GROUPS OF SMALL SYMMETRIC GENUS

by COY L. MAY and JAY ZIMMERMAN

(Received 2 July, 1993)

1. Introduction. Group actions on compact surfaces have received considerable attention during the past century. The surface has often carried an analytic structure and been considered a Riemann surface or, equivalently, a complex algebraic curve.

In connection with group actions on surfaces, there is a natural parameter associated with each finite group. A finite group G can be represented as a group of automorphisms of a compact Riemann surface, that is, G acts on a Riemann surface. The *symmetric genus* $\sigma(G)$ is the minimum genus of any Riemann surface on which G acts (possibly reversing orientation).

The origins of the symmetric genus parameter can be traced to the work of Hurwitz, Poincare, Burnside and others (see [**14**] and [**4**]). The modern terminology was introduced in the important article [**25**]. There is now a considerable body of work on the symmetric genus parameter. Much of this has concentrated on finite simple groups. For example, the symmetric genus of each projective special linear group $\mathrm{PSL}(2,p)$ has been calculated [**12**]. In addition, Conder determined the symmetric genus of all alternating and symmetric groups [**6**], and the genus of most of the sporadic simple groups has now been determined [**8**]. Here also see the survey article [**7**].

Another body of work has concentrated on solvable groups. The symmetric genus of each finite abelian group has been determined [**17**]; also relevant here is the work of Maclachlan [**19**]. The K-metacyclic groups were considered in [**18**].

Here we obtain a good general lower bound for the symmetric genus of a finite group G. We use the standard representation of G as a quotient of a non-euclidean crystallographic group Γ by a surface group K; then G acts on the Riemann surface U/K, where U is the open upper half-plane. We give several examples of infinite families of groups for which the lower bound is attained. In particular, we calculate the symmetric genus of each Hamiltonian group with no odd order part.

We also consider two natural problems connected with groups of small symmetric genus. We complete the determination of the symmetric genus of each group with order less than 48, with the exception of order 32. In addition, we classify the groups of symmetric genus 2. In this classification, we use the correspondence between Riemann surfaces with large automorphism groups and regular maps. The proof does not use Tucker's related result [**26**] on the unique group of graph theoretic genus 2.

In our work on the small order groups, we frequently employ the computer algebra system CAYLEY. This powerful tool was developed by Cannon [**5**].

2. Preliminaries. Non-euclidean crystallographic groups (NEC groups) have been quite useful in investigating group actions on surfaces. We shall assume that all surfaces are compact. Let $\mathscr{L}$ denote the group of automorphisms of the open upper half-plane U,

---

AMS (*MOS*) *subject classification (1992).* Primary 57M60; secondary 05C10, 20H10, 30F99, 57M15.
Key words and phrases. Symmetric genus, Riemann surface, Fuchsian group, non-euclidean crystallographic group, automorphism group, regular map, hamiltonian group, graph theoretic genus.

and let $\mathscr{L}^+$ denote the subgroup of index 2 consisting of the orientation-preserving automorphisms. An NEC group is a discrete subgroup Γ of $\mathscr{L}$ (with the quotient space U/Γ compact). If $\Gamma \subseteq \mathscr{L}^+$, then Γ is called a *Fuchsian* group. Otherwise Γ is called a *proper NEC group*; in this case Γ has a canonical Fuchsian subgroup $\Gamma^+ = \Gamma \cap \mathscr{L}^+$ of index 2.

Associated with the NEC group Γ is its *signature*, which has the form

$$(p; \pm; [\lambda_1, \ldots, \lambda_t]; \{(\nu_{11}, \ldots, \nu_{1s_1}), \ldots, (\nu_{k1}, \ldots, \nu_{ks_k})\}). \tag{2.1}$$

The quotient space $X = U/\Gamma$ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and non-orientable otherwise. Associated with the signature (2.1) is a presentation for the NEC group Γ.

