied

$$d[a] \subset d[b]\,, \tag{1.54}$$

$$a[u] \ge b[u] \quad \text{for all } u \in d[a]\,. \tag{1.55}$$

Note that the case $d[a] = d[b]$ in (1.54) may occur, for instance, if both operators A and B are bounded. Furthermore, the definition is meaningful if we have the special case $a[u] = b[u]$ for all $u \in d[a]$ in (1.55). In applications this

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occurs, for instance, for differential operators where a strict inclusion in (1.54) corresponds to different choices of boundary conditions for A and B.

Trivially, it holds that $A \geq A$. Note further that $A \geq B$ and $B \geq C$ implies $A \geq C$. Finally, if $A \geq B$ and $B \geq A$, then $A = B$. Therefore, this comparison defines a partial order relation.

The following is a consequence of the variational principle (Theorem 1.25).

Proposition 1.29 *Let A and B be self-adjoint, lower semibounded operators satisfying $A \geq B$. Then*

$$N(\mu, A) \leq N(\mu, B) \quad \text{for all } \mu \in \mathbb{R}.$$

In particular, $\inf \sigma_{\mathrm{ess}}(A) \geq \inf \sigma_{\mathrm{ess}}(B)$ *and*

$$\lambda_n(A) \geq \lambda_n(B) \qquad \text{for all } n \leq \min\{N_A, N_B\}.$$

The second application of the variational principle concerns eigenvalues of sums of operators defined in the form sense.

Consider two self-adjoint, lower semibounded operators A and B with corresponding quadratic forms a and b. Assume that $d[a] \cap d[b]$ is dense in $\mathcal{H}$, and let c be the quadratic form $c[u] := a[u] + b[u]$ with domain $d[c] := d[a] \cap d[b]$. This form is lower semibounded and closed. It induces a corresponding self-adjoint operator C, which we formally write as $A + B$. The identification of C with $A + B$ has to be understood in the form sense, not in the sense of a sum of operators.

Corollary 1.30 *Let A and B be self-adjoint, lower semibounded operators and assume that $d[a] \cap d[b]$ is dense in $\mathcal{H}$. Then*

$$N(0, A + B) \leq N(0, A) + N(0, B).$$

Proof Let $P(A)$ and $P(B)$ denote the spectral measures for A and B, respectively, and let $L := \operatorname{ran} P_{(-\infty,0)}(A)$ and $M := \operatorname{ran} P_{(-\infty,0)}(B)$. By the spectral theorem,

$$a[u] \geq 0 \text{ for all } u \in d[a] \cap L^{\perp} \qquad \text{and} \qquad b[v] \geq 0 \text{ for all } v \in d[b] \cap M^{\perp}.$$

Thus,

$$c[w] = a[w] + b[w] \geq 0 \text{ for all } w \in d[a] \cap d[b] \cap (L + M)^{\perp}.$$

By Theorem 1.26, we deduce

$$N(0, C) \leq \dim(L + M) \leq \dim L + \dim M = N(0, A) + N(0, B),$$

as claimed. □

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We conclude this subsection with two somewhat technical applications of the variational principle that will be needed later on.

Let A be a self-adjoint, lower semibounded operator and let a be the corresponding quadratic form. Consider a bounded operator T such that $\{u \in \mathcal{H} : Tu \in d[a]\}$ is dense in $\mathcal{H}$, and define

$$a_T[u] := a[Tu] \qquad \text{for } u \in d[a_T] := \{v \in \mathcal{H} : Tv \in d[a]\} .$$

This form is lower semibounded and closed. It induces a self-adjoint operator, which we write formally as T^*AT. We emphasize that this notation is not understood in the sense of a product of operators.

Corollary 1.31 *If A is self-adjoint and lower semibounded and if T is bounded with $\{u \in \mathcal{H} : Tu \in d[a]\}$ dense in $\mathcal{H}$, then $N(0, T^*AT) \leq N(0, A)$.*

Proof Let P be the spectral measure of A and set $L := \operatorname{ran} P_{(-\infty,0)}$. Then, for all $u \in (T^*L)^\perp \cap d[a_T]$, one has $Tu \in L^\perp \cap d[a]$. Thus, by the spectral theorem, we have $a_T[u] = a[Tu] \geq 0$. By the variational principle (Theorem 1.26), this implies

$$N(0, T^*AT) \leq \dim T^*L \leq \dim L = N(0, A) ,$$

as claimed. □

We shall also need another version of this corollary. Let P be an orthogonal projection and, as before, let A be a self-adjoint, lower semibounded operator with quadratic form a. Assuming that $d[a] \cap \operatorname{ran} P$ is dense in $\operatorname{ran} P$, the set $\{u \in \mathcal{H} : Pu \in d[a]\}$ is dense in $\mathcal{H}$, and then PAP is defined as above. Let $\tilde{A}_P$ be the restriction of PAP to the Hilbert space $\operatorname{ran} P$. The following corollary compares the spectrum of A on $\mathcal{H}$ with that of $\tilde{A}_P$ on $\operatorname{ran} P$.

Corollary 1.32 *If A is self-adjoint and lower semibounded and if P is an orthogonal projection with $d[a] \cap \operatorname{ran} P$ dense in $\operatorname{ran} P$, then $N(\lambda, \tilde{A}_P) \leq N(\lambda, A)$ for all $\lambda \in \mathbb{R}$.*

Proof Let $B := A - \lambda$ in $\mathcal{H}$. Then, by Corollary 1.31, we have $N(\lambda, A) = N(0, B) \geq N(0, PBP)$. Since $N(0, PBP)$ counts the strictly negative eigenvalues and since $PBP = \tilde{B}_P \oplus 0$ on $\mathcal{H} = \operatorname{ran} P \oplus (\operatorname{ran} P)^\perp$, we get $N(0, PBP) = N(0, \tilde{B}_P)$. On $\operatorname{ran} P$ we have $\tilde{B}_P = \tilde{A}_P - \lambda$, and thus $N(0, \tilde{B}_P) = N(\lambda, \tilde{A}_P)$. □

1.2.5 Variational principle for sums of eigenvalues and the trace

In the previous two subsections, we have discussed a variational principle for the eigenvalues of a self-adjoint, lower semibounded operator. In this subsection, we prove a corresponding principle for sums of eigenvalues and, along the way, introduce the trace of a self-adjoint, non-negative operator.

We recall that, for a self-adjoint, lower semibounded operator A, we denote by $\lambda_j(A)$ the eigenvalues below the bottom of its essential spectrum, in non-decreasing order and repeated according to multiplicities. This list contains N_A elements and may be empty, finite or infinite.

Here is a variational characterization for partial sums of these eigenvalues.

Proposition 1.33 *Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. Then, for any finite number $N \le N_A$,*

$$\sum_{j=1}^{N} \lambda_j(A)$$
$$= \inf\left\{ \sum_{n=1}^{N} a[u_n] \colon (u_n)_{n=1}^N \subset d[a] \text{ and } (u_n, u_m) = \delta_{n,m} \text{ for all } 1 \le n, m \le N \right\}.$$

If $\mathcal{F}$ is a form core of a, then in the infimum on the right side it suffices to consider $(u_n)_{n=1}^N \subset \mathcal{F}$.

Proof We first prove that the left side $\ge$ the right side. In fact this follows immediately by choosing u_n to be orthonormal eigenvectors corresponding to $\lambda_n(A)$.

Let us prove the opposite bound. For a given finite number $N \le N_A$ and given $u_1, \ldots, u_N \in d[a]$ with $(u_n, u_m) = \delta_{n,m}$, we define the orthogonal projection $P := \sum_{n=1}^N (\cdot, u_n) u_n$. Then, according to Corollary 1.32,

$$\lambda_j(A) \le \lambda_j(\tilde{A}_P) \quad \text{for all } 1 \le j \le N\,.$$

Thus,

$$\sum_{j=1}^N \lambda_j(A) \le \sum_{j=1}^N \lambda_j\left(\tilde{A}_P\right) = \sum_{n=1}^N a[u_n]\,,$$

where we used the fact from linear algebra that the sum of all eigenvalues of a matrix in a finite-dimensional space can be evaluated by the sum of the diagonal entries.

The statement concerning a form core follows by a simple approximation argument in the form norm. This proves the proposition. □

Proposition 1.33 implies the following somewhat technical result, which will be of importance in §§5.3 and 8.2. Let Ξ be a probability space. We denote integration with respect to the underlying probability measure by $d\xi$. Let $\Xi \ni \xi \mapsto U(\xi)$ be a measurable function taking values in the unitary operators on $\mathcal{H}$, where measurability is understood in the form sense. Let A be a self-adjoint, lower semibounded operator with corresponding quadratic form a. We assume

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that ran $U(\xi) \subset d[a]$ for a.e. $\xi \in \Xi$ and that $\{u \in \mathcal{H} : \int_\Xi a[U(\xi)u]\, d\xi < \infty\}$ is dense in $\mathcal{H}$. Then the quadratic form

$$a^U[u] := \int_\Xi a[U(\xi)u]\, d\xi \quad \text{for } u \in d[a^U] := \left\{v \in \mathcal{H}\colon \int_\Xi a[U(\xi)v]\, d\xi < \infty\right\}$$

is lower semibounded and closed. It defines a self-adjoint, lower semibounded operator, which we denote by A^U and sometimes also by $\int_\Xi U(\xi)^* A U(\xi)\, d\xi$.