Let Γ be an NEC group with signature (2.1). The non-euclidean area $\mu(\Gamma)$ of a fundamental region Γ can be calculated directly from its signature [**23**, p. 235]:

$$\mu(\Gamma)/2\pi = \alpha p + k - 2 + \sum_{i=1}^{t}\left(1 - \frac{1}{\lambda_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{\nu_{ij}}\right), \tag{2.2}$$

where $\alpha = 2$ if the plus sign is used and $\alpha = 1$ otherwise.

An NEC group K is called a *surface group* if the quotient map from U to U/K is unramified. Let X be a Riemann surface of genus $g \geq 2$. Then X can be represented as U/K where K is a Fuchsian surface group with $\mu(K) = 4\pi(g-1)$. Let G be a group of dianalytic automorphisms of the Riemann surface X. Then there is an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$.

If Λ is a subgroup of finite index in Γ, then $[\Gamma : \Lambda] = \mu(\Lambda)/\mu(\Gamma)$. It follows that the genus of the surface U/K on which $G \cong \Gamma/K$ acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \tag{2.3}$$

Minimizing g is therefore equivalent to minimizing $\mu(\Gamma)$.

3. Groups of genera zero and one. For groups of genus zero or one, there is a close connection between the symmetric genus and the graph theoretic genus $\gamma(G)$ [**27**] of a group G. We always have $\gamma(G) \leq \sigma(G)$ [**25**, p. 90].

The groups of symmetric genus zero and graph theoretic genus zero are the same; these groups are quite well-known [**27**, p. 84]. The groups of genus one are also known, at least in a sense. In 1978 Proulx completed the classification of the groups of graph theoretic genus one [**21**]. These groups fall into thirty classes, many of which are infinite. Each class is characterized by a presentation, typically a partial one. The two genus parameters usually agree in genus one. If $\sigma(G) = 1$, then also $\gamma(G) = 1$. On the other hand, if $\gamma(G) = 1$, then $\sigma(G) = 1$ unless G is one of three exceptional groups, the groups with presentations (2.8), (2.9), and (2.19) in Proulx's classification [**13**, §§ 6.4.3, 6.4.6]. The groups with presentations (2.19) and (2.8) are SL(2, 3) and GL(2, 3), respectively. The group SL(2, 3) is also called the binary tetrahedral group (the notation $\langle 2, 3, 3\rangle$ is used in [**10**]). The third exceptional group, which we denote by P_{48}, has presentation

$$R^3 = S^2 = (RS)^3(R^{-1}S)^3 = 1. \tag{3.1}$$

It is another group of order 48 that contains SL(2, 3) [**13**, p. 313].

The graph theoretic genus $\gamma(G)$ of each group G with order less than 32 has been determined. All groups in this range have graph theoretic genus zero or one, except two groups of order 27 and $Z_2 \times Q$ where Q is the quaternions. First $\gamma(Z_2 \times Q) = 3$ [**27**, *p.* 92]. The abelian group $(Z_3)^3$ has graph theoretic genus 7 [**1**]. The third exceptional group in this range is the semi-direct product $Z_9 \times_\varphi Z_3$, and $\gamma(Z_9 \times_\varphi Z_3) = 4$ [**2**].

The symmetric genus of the groups of order less than 32 is probably known, as well, although we do not know a reference. To give $\sigma(G)$ for each group with $|G| < 32$, it is only necessary to consider $Z_2 \times Q$, SL(2, 3), $Z_9 \times_\varphi Z_3$, and $(Z_3)^3$.

4. Riemann surfaces and regular maps. Especially important in the study of automorphisms of Riemann surfaces are the triangle groups. A triangle group is a Fuchsian group with signature

$$(0; +; [\ell, m, n]; \{\}) \quad \text{where} \quad 1/\ell + 1/m + 1/n < 1.$$

We shall denote a group with this signature by $\Gamma(\ell, m, n)$. Large groups of orientation-preserving automorphisms are quotients of triangle groups. Singerman first made this idea precise in [**24**, p. 22]. We expand his result slightly.