Corollary 1.34 *Let A be a self-adjoint, lower semibounded operator and let U be as above. Then for any finite number $N \le \min\{N_A, N_{A^U}\}$,*

$$\sum_{j=1}^{N} \lambda_j(A^U) \ge \sum_{j=1}^{N} \lambda_j(A)\,.$$

Proof Let $(u_n)_{n=1}^N$ be a orthonormal system of eigenvectors of A^U corresponding to the eigenvalues $\lambda_1(A^U), \ldots, \lambda_N(A^U)$. Then

$$\sum_{j=1}^{N} \lambda_j(A^U) = \sum_{n=1}^{N} a^U[u_n] = \sum_{n=1}^{N} \int_\Xi a[U(\xi)u_n]\, d\xi = \int_\Xi \sum_{n=1}^{N} a[U(\xi)u_j]\, d\xi\,.$$

Since for a.e. $\xi \in \Xi$ the system $(U(\xi)u_n)_{n=1}^N$ belongs to $d[a]$ and is orthonormal, the variational principle from Proposition 1.33 implies

$$\sum_{n=1}^{N} a[U(\xi)u_n] \ge \sum_{j=1}^{N} \lambda_j(A)\,.$$

Since $d\xi$ is a probability measure, it remains to integrate both sides in this inequality to obtain the desired conclusion. □

The next result is a version of Proposition 1.33 where N is not fixed. For this purpose, the following notion of the trace of a non-negative bounded operator T will be useful. If the essential spectrum of T is empty or consists only of the point 0, the negative spectrum of $-T$ is discrete and consists of the eigenvalues $\lambda_j(-T)$, which are enumerated counting multiplicities as explained above. We set

$$\operatorname{Tr} T := \begin{cases} +\infty & \text{if } \sup \sigma_{\mathrm{ess}}(T) > 0\,, \\ -\sum_j \lambda_j(-T) & \text{otherwise}\,. \end{cases}$$

Even in the second case, this value can be infinite. On the other hand, if $\operatorname{Tr} T$ is finite, then Lemma 1.15 implies that T is compact.

In the statement of the following corollary, the operator A_- is defined by the functional calculus as $\varphi(A)$ with $\varphi(\lambda) = \lambda_- = \max\{-\lambda, 0\}$. Clearly, $A_- \ge 0$.

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Corollary 1.35 *Let A be a self-adjoint, lower semibounded operator. Then*

$$-\operatorname{Tr} A_- = \inf\left\{ \sum_{n=1}^N a[u_n] \colon N \in \mathbb{N}, \, (u_n)_{n=1}^N \subset d[a] \right.$$
$$\left. \text{and } (u_n, u_m) = \delta_{n,m} \text{ for all } 1 \le n, m \le N \right\}.$$

If $\mathcal{F}$ is a form core of a, then in the infimum on the right side it suffices to consider $(u_n)_{n=1}^N \subset \mathcal{F}$.

Proof If $\sigma_{\mathrm{ess}}(A) = \emptyset$ or if $\inf \sigma_{\mathrm{ess}}(A) \geq 0$, then

$$-\operatorname{Tr} A_- = \sum_j \lambda_j(-A_-) = -\sum_j (\lambda_j(A))_- = \inf_{N \in \mathbb{N}} \sum_{j=1}^N \lambda_j(A),$$

and therefore the corollary follows from Proposition 1.33. By contrast, if $\kappa := \inf \sigma_{\mathrm{ess}}(A) < 0$, then $\operatorname{Tr} A_- = \infty$ and, with the spectral measure P of A, $\dim \operatorname{ran} P_{(-\infty, \kappa+\varepsilon)} = \infty$ for any $\varepsilon > 0$. In particular, if $\kappa + \varepsilon < 0$, there is an infinite sequence of orthonormal (u_n) with $a[u_n] \le \kappa + \varepsilon$, which can also be assumed to lie in a form core. Thus, the right side in the corollary is also equal to $-\infty$. □

As a consequence of Corollary 1.35, we now prove a fundamental property of the trace. Note that we do not assume that $\operatorname{Tr} T$ is finite or even that T is compact.

Lemma 1.36 *Let T be a bounded and non-negative operator. Then for any complete orthonormal system (u_n),*

$$\operatorname{Tr} T = \sum_n (T u_n, u_n).$$

Proof We first assume that $\operatorname{Tr} T < \infty$ and we choose a complete orthonormal system (v_j) such that $T v_j = -\lambda_j(-T) v_j$. Then

$$\operatorname{Tr} T = -\sum_j \lambda_j(-T) = \sum_j \|T^{1/2} v_j\|^2 = \sum_j \sum_n |(T^{1/2} v_j, u_n)|^2$$
$$= \sum_n \sum_j |(v_j, T^{1/2} u_n)|^2 = \sum_n \|T^{1/2} u_n\|^2 = \sum_n (T u_n, u_n),$$

as claimed. (The interchange of summations here is justified since all terms are non-negative.) On the other hand, when $\operatorname{Tr} T = \infty$, by Corollary 1.35 applied to $A = -T$, for any $M > 0$ there is an $N \in \mathbb{N}$ and an orthonormal system $(v_j)_{j=1}^N$

such that $\sum_{j=1}^N \|T^{1/2}v_j\|^2 \geq M$. Similar to before, we have

$$\sum_{j=1}^N \|T^{1/2}v_j\|^2 = \sum_{j=1}^N \sum_n |(T^{1/2}v_j, u_n)|^2 = \sum_n \sum_{j=1}^N |(v_j, T^{1/2}u_n)|^2$$
$$\leq \sum_n \|T^{1/2}u_n\|^2 = \sum_n (Tu_n, u_n)\,.$$

Since M is arbitrary this means that $\sum_n (Tu_n, u_n) = \infty$. □

The next corollary is the analogue of Corollary 1.30 for sums of eigenvalues.

Corollary 1.37 *Let A and B be self-adjoint, lower semibounded operators with corresponding quadratic forms a and b. Assume that $d[a] \cap d[b]$ is dense in $\mathcal{H}$ and let $A + B$ be defined in the form sense. Then*

$$\mathrm{Tr}(A + B)_- \leq \mathrm{Tr}\, A_- + \mathrm{Tr}\, B_-\,.$$

Proof Let $N \in \mathbb{N}$ and let $(u_n)_{n=1}^N \subset d[a] \cap d[b]$ be orthonormal functions. Then, by Corollary 1.35 for the operators A and B,

$$\sum_{n=1}^N (a + b)[u_n] = \sum_{n=1}^N a[u_n] + \sum_{n=1}^N b[u_n] \geq -\mathrm{Tr}\, A_- - \mathrm{Tr}\, B_-\,.$$

Taking the infimum over all N and $(u_n)_{n=1}^N$ as above and, using again Corollary 1.35, we obtain the claimed inequality. □

As a brief digression, we will use Corollary 1.35 to compute the trace of a certain class of operators that appear frequently in applications. We recall the notion of a separable measure space from §1.1.4 needed in the following result.

Lemma 1.38 *Let X, Y be separable, sigma-finite measure spaces and let $N, M \in \mathbb{N}$. Let K be a measurable function on $X \times Y$ taking values in the complex $N \times M$ matrices such that*

$$\iint_{X\times Y} \mathrm{Tr}_{\mathbb{C}^M}\left((K(x,y))^* K(x,y)\right) dx\, dy < \infty\,.$$

Then the operator K from $L^2(Y, \mathbb{C}^M)$ to $L^2(X, \mathbb{C}^N)$, defined by

$$(Kv)(x) := \int_Y K(x,y)v(y)\, dy \qquad \textit{for a.e. } x \in X\,,\ v \in L^2(Y, \mathbb{C}^M)\,,$$

is compact and satisfies

$$\mathrm{Tr}\, K^*K = \iint_{X\times Y} \mathrm{Tr}_{\mathbb{C}^M}\left((K(x,y))^* K(x,y)\right) dx\, dy\,.$$

Proof For the sake of simplicity, we begin with the case $N = M = 1$. For any $u \in L^2(X)$, $v \in L^2(Y)$, we have, by the Cauchy–Schwarz inequality,

$$\begin{aligned}|(Kv, u)| &= \left| \iint_{X\times Y} K(x,y)v(y)\overline{u(x)}\,dx\,dy \right| \\ &\le \left(\iint_{X\times Y} |K(x,y)|^2\,dx\,dy \right)^{1/2} \left(\iint_{X\times Y} |v(y)\overline{u(x)}|^2\,dx\,dy \right)^{1/2} \\ &= \|K\|_{L^2(X\times Y)}\|v\|\|u\|\,.\end{aligned}$$

This implies that K is a bounded operator from $L^2(Y)$ to $L^2(X)$.