LEMMA A [**24**]. *Let G be a group of orientation-preserving automorphisms of a Riemann surface of genus $g \geq 2$. If $|G| > 12(g-1)$, then G is a quotient of a triangle group. If $|G| > 24(g-1)$, then, furthermore, one of the periods of the triangle group is* 2. *If $24(g-1) \geq |G| > 12(g-1)$, then one of the periods of the triangle group is* 2 *except in the following cases; in each case G is a quotient of the triangle group listed.*

1) $|G| = 24(g-1)$ $\quad\Gamma(3, 3, 4)$

2) $|G| = 15(g-1)$ $\quad\Gamma(3, 3, 5)$

There is a strong connection here with the theory of regular maps [**10**] on surfaces. For the basic definitions on regular maps, see [**10**, pp. 20, 101–103]. A map is said to be of type $\{r, q\}$ if it is composed of r-gons, q meeting at each vertex. Large groups of automorphisms of Riemann surfaces correspond in a general way to groups of regular maps. A Riemann surface that has an orientation-reversing involution is said to be *symmetric* [**24**].

Let G be a group of orientation-preserving automorphisms of a Riemann surface W. If G is a quotient of a triangle group $\Gamma(2, n, k)$, then there is a regular map of type $\{n, k\}$ on the topological surface W. If the map is reflexible, then W is a symmetric Riemann surface, and further the automorphism group of W is isomorphic to the full automorphism group of the map.

Conversely, if G is the rotation group of a regular map of type $\{n, k\}$ on a surface S, then G is a quotient of a triangle group $\Gamma(2, n, k)$ and G acts as a group of orientation-preserving automorphisms of a Riemann surface homeomorphic to S. The symmetry of the Riemann surface need not imply the reflexibility of the map. Indeed, there are symmetric Riemann surfaces that correspond to irreflexible maps [**24**, p. 30]. This is not a consideration for surfaces of low genera, however, since Garbe [**11**, p. 42] has shown that for $2 \leq g \leq 6$, there are no irreflexible maps at all. For more details on this correspondence, see [**24**, pp. 27, 28].

Hence, for $2 \leq g \leq 6$, symmetric Riemann surfaces with large automorphisms groups correspond to reflexible regular maps. A great deal is known about the automorphism groups of Riemann surfaces and regular maps of these genera. In particular, a table of the regular maps of genus 2 is in [**10**, p. 140]. In addition, Bujalance and Singerman determined the possibilities for the full automorphism group of a symmetric Riemann surface of genus 2 [**3**] (The table [**3**, p. 518] does have a missing entry [**15**, p. 70]). We shall classify the groups of symmetric genus 2 by using the completed classification of the regular maps of genus 2.

5. A lower bound. Here we establish a useful lower bound for the symmetric genus of a finite group.

Let G be a finitely presented group and S a generating set for G. For $q=2$ or $q=3$, let $t_q(S)$ denote the number of generators in S of order q. Also let $t_h(S)$ be the number of generators of order larger than 3 (using "h" for high order). We will write simply t_q and t_h if the generating set is obvious. Then $|S|=t_2+t_3+t_h$. We define

$$\psi(G)=\text{minimum }\{9t_h(S)+8t_3(S)+3t_2(S) \mid S \text{ a generating set for } G\}.$$

A generating set for which $\psi(G)$ is attained is said to be *ψ-minimal.* The parameter $\psi(G)$ appears in our lower bound for the symmetric genus of a finite group. The same parameter also appears in a general lower bound for the real genus of a group [**16**]. A similar parameter is used to study the graph theoretic genus of a group in [**20**]. The following result is basic; for a proof, see [**16**, § 3].

LEMMA 1. *Let G' be a quotient group of the finitely presented group G. Then*

$$\psi(G)\geq\psi(G').$$

Next we find a useful upper bound for $\psi(\Gamma)$ for an NEC group Γ; the bound depends upon the topological type of the quotient space U/Γ, however. To establish the following, simplify the canonical presentation for Γ and apply the definition of $\psi(\Gamma)$.