Let $(v_n)_{n=1}^N$ be an orthonormal system in $L^2(Y)$ and let (u_j) be a complete orthonormal system in $L^2(X)$. Then

$$\sum_{n=1}^N \|Kv_n\|^2 = \sum_{n=1}^N \sum_j \left|(Kv_n, u_j)\right|^2 = \sum_{n=1}^N \sum_j \left|(K, u_j \otimes \overline{v_n})\right|^2,$$

where the inner product on the right side is in $L^2(X\times Y)$ and where the function $u_j \otimes \overline{v_n}$ is defined by

$$\left(u_j \otimes \overline{v_n}\right)(x,y) = u_j(x)\overline{v_n(y)} \qquad \text{for all } (x,y) \in X \times Y\,.$$

Since the $u_j \otimes \overline{v_n}$ are orthonormal in $L^2(X\times Y)$, we find, by Bessel's inequality, that

$$\sum_{n=1}^N \sum_j \left|(K, u_j \otimes \overline{v_n})\right|^2 \le \|K\|^2_{L^2(X\times Y)}\,.$$

By the variational principle in Corollary 1.35 (with $A = -K^*K$) we conclude that

$$\operatorname{Tr} K^*K \le \|K\|^2_{L^2(X\times Y)}\,.$$

In particular, finiteness of the trace implies that K^*K is compact, so, by Lemma 1.2, K is compact. Repeating the argument with a *complete* orthonormal system (v_n) and using Parseval's identity instead of Bessel's inequality we find that, in fact, $\operatorname{Tr} K^*K = \|K\|^2_{L^2(X\times Y)}$. The proves the lemma for $N = M = 1$.

The case of arbitrary N and M can be reduced to the previous case by identifying $L^2(X, \mathbb{C}^N)$ with $L^2(X \times \{1, \ldots, N\})$ and $L^2(Y, \mathbb{C}^M)$ with $L^2(Y \times \{1, \ldots, M\})$. Indeed, fixing a basis $(e_n)_{n=1}^N$ in $\mathbb{C}^N$, the operator U, defined by $(Uu)(x,n) := (u(x), e_n)_{\mathbb{C}^N}$ from $L^2(X, \mathbb{C}^N)$ to $L^2(X \times \{1, \ldots, N\}, \mathbb{C})$, is unitary. Using a similar unitary operator V from $L^2(Y, \mathbb{C}^M)$ to $L^2(Y \times \{1, \ldots, M\}, \mathbb{C})$ and applying the scalar result to the operator VKU^*, we obtain the lemma for arbitrary N and M. □

1.2.6 Riesz means

In this subsection, we study *Riesz means* of order $\gamma > 0$ of a self-adjoint, lower semibounded operator A; that is, the quantities

$$\operatorname{Tr} A_-^\gamma .$$

If this quantity is finite, then the negative spectrum of A consists of eigenvalues of finite multiplicities and, if $\lambda_1(A) \le \lambda_2(A) \le \cdots$ is an enumeration of those, we have

$$\operatorname{Tr} A_-^\gamma = \sum_j \left(\lambda_j(A)\right)_-^\gamma .$$

A very useful identity connects the Riesz means to the spectral counting function.

Lemma 1.39 *Let A be a self-adjoint, lower semibounded operator and let $\gamma > 0$. Then*

$$\operatorname{Tr} A_-^\gamma = \gamma \int_0^\infty N(-\tau, A)\, \tau^{\gamma-1}\, d\tau .$$

Proof Since

$$a_-^\gamma = \gamma \int_0^\infty \chi_{\{a<-\tau\}} \tau^{\gamma-1}\, d\tau \qquad \text{for } a \in \mathbb{R},$$

the identity follows from Lemma 1.36. □

One of the consequences of this formula is a generalization of Corollary 1.30 to the case of Riesz means.

Proposition 1.40 *Let A and B be self-adjoint, lower semibounded operators with corresponding quadratic forms a and b, assume that $d[a] \cap d[b]$ is dense in $\mathcal{H}$, and let $A + B$ be defined in the form sense. If $\gamma > 0$ then*

$$\left(\operatorname{Tr}(A+B)_-^\gamma\right)^{\frac{1}{\gamma+1}} \le \left(\operatorname{Tr} A_-^\gamma\right)^{\frac{1}{\gamma+1}} + \left(\operatorname{Tr} B_-^\gamma\right)^{\frac{1}{\gamma+1}}$$

and, for all $0 < \theta < 1$,

$$\operatorname{Tr}(A+B)_-^\gamma \le \theta^{-\gamma} \operatorname{Tr} A_-^\gamma + (1-\theta)^{-\gamma} \operatorname{Tr} B_-^\gamma .$$

Proof According to Corollary 1.30, for any $\tau > 0$ and for any $0 < \theta < 1$ we have

$$\begin{aligned} N(-\tau, A+B) &= N(0, (A+\theta\tau) + (B + (1-\theta)\tau)) \\ &\le N(0, A+\theta\tau) + N(0, B + (1-\theta)\tau) \\ &= N(-\theta\tau, A) + N(-(1-\theta)\tau, B) . \end{aligned}$$

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Therefore, by Lemma 1.39,

$$\begin{aligned}\operatorname{Tr}(A+B)_-^\gamma &= \gamma \int_0^\infty N(-\tau, A+B)\tau^{\gamma-1}\, d\tau \\ &\le \gamma \int_0^\infty N(-\theta\tau, A)\tau^{\gamma-1}\, d\tau + \gamma \int_0^\infty N(-(1-\theta)\tau, B)\tau^{\gamma-1}\, d\tau \\ &= \theta^{-\gamma} \operatorname{Tr} A_-^\gamma + (1-\theta)^{-\gamma} \operatorname{Tr} B_-^\gamma\,,\end{aligned}$$

as claimed. The other assertion follows by optimizing over θ. □

The remainder of this subsection contains improvements of the constants appearing in the bound in Proposition 1.40. These are not necessary for the applications in this book and can be omitted in a first reading.

The fact that for $\gamma = 1$ an improvement is possible can be seen from Corollary 1.37, which contains a bound like that in Proposition 1.40, but without the prefactors $\theta^{-\gamma}$ and $(1-\theta)^{-\gamma}$.

The first improvements concern the case $\gamma > 1$. In this case, we start from the following variant of Lemma 1.39.

Lemma 1.41 *Let A be a self-adjoint, lower semibounded operator and let $\gamma > 1$. Then*

$$\operatorname{Tr} A_-^\gamma = \gamma(\gamma-1) \int_0^\infty \operatorname{Tr}(A+\tau)_-\, \tau^{\gamma-2}\, d\tau\,.$$

Proof Since

$$a_-^\gamma = \gamma(\gamma-1) \int_0^\infty (a+\tau)_- \tau^{\gamma-2}\, d\tau \qquad \text{for } a \in \mathbb{R}\,,$$

the identity follows from Lemma 1.36. □

Using this lemma, we obtain the following bound.

Proposition 1.42 *Let A and B be self-adjoint, lower semibounded operators with corresponding quadratic forms a and b. Assume that $d[a] \cap d[b]$ is dense in $\mathcal{H}$ and let $A+B$ be defined in the form sense. If $\gamma > 1$ then*

$$\left(\operatorname{Tr}(A+B)_-^\gamma\right)^{\frac{1}{\gamma}} \le \left(\operatorname{Tr} A_-^\gamma\right)^{\frac{1}{\gamma}} + \left(\operatorname{Tr} B_-^\gamma\right)^{\frac{1}{\gamma}}$$

and, for all $0 < \theta < 1$,

$$\operatorname{Tr}(A+B)_-^\gamma \le \theta^{-\gamma+1} \operatorname{Tr} A_-^\gamma + (1-\theta)^{-\gamma+1} \operatorname{Tr} B_-^\gamma\,.$$

Proof According to Corollary 1.37, for any $\tau > 0$ and for any $0 < \theta < 1$ we

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have

$$\begin{aligned}\operatorname{Tr}(A+B+\tau)_- &= \operatorname{Tr}((A+\theta\tau)+(B+(1-\theta)\tau))_- \\ &\leq \operatorname{Tr}(A+\theta\tau)_- + \operatorname{Tr}(B+(1-\theta)\tau)_- .\end{aligned}$$

Therefore, by Lemma 1.41,

$$\begin{aligned}\operatorname{Tr}(A+B)_-^\gamma &= \gamma(\gamma-1)\int_0^\infty \operatorname{Tr}(A+B+\tau)_-\tau^{\gamma-2}\,d\tau \\ &\leq \gamma(\gamma-1)\int_0^\infty \operatorname{Tr}(A+\theta\tau)_-\tau^{\gamma-2}\,d\tau \\ &\quad + \gamma(\gamma-1)\int_0^\infty \operatorname{Tr}(B+(1-\theta)\tau)_-\tau^{\gamma-2}\,d\tau \\ &= \theta^{-\gamma+1}\operatorname{Tr}A_-^\gamma + (1-\theta)^{-\gamma+1}\operatorname{Tr}B_-^\gamma ,\end{aligned}$$

as claimed. The other assertion follows by optimizing over θ. □

Even for numbers $a, b \in \mathbb{R}$ the constants $\theta^{-\gamma+1}$ and $(1-\theta)^{-\gamma+1}$ in the inequality

$$(a+b)_-^\gamma \leq \theta^{-\gamma+1}a_-^\gamma + (1-\theta)^{-\gamma+1}b_-^\gamma$$

cannot be improved for $\gamma > 1$. On the other hand, for $0 < \gamma < 1$, the inequality for numbers holds without the factors of $\theta^{-\gamma+1}$ and $(1-\theta)^{-\gamma+1}$. This motivates the next result, which is an improvement of Proposition 1.40 in the case $0 < \gamma < 1$. It is a special case of a theorem by Rotfel'd (1967, 1968).

Proposition 1.43 *Let A and B be self-adjoint, lower semibounded operators with corresponding quadratic forms a and b. Assume that $d[a] \cap d[b]$ is dense in $\mathcal{H}$ and let $A + B$ be defined in the form sense. If $0 < \gamma < 1$, then*

$$\operatorname{Tr}(A+B)_-^\gamma \leq \operatorname{Tr}A_-^\gamma + \operatorname{Tr}B_-^\gamma .$$

For the proof of this proposition, we need several preliminary results. The first one is an extension of Corollary 1.31 to Riesz means. We refer to the discussion before that corollary for the precise definition of the operator T^*AT.

Lemma 1.44 *If A is self-adjoint and lower semibounded and if T is bounded with $\{u \in \mathcal{H} : Tu \in d[a]\}$ dense in $\mathcal{H}$, then for any $\gamma > 0$,*

$$\operatorname{Tr}(T^*AT)_-^\gamma \leq \|T\|^{2\gamma}\ \operatorname{Tr}A_-^\gamma .$$

Proof According to Lemma 1.39, we have

$$\operatorname{Tr}(T^*AT)_-^\gamma = \gamma\int_0^\infty N(-\tau, T^*AT)\tau^{\gamma-1}\,d\tau .$$

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For any $\tau > 0$, we have $T^*AT + \tau \geq T^*(A + \|T\|^{-2}\tau)T$, and therefore

$$N(-\tau, T^*AT) = N(0, T^*AT + \tau) \leq N(0, T^*(A + \|T\|^{-2}\tau)T)\,.$$

We now apply Corollary 1.31, which implies that

$$N(0, T^*(A + \|T\|^{-2}\tau)T) \leq N(0, A + \|T\|^{-2}\tau) = N(-\|T\|^{-2}\tau, A)\,,$$

and therefore

$$\operatorname{Tr}(T^*AT)_-^\gamma \leq \gamma \int_0^\infty N(-\|T\|^{-2}\tau, A)\tau^{\gamma-1}\,d\tau = \|T\|^{2\gamma}\ \operatorname{Tr} A_-^\gamma\,,$$

as claimed. □

We now bound the change in eigenvalues under a rank-one perturbation.

Lemma 1.45 *Let A be a self-adjoint, lower semibounded operator and let B be a self-adjoint, non-positive rank-one operator. Then*

$$\lambda_1(A + B) \leq \lambda_1(A) \leq \lambda_2(A + B) \leq \lambda_2(A) \leq \cdots\,,$$

where $\lambda_n(A)$ and $\lambda_n(A+B)$ denote the eigenvalues of A and $A+B$, respectively, below $\inf \sigma_{\mathrm{ess}}(A) = \inf \sigma_{\mathrm{ess}}(A + B)$, in non-decreasing order and counting multiplicities.

Proof The equality $\inf \sigma_{\mathrm{ess}}(A) = \inf \sigma_{\mathrm{ess}}(A+B)$ follows from Weyl's theorem (Theorem 1.14). From the variational principle (Proposition 1.29) we obtain $\lambda_n(A + B) \leq \lambda_n(A)$ for all n. We will show the remaining inequality $\lambda_n(A) \leq \lambda_{n+1}(A + B)$ by proving that, for any $\mu < \inf \sigma_{\mathrm{ess}}(A)$,

$$N(\mu, A + B) \leq N(\mu, A) + 1\,. \tag{1.56}$$

Choosing here $\mu = \lambda_n(A)$, so that $N(\mu, A) \leq n - 1$, we deduce the claimed inequality.

We prove (1.56) by contradiction, assuming $N(\mu, A + B) > N(\mu, A) + 1$. Let $P(A)$ and $P(A + B)$ denote the spectral measures of A and $A + B$. Since the space $\operatorname{ran} P_{(-\infty,\mu)}(A+B) \cap \ker B$ has dimension at least $N(\mu, A+B) - 1$, which, by assumption, is larger than the dimension of the space $\operatorname{ran} P_{(-\infty,\mu)}(A)$, there is a $0 \neq u \in \operatorname{ran} P_{(-\infty,\mu)}(A + B) \cap \ker B \cap (\operatorname{ran} P_{(-\infty,\mu)}(A))^\perp$. The conditions $0 \neq u \in \operatorname{ran} P_{(-\infty,\mu)}(A + B)$ and $u \in (\operatorname{ran} P_{(-\infty,\mu)}(A))^\perp$ imply, respectively,

$$a[u] + b[u] < \mu\|u\|^2 \qquad \text{and} \qquad a[u] \geq \mu\|u\|^2\,.$$

Since $u \in \ker B$ implies $b[u] = 0$, this is a contradiction. □

The final ingredient in the proof of Proposition 1.43 is the following rearrangement inequality.

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Lemma 1.46 *Let* $0 < \gamma < 1$. *Then for any set* $E \subset [0, \infty)$ *of finite measure*

$$\int_E t^{\gamma-1}\,dt \le \int_0^{|E|} t^{\gamma-1}\,dt\,.$$

Proof Since

$$t^{\gamma-1} = (1-\gamma)\int_t^\infty s^{\gamma-2}\,ds = (1-\gamma)\int_0^\infty s^{\gamma-2}\chi_{(0,s)}(t)\,ds\,,$$

we have

$$\begin{aligned}\int_E t^{\gamma-1}\,dt &= (1-\gamma)\int_0^\infty \left(\int_0^\infty \chi_E(t)\chi_{(0,s)}(t)\,dt\right) s^{\gamma-2}\,ds\\ &= (1-\gamma)\int_0^\infty |E\cap(0,s)|s^{\gamma-2}\,ds\,.\end{aligned}$$

Obviously, $|E \cap (0,s)| \le \min\{|E|, s\}$. Note that equality holds here if $E = (0, |E|)$. Inserting this into the above identity, we obtain

$$\int_E t^{\gamma-1}\,dt \le (1-\gamma)\left(\int_0^{|E|} s^{\gamma-1}\,ds + |E|\int_{|E|}^\infty s^{\gamma-2}\,ds\right) = \frac{|E|^\gamma}{\gamma} = \int_0^{|E|} t^{\gamma-1}\,dt\,,$$

as claimed. □

Proof of Proposition 1.43 We may assume that $\operatorname{Tr} A_-^\gamma + \operatorname{Tr} B_-^\gamma < \infty$. We begin with the case where B_- is rank-one and denote its positive eigenvalue by β. Moreover, we write the negative eigenvalues of A and $A+B$ as $-\alpha_1 \le -\alpha_2 \le \cdots$ and $-\lambda_1 \le -\lambda_2 \le \cdots$, respectively. According to Lemma 1.45, we have

$$-\lambda_n \le -\alpha_n \le -\lambda_{n+1} \le -\alpha_{n+1} \le \cdots \qquad \text{for all } n\,. \tag{1.57}$$

Then

$$\begin{aligned}\operatorname{Tr}(A+B)_-^\gamma &= \sum_n \lambda_n^\gamma = \gamma\sum_n \int_0^{\lambda_n} t^{\gamma-1}\,dt\\ &= \gamma\sum_n\int_0^{\alpha_n} t^{\gamma-1}\,dt + \gamma\sum_n\int_{\alpha_n}^{\lambda_n} t^{\gamma-1}\,dt\\ &= \operatorname{Tr} A_-^\gamma + \gamma\int_E t^{\gamma-1}\,dt\,,\end{aligned} \tag{1.58}$$

where $E := \bigcup_n(\alpha_n, \lambda_n)$ and where we used the fact that, by (1.57), these intervals are disjoint. Furthermore, again by the disjointness,

$$|E| = \sum_n \lambda_n - \sum_n \alpha_n = \operatorname{Tr}(A+B)_- - \operatorname{Tr} A_- \le \operatorname{Tr} B_- = \beta\,,$$

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where the inequality comes from Corollary 1.37. Note that the assumed finiteness of $\operatorname{Tr} A_-^\gamma$ implies the finiteness of $\operatorname{Tr} A_-$, and therefore there is no cancellation of infinities in the above computation. Now Lemma 1.46 implies that

$$\gamma \int_E t^{\gamma-1}\,dt \le \gamma \int_0^{|E|} t^{\gamma-1}\,dt \le \gamma \int_0^{\beta} t^{\gamma-1}\,dt = \beta^\gamma = \operatorname{Tr} B_-^\gamma\,.$$

Inserting this into (1.58) yields the claimed inequality in the case where B_- is rank-one.

The case where B_- is of finite rank follows by iterating the inequality in the rank-one case.