LEMMA 2. *Let Γ be an NEC group with signature* (2.1). *For $q=2$ or $q=3$, let r_q denote the number of ordinary periods equal to q; let r_h be the number greater than* 3.

1) *Suppose $k\geq 1$ and exactly ℓ of the k period cycles are empty. Then*

$$\psi(\Gamma)\leq 9(\gamma+r_h)+8r_3+3(r_2+B+\ell), \tag{5.1}$$

where $B=s_1+\ldots+s_k$ and γ is the algebraic genus of the quotient space U/Γ. Also, the number of generators in the simplified presentation is $\gamma+r_h+r_3+r_2+B+\ell$.

2) *Suppose $k=0$. Then*

$$\psi(\Gamma)\leq 9(\alpha p+r_h)+8r_3+3r_2, \tag{5.2}$$

where $\alpha=2$ if the plus sign is used and $\alpha=1$ otherwise. The number of generators in the simplified presentation is $\alpha p+r_h+r_3+r_2$; if $r>0$, then, further, one elliptic generator is redundant.

Now we establish our general lower bound.

THEOREM 1. *Let G be a finite group with $\sigma(G)\geq 2$. Then*

$$\sigma(G)\geq 1+|G|\,[\psi(G)-16]/24. \tag{5.3}$$

Proof. Let G act on the Riemann surface X of genus $g \geq 2$. Represent X as U/K where K is a surface group, and obtain an NEC group Γ and a homomorphism $\phi : \Gamma \to G$ onto G such that kernel $\phi = K$. We distinguish two main cases.

Case I. G acts on X with reflections so that $X/G = U/\Gamma$ is a bordered surface and $k \geq 1$. We shall use the notation of Lemma 2. From (2.2)

$$\mu(\Gamma)/2\pi \geq \gamma - 1 + r_2 \cdot \tfrac{1}{2} + r_3 \cdot \tfrac{2}{3} + r_h \cdot \tfrac{3}{4} + B \cdot \tfrac{1}{4}.$$

Write $M = 12[\mu(\Gamma)/2\pi]$. Then

$$M \geq 12\gamma + 9r_h + 8r_3 + 6r_2 + 3B - 12. \tag{5.4}$$

We always have $\ell \leq k \leq \gamma + 1$, and thus $12\gamma \geq 9\gamma + 3(\ell - 1)$. Now

$$M \geq 9(\gamma + r_h) + 8r_3 + 3(r_2 + B + \ell) - 15.$$

Applying (5.1) and Lemma 1 yields $M \geq \psi(G) - 15$ in this case.

Case II. G acts on X without reflections so that $X/G = U/\Gamma$ is a surface without boundary and Γ has no period cycles ($k = 0$). First assume that the quotient space U/Γ is orientable so that Γ has signature $(p; +; [\lambda_1, \ldots, \lambda_r]; \{\})$. From (2.2)

$$\mu(\Gamma)/2\pi \geq 2p - 2 + r_2 \cdot \tfrac{1}{2} + r_3 \cdot \tfrac{2}{3} + r_h \cdot \tfrac{3}{4}.$$

Therefore

$$M \geq 18p + 9r_h + 8r_3 + 3r_2 + (6p + 3r_2) - 24. \tag{5.5}$$

First suppose $p \geq 2$. Then $6p \geq 12$ and from (5.2) and Lemma 1, we have $M \geq \psi(G) - 12$. Next suppose $r_h > 0$. Then in the simplified presentation for Γ in Lemma 2, there is a redundant elliptic generator of high order. Thus (5.2) can be improved so that

$$\psi(\Gamma) \leq 9(2p + r_h - 1) + 8r_3 + 3r_2.$$

From (5.5) and Lemma 1, we have $M \geq \psi(G) - 15$.

Now assume $p \leq 1$ and $r_h = 0$. Then (5.5) becomes

$$M \geq 24p + 8r_3 + 6r_2 - 24. \tag{5.6}$$

Suppose $r_3 > 0$. In the simplified presentation for Γ in Lemma 2, there is a redundant elliptic generator of order 3. Thus (5.2) (with $r_h = 0$) can be improved so that

$$\psi(\Gamma) \leq 18p + 8(r_3 - 1) + 3r_2.$$

From (5.6) we have

$$M \geq 18p + 8(r_3 - 1) + 6p + 3r_2 - 16.$$

If $p = 1$, then $M \geq \psi(G) - 10$. Assume $p = 0$. If $r_2 > 0$, then $3r_2 \geq 3$ and $M \geq \psi(G) - 13$. If $p = 0$ and $r_2 = 0$, then we still have $M \geq \psi(G) - 16$; in this case Γ has signature $(0; +; [3^r]; \{\})$, and G is generated by elements of order 3.

https://doi.org/10.1017/S0017089500030457 Published online by Cambridge University Press

Continue to assume $p \le 1$ and $r_h = 0$, but now suppose $r_3 = 0$. Since $\mu(\Gamma)$ is positive, we must have $r_2 > 0$. Then (5.5) becomes

$$M \ge 24p + 6r_2 - 24. \tag{5.7}$$

In the simplified presentation for Γ, there is a redundant elliptic generator of order 2. Now (5.2) (with $r_h = r_3 = 0$) can be improved to give

$$\psi(\Gamma) \le 18p + 3(r_2 - 1).$$

From (5.7) we have

$$M \ge 18p + 3(r_2 - 1) + (6p + 3r_2) - 21.$$

We know G is not cyclic. Hence

$$2p + r_2 - 1 \ge \operatorname{rank}(G) \ge 2$$

and $6p + 3r_2 \ge 9$. Thus $M \ge \psi(G) - 12$.

The proof in case the quotient space U/Γ is non-orientable is quite similar and no more difficult. We find that $M \ge \psi(G) - 15$ in all cases here. We omit the details.

A review of the calculations shows that in every case,

$$M \ge \psi(G) - 16.$$

Further, in every case except one (in which G is generated by elements of order 3),

$$M \ge \psi(G) - 15.$$

In any case, applying (2.3) we have the genus $g \ge 1 + |G| \cdot M/24$. Thus $\sigma(G) \ge 1 + |G| . M/24$. This completes the proof.

The lower bound (5.3) is the best possible. We give examples of two infinite families of groups for which the bound is attained. First let $G = (Z_3)^n$. Observe that $\sigma(G) = 1 + 3^{n-1}(n-2)$ [**19**]. But clearly $\psi(G) = 8n$, so that (5.3) gives the genus.

The bound is also attained for a family of non-abelian 3-groups. Let L be the non-abelian group of order 27 with no element of order 9. The group L has presentation [**10**, p. 135]

$$R^3 = S^3 = (RS)^3 = (R^{-1}S)^3 = 1.$$

Corollary. $\sigma(L^n) = 1 + 2 \cdot 3^{3n-1}(n-1)$.

Proof. The group L is generated by two elements of order 3, and it is clear that $\psi(L) = 16$ and $\psi(L^n) = 16n$. Now (5.3) gives $\sigma(L^n) \ge 1 + 2 \cdot 3^{3n-1}(n-1)$.

Let Γ be a Fuchsian group with signature $(0; +; [3^{2n+1}]; \{\})$. It is easy to construct a homomorphism from Γ onto L^n such that the kernel is a surface group. A calculation using (2.2) and (2.3) shows that L^n acts on a surface of genus $1 + 2 \cdot 3^{3n-1}(n-1)$. Hence $\sigma(G) \le 1 + 2 \cdot 3^{3n-1}(n-1)$.

However, the proof of Theorem 1 shows that in most cases, the lower bound can be improved slightly. The following version is perhaps more useful, in general.