We finally deal with the case of an arbitrary operator B with $\operatorname{Tr} B_-^\gamma < \infty$. For $\varepsilon > 0$ we write $P_\varepsilon := \chi_{(-\infty,-\varepsilon)}(A+B)$. According to Lemma 1.44, we have

$$\operatorname{Tr}(P_\varepsilon A P_\varepsilon)_-^\gamma \le \operatorname{Tr} A_-^\gamma < \infty \qquad \text{and} \qquad \operatorname{Tr}(P_\varepsilon B P_\varepsilon)_-^\gamma \le \operatorname{Tr} B_-^\gamma < \infty\,.$$

We know from the non-sharp bound of Proposition 1.40 that under the assumption $\operatorname{Tr} A_-^\gamma + \operatorname{Tr} B_-^\gamma < \infty$ we have $\operatorname{Tr}(A+B)_-^\gamma < \infty$, and therefore, in particular, P_ε has finite rank. Thus, $P_\varepsilon B P_\varepsilon$ has finite rank as well and, by what we have shown so far, we know

$$\operatorname{Tr}(P_\varepsilon A P_\varepsilon + P_\varepsilon B P_\varepsilon)_-^\gamma \le \operatorname{Tr}(P_\varepsilon A P_\varepsilon)_-^\gamma + \operatorname{Tr}(P_\varepsilon B P_\varepsilon)_-^\gamma \le \operatorname{Tr} A_-^\gamma + \operatorname{Tr} B_-^\gamma\,.$$

On the other hand,

$$\operatorname{Tr}(P_\varepsilon A P_\varepsilon + P_\varepsilon B P_\varepsilon)_-^\gamma = \operatorname{Tr}(P_\varepsilon (A+B) P_\varepsilon)_-^\gamma = \sum_{|\lambda_n(A+B)|>\varepsilon} |\lambda_n(A+B)|^\gamma$$

and, by monotone convergence, this converges to $\sum_n |\lambda_n(A+B)|^\gamma = \operatorname{Tr}(A+B)_-^\gamma$ as $\varepsilon \to 0$. This proves the claimed inequality. □

1.2.7 Perturbations of quadratic forms

In this subsection, we take on a perturbation-theoretic point of view. That is, there will be a quadratic form a that is densely defined, lower semibounded and closed, and corresponds to an operator A, and we will study self-adjointness of $A + B$, where B is, in a sense to be made precise, small with respect to A. We formulate this smallness in the sense of quadratic forms.

The following simple lemma is sometimes useful for verifying that a perturbation of a lower semibounded, closed quadratic form is also lower semibounded and closed.

Lemma 1.47 *Assume that a is a lower semibounded, closed quadratic form*

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with domain $d[a]$ and assume that b is a real-valued quadratic form on $d[a]$ such that for some $\theta \in [0,1)$ and some $C \in \mathbb{R}$,

$$|b[u]| \le \theta\, a[u] + C\,\|u\|^2 \qquad \textit{for all } u \in d[a]\,. \tag{1.59}$$

Then the quadratic form $a + b$ with domain $d[a]$ is lower semibounded and closed. Moreover, any form core of a is also a form core of $a + b$.

Proof By assumption, we have the inequalities

$$(1-\theta)a[u] - C\|u\|^2 \le a[u] + b[u] \le (1+\theta)a[u] + C\|u\|^2\,.$$

The inequality on the left shows the lower semiboundedness of $a + b$, and the proof of closedness, as well as the form core property, follows easily from this two-sided bound. □

As a consequence of Lemma 1.47 and Theorem 1.18, when $d[a]$ is dense, the quadratic form $a + b$ with domain $d[a]$ generates a self-adjoint operator in $\mathcal{H}$. Often we shall denote this operator by $A + B$, but we emphasize that this is an abuse of notation since, in general, there need not be a well-defined, self-adjoint operator B given by the difference of $A + B$ and A.

Next, we will discuss a version of the resolvent identity for operators $A + B$ defined via quadratic forms. To motivate the formula we are seeking, we recall that, if the real quadratic form b corresponds to a bounded self-adjoint operator B, then, according to (1.32),

$$(A+B-z)^{-1} - (A-z)^{-1} = -\,(A-z)^{-1}\,B\,(A+B-z)^{-1}\ .$$

We can write the right side as

$$-\left[(A+m)^{1/2}\,(A-z)^{-1}\right]\left[(A+m)^{-1/2}\,B\,(A+m)^{-1/2}\right] \times\left[(A+m)^{1/2}\,(A+B-z)^{-1}\right]$$

for some $m > -m_a$. As we will show, the analogue of each of the three factors in square brackets is well defined under assumption (1.59), and therefore will lead to a version of the resolvent formula when $A + B$ is defined via quadratic forms.

It follows from (1.59) that for any $m > -m_a$ there is a constant $C' < \infty$ such that

$$|b[u]| \le C'\left(a[u] + m\|u\|^2\right) \qquad \text{for all } u \in d[a]\,.$$

Let us fix an $m > -m_a$ and a corresponding norm $\left(a[u] + m\|u\|^2\right)^{1/2}$ on $d[a]$.

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By the Riesz representation theorem, there is a bounded operator $\mathcal{B}_a$ on $d[a]$ such that

$$b[u,v] = a[\mathcal{B}_a u, v] + m(\mathcal{B}_a u, v) \qquad \text{for all } u, v \in d[a]\,. \tag{1.60}$$

Clearly, the operator $\mathcal{U} : \mathcal{H} \to d[a]$, $f \mapsto (A+m)^{-1/2} f$ is unitary. Thus,

$$\hat{\mathcal{B}}_a := \mathcal{U}^* \mathcal{B}_a \mathcal{U}$$

is a bounded operator on $\mathcal{H}$. Note that in the case where b comes from a bounded operator B in $\mathcal{H}$, we have

$$\hat{\mathcal{B}}_a = (A+m)^{-1/2} B (A+m)^{-1/2}\,.$$

This operator appears in the formula for the resolvent difference.

Proposition 1.48 *Let a be a densely defined, lower semibounded and closed quadratic form and let b be a real quadratic form satisfying* (1.59) *for some $\theta \in [0,1)$ and $C \in \mathbb{R}$. Then, for all $z \in \rho(A) \cap \rho(A+B)$,*

$$\begin{aligned}
&(A+B-z)^{-1} - (A-z)^{-1} \\
&\quad = -\left[(A+m)^{1/2}(A-z)^{-1}\right] \hat{\mathcal{B}}_a \left[(A+m)^{1/2}(A+B-z)^{-1}\right].
\end{aligned}$$

Each one of the three factors on the right side is a bounded operator.

Proof *Step 1.* We begin by showing the last assertion, namely, that each one of the three factors on the right side is a bounded operator. For the first factor this is clear from the spectral theorem, and for the second factor this was discussed before the proposition. For any $M > -m_a$ we write the third factor as

$$\begin{aligned}
&(A+m)^{1/2}(A+B-z)^{-1} \\
&\quad = \left[(A+m)^{1/2}(A+M)^{-1/2}\right]\left[(A+M)^{1/2}(A+B+M)^{-1/2}\right] \\
&\qquad \times \left[(A+B+M)^{1/2}(A+B-z)^{-1}\right].
\end{aligned}$$

Here the first and the third factors are bounded by the same arguments as before. In the remainder of this step we will show that for any $M > C/\theta$ the operator $(A+M)^{1/2}(A+B+M)^{-1/2}$ is bounded with

$$\left\|(A+M)^{1/2}(A+B+M)^{-1/2}\right\| \le (1-\theta)^{-1/2}\,. \tag{1.61}$$

This proves the boundedness of the third factor in the proposition.

To prove (1.61), we first note that, since the left side of (1.59) is non-negative, we have $a[u] + (C/\theta)\|u\|^2 \geq 0$ for all $u \in d[a]$, and therefore $C/\theta \geq -m_a$. Again by (1.59), for any $M > C/\theta$ and any $u \in d[a]$ we have

$$a[u]+b[u]+M\|u\|^2 \geq (1-\theta)\left(a[u] + \frac{M-C}{1-\theta}\|u\|^2\right) \geq (1-\theta)\left(a[u] + M\|u\|^2\right).$$

Since $M > C/\theta \geq -m_a$, the right side is not smaller than a positive constant times $\|u\|^2$. This proves that the operator $(A+B+M)^{-1/2}$ exists and is bounded. Setting $u = (A+B+M)^{-1/2} f$ with $f \in \mathcal{H}$, the previous inequality becomes

$$\|f\|^2 \geq (1-\theta)\|(A+M)^{1/2}(A+B+M)^{-1/2} f\|^2 .$$

This implies (1.61).