THEOREM 2. *Let G be a finite group with $\sigma(G) \geq 2$. If G is not generated by elements of order three, then*

$$\sigma(G) \geq 1 + |G|\,[\psi(G) - 15]/24. \tag{5.8}$$

There are also infinite families for which the bound (5.8) is attained. First let $G = (Z_4)^n$. Then $\sigma(G) = 1 + 2^{2n-3}(3n - 5)$ [**17**]. But clearly $\psi(G) = 9n$, so that the genus is given by (5.8).

This bound is attained for a family of non-abelian groups, as well. Let Q be the quaternion group of order 8.

COROLLARY. $\sigma(Q^n) = 1 + 8^{n-1}(6n - 5)$.

Proof. The group Q is well-known, of course, Clearly, $\psi(Q) = 18$ and $\psi(Q^n) = 18n$. Applying (5.8) gives $\sigma(Q^n) \geq 1 + 8^{n-1}(6n - 5)$.

Let Γ be a Fuchsian group with signature $(0; +; [4^{2n+1}]; \{\})$. It is not hard to construct a homomorphism from Γ onto Q^n such that the kernel is a surface group. A calculation using (2.2) and (2.3) shows that $\sigma(G) \leq 1 + 8^{n-1}(6n - 5)$.

6. Hamiltonian groups. A *hamiltonian* group is a non-abelian group in which every subgroup is normal. The finite hamiltonian groups have the form

$$Q \times A \times B,$$

where A is an elementary abelian 2-group and B is an abelian group of odd order [**10**, p. 8]. We shall consider the hamiltonian groups with no odd order part. The following is not hard; for a proof, see [**16**, Lemma 4].

LEMMA 3. *Let $G = (Z_2)^a \times Q$, where $a \geq 1$. We have that* $\text{rank}(G) = a + 2$ *and* $\psi(G) = 3a + 18$.

For these groups, it is usually possible to improve the lower bound (5.8).

LEMMA 4. *Let $G = (Z_2)^a \times Q$. If $a \geq 4$, then*

$$\sigma(G) \geq 1 + |G|\,[\psi(G) - 12]/24.$$

Proof. First $\sigma(Z_2 \times Q) \geq \gamma(Z_2 \times Q) = 3$. Since G has $Z_2 \times Q$ as a quotient group, $\sigma(G) \geq 3$. We reexamine the proof of Theorem 1 and use the same notation. Each nonidentity element of G has order 2 or 4. In particular, there are no elements of order 3; thus Γ has no elliptic generators of order 3, that is, $r_3 = 0$. Also, there must be at least two generators of order 4 in any generating set for G. Again write $M = 12[\mu(\Gamma)/2\pi]$.

Case I. Now (5.4) becomes

$$M \geq 12\gamma + 9r_h + 6r_2 + 3B - 12. \tag{6.1}$$

We always have $\ell \leq k \leq \gamma + 1$.

https://doi.org/10.1017/S0017089500030457 Published online by Cambridge University Press

First suppose $\ell \leq \gamma$ or $r_2 \geq 1$. Then easily, $2r_2 + \gamma \geq r_2 + \ell$. From (6.1)

$$\begin{aligned} M &\geq 9(\gamma + r_h) + 3(2r_2 + \gamma + B) - 12 \\ &\geq 9(\gamma + r_h) + 3(r_2 + \ell + B) - 12 \\ &\geq \psi(G) - 12, \end{aligned}$$

using (5.1) and Lemma 1.

Now assume $\ell = k = \gamma + 1$ and $r_2 = 0$. In this case all of the period cycles are empty so that $B = 0$. With $r_2 = B = 0$, from (6.1) we have

$$M \geq 12\gamma + 9r_h - 12.$$

As in Lemma 2, we must have $\gamma + r_h + \ell = 2\gamma + r_h + 1 \geq a + 2 = \text{rank}(G)$. Thus $M \geq 6(2\gamma + r_h) - 12 \geq 6(a+1) - 12 = 3a + 3a - 6 \geq 3a + 6$, since $a \geq 4$. Thus $M \geq \psi(G) - 12$.

Case II. Assume that the quotient space U/Γ is orientable. Since there must be at least two generators of order 4 in any generating set for G, it follows in this case that if $p = 0$, then $r_h \geq 2$. If $p \geq 2$, then we already have $M \geq \psi(G) - 12$. We may assume, then, that $p \leq 1$. From (5.5) we have

$$M \geq 24p + 9r_h + 6r_2 - 24. \tag{6.2}$$

First suppose $p = 1$. Counting generators in the simplified presentation for Γ, we have

$$2p + r_h + r_2 \geq a + 2 = \text{rank}(G),$$

so that $r_h + r_2 \geq a$. From (6.2) $M \geq 6(r_h + r_2) \geq 6a \geq 3a + 12$, since $a \geq 4$. Hence $M \geq \psi(G) - 6$.

Now suppose $p = 0$. In the simplified presentation for Γ in Lemma 2, there is a redundant elliptic generator of high order. With $p = 0$, we have

$$r_h - 1 + r_2 \geq a + 2 = \text{rank}(G).$$

Now from (6.2) $M \geq 6(a+3) + 3r_h - 24$. Since $r_h \geq 2$, $M \geq 6a$ and again $M \geq \psi(G) - 6$.

Again the proof is similar in the non-orientable case. In all cases, we have $M \geq \psi(G) - 12$. Now apply (2.3) as in Theorem 1.

THEOREM 3. *Let $G = (Z_2)^a \times Q$. Then*

$$\sigma(G) = \begin{cases} 1 + 2^a(a+1) & \text{if } 1 \leq a \leq 3 \\ 1 + 2^a(a+2) & \text{if } a \geq 4 \end{cases}$$

Proof. First suppose $a \geq 4$. Then Lemmas 3 and 4 yield $\sigma(G) \geq 1 + 2^a(a+2)$. Let Γ be an NEC group with signature $(0; +; [4,4]; \{(2^a)\})$. It is easy to show that G is a quotient of Γ and the kernel is a surface group. This gives the genus of G quoted above.

For $1 \leq a \leq 3$, (5.8) provides the lower bound. If $a = 1$, then G is a quotient of an NEC group with signature $(0; +; [4,4]; \{(\,)\})$ such that the kernel is a surface group. If a

TABLE I. THE SYMMETRIC GENUS OF GROUPS OF ORDER 36

	Groups	Genus	Reason
1.	D_{18}	0	*
2.	$Z_6 \times D_3$	1	(3.10)
3.	$Z_2 \times G_{18}$	1	(3.2)
4.	$Z_3 \times A_4$	1	(2.18)
5.	$Z_3 \times DC_3$	1	(2.13)
6.	DC_9	1	(2.13)
7.	$S_3 \times S_3$	1	(4.1)
8.	Gp(1)	1	(2.15)
9.	Gp(2)	6	$\Gamma(2,9,9)$
10.	Gp(3)	16	$\Gamma(3,3,4,4)$

$\text{Gp}(1) = \langle a, b, c \mid a^3 = b^3 = c^4 = [a,b] = 1, c^{-1}ac = b, c^{-1}bc = a^{-1} \rangle$
$\text{Gp}(2) = \langle a, b, c \mid a^2 = b^2 = c^9 = [a,b] = 1, c^{-1}ac = b, c^{-1}bc = ab \rangle$
$\text{Gp}(3) = \langle a, b, c \mid a^3 = b^3 = c^4 = [a,b] = 1, c^{-1}ac = a^{-1}, c^{-1}bc = b^{-1} \rangle$

is 2 or 3, then use an NEC group with signature $(0; +; [4]; \{(\,)^2\})$ or $(0; +; [\,]; \{(\,)^3\})$, respectively.