Step 2. We show that for any $f, g \in \mathcal{H}$,

$$\begin{aligned}&\left(\left((A+B-z)^{-1} - (A-z)^{-1}\right) f, g\right)\\ &\qquad = -\left(\left[(A+m)^{1/2}(A-z)^{-1}\right] \hat{\mathcal{B}}_a \left[(A+m)^{1/2}(A+B-z)^{-1}\right] f, g\right).\end{aligned}$$

This proves the formula in the proposition. Since $z \in \rho(A) \cap \rho(A+B)$, we can define

$$u := (A+B-z)^{-1} f, \qquad v := (A-\overline{z})^{-1} g .$$

Then $u \in \operatorname{dom}(A+B)$ and $v \in \operatorname{dom} A$ and, in particular, $u, v \in d[a]$. Thus, by the definition of an operator corresponding to a quadratic form, the left side above is

$$\begin{aligned}\left(\left((A+B-z)^{-1} - (A-z)^{-1}\right) f, g\right) &= (u, (A-\overline{z})\, v) - ((A+B-z)\, u, v)\\ &= a[u,v] - (a[u,v] + b[u,v]) = -b[u,v] .\end{aligned}$$

For the right side above, using

$$\left((A+m)^{1/2}(A-z)^{-1}\right)^* = (A+m)^{1/2}(A-\overline{z})^{-1},$$

we have

$$\begin{aligned}&\left(\left[(A+m)^{1/2}(A-z)^{-1}\right]\hat{\mathcal{B}}_a\left[(A+m)^{1/2}(A+B-z)^{-1}\right] f, g\right)\\ &\quad = \left(\hat{\mathcal{B}}_a\,(A+m)^{1/2}\, u, (A+m)^{1/2}\, v\right) = \left(\mathcal{U}^*\mathcal{B}_a\mathcal{U}(A+m)^{1/2} u, (A+m)^{1/2}\, v\right)\\ &\quad = (\mathcal{U}^*\mathcal{B}_a u, \mathcal{U}^* v) = a[\mathcal{B}_a u, v] + m(\mathcal{B}_a u, v) = b[u,v] .\end{aligned}$$

Thus, both sides of the identity coincide, which completes the proof. □

We now introduce a relative compactness condition that is important in applications. Let $\mathcal{G}$ be a Hilbert space with norm $\|\cdot\|_*$ and let b be a real-valued quadratic form with $d[b] = \mathcal{G}$. Then b is said to be *compact* in $\mathcal{G}$ if there is a constant C such that $|b[u]| \leq C\|u\|_*^2$ for all $u \in \mathcal{G}$ and if the bounded operator in $\mathcal{G}$ induced by the form b is compact. Here we apply the Riesz representation theorem.

Lemma 1.49 *Let $\mathcal{G}$ be a Hilbert space with norm $\|\cdot\|_*$ and let b be a real-valued quadratic form with $d[b] = \mathcal{G}$ such that, for a constant C, one has $|b[u]| \le C\|u\|_*^2$ for all $u \in \mathcal{G}$. If b is compact in $\mathcal{G}$, then, for any sequence $(u_n) \subset \mathcal{G}$ that converges weakly in $\mathcal{G}$ to zero, we have $b[u_n] \to 0$. Conversely, if $b[u] \ge 0$ for all $u \in \mathcal{G}$ and if, for any sequence $(u_n) \subset \mathcal{G}$ that converges weakly in $\mathcal{G}$ to zero, we have $b[u_n] \to 0$, then b is compact in $\mathcal{G}$.*

Proof Let $(u_n) \subset \mathcal{G}$ be a sequence converging weakly to zero in $\mathcal{G}$. Denote by $\mathcal{B}$ the compact operator in $\mathcal{G}$ induced by the form b. Then, by Lemma 1.1, $(\mathcal{B}u_n)$ converges strongly to zero in $\mathcal{G}$, and therefore $b[u] = (\mathcal{B}u_n, u_n)_* \to 0$, as claimed.

To prove the converse, assume that $b[u] \ge 0$ for all $u \in \mathcal{G}$. By assumption, if $(u_n) \subset \mathcal{G}$ converges weakly to zero in $\mathcal{G}$, then $\|\mathcal{B}^{1/2}u_n\|_*^2 = b[u_n] \to 0$; that is, $\mathcal{B}^{1/2}u_n$ tends strongly to zero in $\mathcal{G}$. By Lemma 1.1, this implies that $\mathcal{B}^{1/2}$ is compact in $\mathcal{G}$. Thus $\mathcal{B} = (\mathcal{B}^{1/2})^2$ is compact, as claimed. □

The most frequent use of this relative compactness notion is when $\mathcal{G} = d[a]$, endowed with the norm $(a[u] + m\|u\|^2)^{1/2}$ for some $m > -m_a$.

Lemma 1.50 *Let a be a lower semibounded, closed quadratic form and assume that b is a real-valued quadratic form that is compact in $d[a]$ with the norm $(a[u] + m\|u\|^2)^{1/2}$ for some $m > -m_a$. Then for any $\theta > 0$ there is a C such that* (1.59) *holds.*

Proof We argue by contradiction. If the assertion of the lemma was false, there would be a sequence $(u_n) \subset d[a]$ with $a[u_n] + m\|u_n\|^2 = 1$ (for some $m > -m_a$) and

$$|b[u_n]| \ge \theta + n\|u_n\|^2 . \tag{1.62}$$

By weak compactness of the unit ball in $d[a]$, there is a subsequence (u_{n_m}) that converges weakly in $d[a]$ to some u. Since b is compact in $d[a]$, we have, by Lemma 1.1, $\mathcal{B}_a u_{n_m} \to \mathcal{B}_a u$ strongly in $d[a]$ and therefore $b[u_{n_m}] \to b[u]$. Thus, (1.62) implies that $n_m\|u_{n_m}\|^2$ is bounded, and hence $u_{n_m} \to 0$ in $\mathcal{H}$. Since $d[a]$ is continuously embedded into $\mathcal{H}$, we deduce that $u = 0$, and then (1.62) leads to a contradiction since $\theta > 0$. □

We end this subsection with a variant of Weyl's theorem (Theorem 1.14) for operators defined via quadratic forms.

Theorem 1.51 *Let a be a densely defined, lower semibounded and closed*

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quadratic form and let b be a real-valued quadratic form that is compact in $d[a]$. Then the operators A and $A + B$ corresponding to a and $a + b$ satisfy $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A + B)$.

Proof We want to apply Theorem 1.14 and therefore have to show that

$$(A + B - z)^{-1} - (A - z)^{-1}$$

is compact. This follows immediately from the resolvent identity in Proposition 1.48 since the middle factor $\hat{\mathcal{B}}_a$ is compact by assumption and the outer factors are bounded. This completes the proof. □

1.2.8 The Birman–Schwinger principle

In this subsection, we work under the following assumptions, depending on a number $\alpha > 0$.

(H_α) Let a be a densely defined, non-negative, closed quadratic form in a Hilbert space $\mathcal{H}$ with domain $d[a]$ satisfying

$$a[u] > 0 \quad \text{for all } 0 \neq u \in d[a]\,. \tag{1.63}$$

Let b be a real-valued quadratic form satisfying, for some $M < \infty$,

$$d[b] \supset d[a] \quad \text{and} \quad |b[u]| \leq M\, a[u] \quad \text{for all } u \in d[a]\,. \tag{1.64}$$

Assume that the quadratic form $a - \alpha b$ is lower semibounded and closed in $\mathcal{H}$, and denote by $A - \alpha B$ the corresponding self-adjoint operator.

Assumption (1.63) implies that $\sqrt{a[\cdot]}$ is a norm in $d[a]$. Let $\mathcal{H}_a$ be the completion of $d[a]$ with respect to this norm and let $\hat{a}$ be the extension of a by continuity to $\mathcal{H}_a$.

If (1.63) is replaced by the stronger assumption that there is an $\varepsilon > 0$ with

$$a[u] \geq \varepsilon \|u\|^2 \qquad \text{for all } u \in d[a]\,, \tag{1.65}$$

then $\mathcal{H}_a$ coincides with $d[a]$ (with equivalent norms). In general, this need not be the case and $\mathcal{H}_a$ may not be a subset of $\mathcal{H}$.

For practical purposes it is useful to note that, if $\mathcal{F} \subset d[a]$ is dense in $d[a]$ (with respect to the norm $\sqrt{a[\cdot] + m\| \cdot \|^2}$ for some $m > -m_a$), then $\mathcal{H}_a$ coincides with the completion of $\mathcal{F}$ with respect to $\sqrt{a[\cdot]}$.

Assumption (1.64) implies that b can be extended by continuity to a quadratic form $\hat{b}$ on $\mathcal{H}_a$. This extended quadratic form defines a bounded, self-adjoint operator $\mathcal{B}_a$ on $\mathcal{H}_a$. We emphasize that the operator $\mathcal{B}_a$ is related to, but different from, the operator $\mathcal{B}_a$ appearing in (1.60). Indeed, the operator in

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(1.60) is defined on $d[a]$ with norm $\sqrt{a[\cdot] + m\|\cdot\|^2}$, where $m > -m_a$. Now we allow for $m = -m_a$, which is why we have to introduce the possibly larger space $\mathcal{H}_a$.

The parameter $\alpha > 0$ in assumption (H_α) is not really necessary, since (H_α) holds if and only if (H_1) holds with b replaced by αb. Nevertheless, in applications it is sometimes convenient to have this parameter present. Conditions that guarantee that $a - \alpha b$ is lower semibounded and closed in $\mathcal{H}$ are given in Lemmas 1.47 and 1.50. The notation $A - \alpha B$ is, in general, an abuse of notation since there need not be a well-defined, self-adjoint operator αB given by the difference of A and $A - \alpha B$.

We denote the spectral measures of $A - \alpha B$ and $\mathcal{B}_a$ by $P(A - \alpha B)$ and $P(\mathcal{B}_a)$, respectively.

The following result is called the *Birman–Schwinger principle*.

Theorem 1.52 *Assume* (H_α) *for some* $\alpha > 0$. *Then*

$$\dim P_{(-\infty,0)}(A - \alpha B)\mathcal{H} = \dim P_{(\alpha^{-1},\infty)}(\mathcal{B}_a)\mathcal{H}_a\,. \tag{1.66}$$

If, in addition, (1.65) *holds, then*

$$\dim P_{(-\infty,0]}(A - \alpha B)\mathcal{H} = \dim P_{[\alpha^{-1},\infty)}(\mathcal{B}_a)\mathcal{H}_a \tag{1.67}$$

and

$$\dim\ker(A - \alpha B) = \dim\ker(\mathcal{B}_a - \alpha^{-1})\,. \tag{1.68}$$

Proof It follows from Glazman's lemma (Theorem 1.25) applied to the operator $-\mathcal{B}_a$ in the Hilbert space $\mathcal{H}_a$ that, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}&\dim P_{(\lambda,\infty)}(\mathcal{B}_a)\mathcal{H}_a\\ &\quad = \sup\{\dim F\colon\ F \subset d[a] \text{ and } b[u] > \lambda a[u] \text{ for all } 0 \neq u \in F\}\ .\end{aligned}$$

Here we used the fact that $d[a]$ is dense in $\mathcal{H}_a$. On the other hand, applying Theorem 1.25 to the operator $A - \alpha B$ in the Hilbert space $\mathcal{H}$, we find that

$$\begin{aligned}&\dim P_{(-\infty,0)}(A - \alpha B)\mathcal{H}\\ &\quad = \sup\left\{\dim F\colon\ F \subset d[a] \text{ and } b[u] > \alpha^{-1}a[u] \text{ for all } 0 \neq u \in F\right\}\ .\end{aligned}$$

Choosing $\lambda = \alpha^{-1}$, we obtain (1.66).

Now, assuming (1.65), let us prove (1.68). Note that, once this is proved, (1.67) follows by adding (1.66) and (1.68).

To show (1.68), we show that $\ker(A - \alpha B) = \ker(\mathcal{B}_a - \alpha^{-1})$. Note that, under (1.65), $\mathcal{H}_a$ can be considered as a subset of $\mathcal{H}$. We have $u \in \ker(A - \alpha B)$ if and only if

$$b[u, v] = \alpha^{-1}a[u, v] \quad \text{for all } v \in d[a]\,.$$

By the definition of $\mathcal{B}_a$, this means

$$(\mathcal{B}_a u, v)_{\mathcal{H}_a} = \alpha^{-1}(u, v)_{\mathcal{H}_a} \quad \text{for all } v \in \mathcal{H}_a\,,$$

which is equivalent to $u \in \ker(\mathcal{B}_a - \alpha^{-1})$, as claimed.

As an aside, we mention that (1.67) can be proved using the second identity in Theorem 1.25. Here it is important that under assumption (1.65), $d[a]$ is not only dense in $\mathcal{H}_a$, but actually equal to $\mathcal{H}_a$. If $\dim P_{(-\infty,0)}(A - \alpha B)\mathcal{H} < \infty$, then (1.68) follows by subtracting (1.66) from (1.67). $\square$

We emphasize that (1.67) and (1.68) may fail if assumption (1.65) does not hold. Elements of $\ker(\mathcal{B}_a - \alpha^{-1})$ may belong to $\mathcal{H}_a$ without belonging to $\mathcal{H}$; see the reference in §1.3.

The reason why Theorem 1.52 is useful is that it relates the number of negative eigenvalues of a lower semibounded operator to the number of eigenvalues of a bounded (and typically compact) operator. For example, by Lemma 1.15, an immediate consequence of Theorem 1.52 is the following.

Corollary 1.53 *Assume* (H$_\alpha$) *for every* $\alpha > 0$. *Then*

$$\dim P_{(-\infty,0)}(A - \alpha B)\mathcal{H} < \infty \qquad \textit{for all } \alpha > 0$$

if and only if $(\mathcal{B}_a)_+$ *is compact in* $\mathcal{H}_a$. *For this, it is sufficient that* $\hat{b}$ *is compact in* $\mathcal{H}_a$. *If* $b \geq 0$, *the compactness of* $\hat{b}$ *in* $\mathcal{H}_a$ *is also necessary.*

For applications it is useful to reformulate Theorem 1.52 in terms of operators acting on the original Hilbert space $\mathcal{H}$ rather than on $\mathcal{H}_a$. We note that assumption (1.63) guarantees that $A^{-1/2}$ is a densely defined operator in $\mathcal{H}$.

Lemma 1.54 *Assume* (1.63) *and* (1.64). *Then the quadratic form*

$$b[A^{-1/2} f]\,, \qquad f \in \operatorname{dom} A^{-1/2}\,,$$

is real-valued, densely defined and bounded in $\mathcal{H}$. *The corresponding bounded self-adjoint operator in* $\mathcal{H}$ *is unitarily equivalent to the operator* $\mathcal{B}_a$ *in* $\mathcal{H}_a$.

In the formulation and proof of the lemma we use the fact that a real-valued, densely defined and bounded quadratic form generates a bounded, self-adjoint operator. This is a consequence of the Riesz representation theorem.

Proof We consider the operator $A^{-1/2}$, defined on $\operatorname{dom} A^{-1/2}$, as a mapping from a dense subset of $\mathcal{H}$ into $d[a] = \operatorname{dom} A^{1/2}$ equipped with the norm $\sqrt{a[\cdot]}$. This mapping is isometric since $a[A^{-1/2} f] = \|f\|^2$ for all $f \in \operatorname{dom} A^{-1/2}$. Moreover, $\operatorname{ran} A^{-1/2} = d[a]$. Since $\mathcal{H}_a$ is the completion of $d[a]$ with respect to $\sqrt{a[\cdot]}$, the above mapping extends to a unitary operator $U : \mathcal{H} \to \mathcal{H}_a$.

By definition of $\mathcal{B}_a$, we have (recall $\hat{a}$, $\hat{b}$ from the discussion of (H_α))

$$\hat{a}[\mathcal{B}_a\varphi,\varphi] = \hat{b}[\varphi] \qquad \text{for all } \varphi \in \mathcal{H}_a\,.$$

We apply this identity to $\varphi = Uf$ with $f \in \operatorname{dom} A^{-1/2}$. Then, since $Uf = A^{-1/2}f \in d[a] \subset d[b]$ for $f \in \operatorname{dom} A^{-1/2}$, we have $\hat{b}[\varphi] = b[A^{-1/2}f]$, and therefore

$$\hat{a}[\mathcal{B}_a Uf, Uf] = b[A^{-1/2}f] \quad \text{for all } f \in \operatorname{dom} A^{-1/2}\,.$$

Since $U : \mathcal{H} \to \mathcal{H}_a$ is unitary, the previous identity implies that

$$(U^*\mathcal{B}_a Uf, f) = b[A^{-1/2}f] \qquad \text{for all } f \in \operatorname{dom} A^{-1/2}\,.$$

By (1.64), the operator $\mathcal{B}_a$ is bounded on $\mathcal{H}_a$, and therefore $U^*\mathcal{B}_a U$ is bounded on $\mathcal{H}$. This implies the assertions of the lemma. □

We shall denote the operator in Lemma 1.54 by $A^{-1/2}BA^{-1/2}$ and its spectral measure by $P(A^{-1/2}BA^{-1/2})$.

Writing $A^{-1/2}BA^{-1/2}$ is an abuse of notation, which, however, is motivated by the following. Under assumption (1.65), the operator $A^{-1/2}$ is bounded and, under the additional assumption that the quadratic form b is bounded in $\mathcal{H}$, it generates a bounded operator B in $\mathcal{H}$. In this case, the quadratic form $b[A^{-1/2}f]$, $f \in \operatorname{dom} A^{-1/2} = \mathcal{H}$, is closed and the corresponding operator is simply the product of the three bounded operators $A^{-1/2}BA^{-1/2}$.

In view of Lemma 1.54, Theorem 1.52 can be reformulated as follows.

Theorem 1.55 *Assume* (H_α) *for some* $\alpha > 0$. *Then*

$$\dim P_{(-\infty,0)}(A - \alpha B)\mathcal{H} = \dim P_{(\alpha^{-1},\infty)}(A^{-1/2}BA^{-1/2})\mathcal{H}\,.$$

If, in addition, (1.65) *holds, then*

$$\dim P_{(-\infty,0]}(A - \alpha B)\mathcal{H} = \dim P_{[\alpha^{-1},\infty)}(A^{-1/2}BA^{-1/2})\mathcal{H}$$

and

$$\dim\ker(A - \alpha B) = \dim\ker(A^{-1/2}BA^{-1/2} - \alpha^{-1})\,.$$

Finally, we discuss the case where b is non-negative and where there is an operator Q in $\mathcal{H}$ such that

$$\operatorname{dom} Q \supset \operatorname{dom} A^{-1/2} \quad \text{and} \qquad b[A^{-1/2}f] = \|Qf\|^2 \ \text{ for all } f \in \operatorname{dom} A^{-1/2}\,. \tag{1.69}$$

Assuming (1.63) and (1.64), it follows from Lemma 1.54 that any operator Q satisfying (1.69) is densely defined and bounded. In particular, it has a unique extension to a bounded operator defined on all of $\mathcal{H}$. Therefore, we may assume without loss of generality that Q is closed and defined on all of $\mathcal{H}$.

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By (1.69), the operator $A^{-1/2}BA^{-1/2}$ is equal to Q^*Q. We denote the operator QQ^* by $B^{1/2}A^{-1}B^{1/2}$ and its spectral measure by $P(B^{1/2}AB^{1/2})$. Writing $B^{1/2}A^{-1}B^{1/2}$ is an abuse of notation that can be motivated as in the discussion preceding Theorem 1.55. The operator $B^{1/2}A^{-1}B^{1/2}$ is called the *Birman–Schwinger operator*.

By Proposition 1.23 and Corollary 1.24, Theorem 1.55 yields the following.

Theorem 1.56 *Assume* (H_α) *for some* $\alpha > 0$ *and* (1.69)*. Then*

$$\dim P_{(-\infty,0)}(A - \alpha B)\mathcal{H} = \dim P_{(\alpha^{-1},\infty)}(B^{1/2}A^{-1}B^{1/2})\mathcal{H}\,. \tag{1.70}$$

If, in addition, (1.65) *holds, then*

$$\dim P_{(-\infty,0]}(A - \alpha B)\mathcal{H} = \dim P_{[\alpha^{-1},\infty)}(B^{1/2}A^{-1}B^{1/2})\mathcal{H} \tag{1.71}$$

and

$$\dim\ker(A - \alpha B) = \dim\ker(B^{1/2}A^{-1}B^{1/2} - \alpha^{-1})\,. \tag{1.72}$$

Corollary 1.57 *Assume* (H_α) *for some* $\alpha > 0$*,* (1.69) *and that the negative spectrum of* $A - \alpha B$ *is discrete. Let* $(-E_n)$ *be its negative eigenvalues, in nondecreasing order and repeated according to multiplicities. For fixed n, let* (μ_m) *be the eigenvalues of* $B^{1/2}(A + E_n)B^{1/2}$ *greater than or equal to* α^{-1} *in nonincreasing order and repeated according to multiplicities. Then* $\mu_n = \alpha^{-1}$*.*

Note that, by (1.71) with $A + E_n$ instead of A, the total spectral multiplicity of $B^{1/2}(A + E_n)B^{1/2}$ in $[\alpha^{-1},\infty)$ is finite. Therefore, the μ_m in the corollary are well defined. Here we use the fact that for $A + E_n$, assumption (1.65) holds (with $\varepsilon = E_n$).

Proof Let K be the multiplicity of the eigenvalue $-E_n$ of $A - \alpha B$ and let k be such that $E_n = E_k = \cdots = E_{k+K-1}$. That is, $k = n$ if $-E_n$ is simple, and otherwise k is the minimal index for which $E_k = E_n$. Note also that $k \le n \le k + K - 1$.

By (1.72) (applied with $A+E_n$ instead of A), the operator $B^{1/2}(A+E_n)^{-1}B^{1/2}$ has an eigenvalue α^{-1} and its multiplicity is K. Thus, there is an ℓ such that $\alpha^{-1} = \mu_\ell = \cdots = \mu_{\ell+K-1}$ and, if $\ell \ge 2$, $\mu_{\ell-1} > \alpha^{-1}$.

We observe that $\ell = k$ since by (1.70) (applied with $A + E_n$ instead of A),

$$k-1 = \dim P_{(-\infty,-E_n)}(A-\alpha B)\mathcal{H} = \dim P_{(\alpha^{-1},\infty)}(B^{1/2}(A+E_n)B^{1/2})\mathcal{H} = \ell-1\,.$$

Since $k \le n \le k+K-1$, this shows $\ell \le n \le \ell+K-1$, and thus $\mu_n = \alpha^{-1}$. □

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1.3 Comments

We have not given a full account of spectral theory in this chapter, but rather selected material that is needed for our applications to Laplace or Schrödinger operators. Further material can be found in such textbooks as Akhiezer and Glazman (1963), Birman and Solomjak (1987), Davies (1995), Helffer (2013), Reed and Simon (1972, 1975, 1978, 1979), and Teschl (2014), as well as in the monograph of Kato (1980).

Section 1.1: Hilbert spaces, self-adjoint operators, and the spectral theorem
For the history of the development of the notion of unbounded operators we refer to Simon (2015c, §7.1). A first version of the spectral theorem for bounded self-adjoint operators goes back to Hilbert (1906). For further references on early developments, see Simon (2015c, §5.1). The spectral theorem in the unbounded case is due to von Neumann (1930). The above-mentioned references give several different proofs of this theorem. We also refer to the above-mentioned textbooks for the notion of a normal operator and for the corresponding spectral theorem.

There is a proof of the formula (1.24) for the norm of the resolvent that does not use the spectral theorem, but instead the fact that for a bounded, normal operator, the spectral radius is equal to the norm (Simon, 2015c, Theorems 2.2.10 and 2.2.11); see also Kato (1980, (V.3.16)) and Edmunds and Evans (2018, Lemma 3.4.3).

Theorem 1.14 on the stability of the essential spectrum is from Weyl (1909). The quadratic form version in Theorem 1.51 appears in Birman (1959).

Section 1.2: Semibounded operators and forms and the variational principle
The proof of Theorem 1.16 that we presented uses the square root of a non-negative self-adjoint operator, which we defined using the spectral theorem and the functional calculus. For an alternative construction of the square root, see Kato (1980, §V.3.11).

Theorem 1.18, which associates to each lower semibounded, closed quadratic form a self-adjoint operator, is due to Friedrichs (1934a). Applications to second-order differential operators are given in Friedrichs (1934b). For a proof that does not require self-adjointness, see Kato (1980, §VI.2).

Lemma 1.20 is well known to specialists, but hard to locate in the literature; see Lewin (2022, Théorème 5.14). One half of Corollary 1.21, namely that compactness of an embedding implies discreteness of the spectrum, appears in

Friedrichs (1934a, Zusatz 18) abstracting the arguments in Rellich (1930). An operator-domain version of the lemma appears in Rellich (1942).

Different versions of the variational principle go back to Rayleigh (1877), Fischer (1905), Ritz (1908), Weyl (1912a,b) and Courant (1920). An operator version of Glazman's lemma (Theorem 1.25) appears in Glazman (1966, §3, Theorem 9bis). The version for quadratic forms is mentioned, for instance, in Birman (1961).

Proposition 1.33 (at least in finite dimension) is due to Fan (1949). Proposition 1.33 and Corollary 1.35 remain valid when the orthogonality condition $(u_n, u_m) = \delta_{nm}$ is replaced by the sub-orthogonality condition that the eigenvalues of the matrix (u_n, u_m) are between 0 and 1; see, for instance, Simon (2011, Proposition 15.13). This is sometimes useful in applications and appears implicitly in §7.4.

Proposition 1.43 is from Rotfel'd (1967, 1968).

The Birman–Schwinger principle, discussed in various forms in §1.2.8, appears in Birman (1961) and Schwinger (1961); for a detailed presentation, see also Birman and Solomyak (1992). This abstract principle will be spelled out for Schrödinger operators in §4.3.3. In §4.9 we give references for concrete applications of this principle.

Let us already here give a typical example of the setting of §1.2.8, using freely the notation in the next chapter. In the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \geq 1$, and with a parameter $\tau \geq 0$ we consider the quadratic form $a[u] := \int_{\mathbb{R}^d} \left(|\nabla u|^2 + \tau |u|^2\right) dx$ with form domain $d[a] := H^1(\mathbb{R}^d)$. Then (1.63) is satisfied, and (1.65) is satisfied if and only if $\tau > 0$. For $\tau > 0$, one has $\mathcal{H}_a = H^1(\mathbb{R}^d)$ (with equivalent norms). For $\tau = 0$ and $d \geq 3$, one has $\mathcal{H}_a = \dot{H}^1(\mathbb{R}^d)$, the homogeneous Sobolev space, while for $\tau = 0$ and $d = 1, 2$, the space $\mathcal{H}_a$ is not a space of (a.e. equivalence classes of) functions and is typically avoided; see §2.7.2. In applications to Schrödinger operators with sufficiently regular $V \geq 0$, one chooses $b[u] := \int_{\mathbb{R}^d} V|u|^2\, dx$. Then the Birman–Schwinger principle, for instance in its form in Theorem 1.56, relates the number of eigenvalues less than $-\tau < 0$ of $-\Delta - V$ to the number of eigenvalues greater than 1 of the Birman–Schwinger operator $V^{1/2}(-\Delta + \tau)^{-1}V^{1/2}$. As mentioned after Theorem 1.52, the equalities (1.68) (or, equivalently, (1.72)) need not be true if (1.65) fails. In the concrete case of Schrödinger operators in dimensions $d \geq 3$, this occurs if there is a solution $0 \neq u \in \dot{H}^1(\mathbb{R}^d)$ of $-\Delta u - Vu = 0$ that does not belong to $L^2(\mathbb{R}^d)$. Such solutions are counted in the right side of (1.72), but not in the left side. One speaks of a 'zero energy resonance' or a 'virtual level'; see, for instance, Jensen and Kato (1979), Jensen (1980, 1984) and Klaus and Simon (1980) for more on this phenomenon.