7. The groups of order 36. There are 14 groups of order 36; 4 of these are abelian. The symmetric genus of each abelian group has been calculated [**17**]. Table I gives the symmetric genus of each non-abelian group G. If $\sigma(G) = 1$ (and $\gamma(G) = 1$ as well), then we also give the Proulx class of G. If $\sigma(G) > 1$, then we give an NEC group with minimal area that maps onto G. The dicyclic group of order $4m$ is denoted DC_m (the notation $\langle 2, 2, m \rangle$ is used in [**10**]). This group has presentation

$$X^{2m} = 1, \qquad X^m = Y^2, \qquad Y^{-1}XY = X^{-1}.$$

Also, let G_{18} denote the non-abelian group of order 18 that is not D_9 and not $Z_3 \times D_3$; G_{18} is denoted $((3,3,3;2))$ in [**10**].

Let's examine the group $Gp(2)$ of order 36. It is easy, using CAYLEY, to show that this group has a unique normal subgroup N of order 12 and that all elements of orders $1, 2, 3$, or 6 are contained in N. Furthermore, all elements not in N have order 9. Also notice that this group has no normal subgroups of index 2 or 4 and so it cannot act on a surface in an orientation-reversing manner. It is not hard to see that the genus of $Gp(2)$ is greater than 1.

Now $\Gamma_1 = \Gamma(2,9,9)$ maps onto $Gp(2)$ and the area is given by $\mu(\Gamma_1))/2\pi = 5/18$. Suppose that Γ is any other NEC group which maps onto $Gp(2)$ and for which $\mu(\Gamma) < \mu(\Gamma_1)$. Since $Gp(2)$ acts in an orientation-preserving fashion, Γ must be a triangle group $\Gamma(\lambda_1, \lambda_2, \lambda_3)$. Since any set of generators must contain one of order 9, we see that $\lambda_2 = \lambda_3 = 9$. Clearly, Γ_1 has minimal area and $Gp(2)$ has symmetric genus 6.

Next we consider the group $Gp(3)$ and let g be its symmetric genus. Now define $N = \langle a, b, c^2 \rangle$. Clearly, N has order 18 and is normal in $Gp(3)$. Let Λ be a minimal set of generators for $Gp(3)$. Therefore, there exists $z \in \Lambda$ such that $z \notin N$. Since $z \in cN$, it is of the form ca^lb^k or $c^{-1}a^lb^k$ for integers k and l and has order 4. If $x \in \Lambda$ and $x \in N$, then z acts on x by inverting it. Therefore, $\langle x, z \rangle$ has order 12. Similarly, if $y \in cN$, then

https://doi.org/10.1017/S0017089500030457 Published online by Cambridge University Press

TABLE II. THE SYMMETRIC GENUS OF GROUPS OF ORDER 40

	Group	Genus	Reason
1.	D_{20}	0	*
2.	$Z_2 \times D_{10}$	0	*
3.	$Z_5 \times D_4$	1	(3.10)
4.	$Z_5 \times Q$	1	(2.14)
5.	$Z_4 \times D_5$	1	(3.10)
6.	$Z_2 \times DC_5$	1	(2.13)
7.	$Z_2 \times \langle 5, 4, 2\rangle$	1	(2.6)
8.	$Z_5 \times_\varphi Q$	1	(2.14)
9.	$Z_5 \times_\varphi Z_8$, $(\mathrm{Ker}\ \varphi = Z_4)$	1	(2.14)
10.	Gp(1)	1	(2.3)
11.	$Z_5 \times_\varphi Z_8$, $(\mathrm{Ker}\ \varphi = Z_2)$	11	$\Gamma(4, 8, 8)$

$\mathrm{Gp}(1) = \langle a, b, c \mid a^2 = b^4 = c^5 = (ab)^2 = [a, c] = 1, b^{-1}cb = c^{-1}\rangle$

$\langle y, z\rangle = \langle z, yz\rangle$ has order less than or equal to 12. It follows that any generating set has at least 3 generators. An easy calculation shows that c^2 is in the Frattini subgroup of $Gp(3)$ and so is a non-generator. Since c^2 is the only element of order 2, any element in Λ must have order greater than 2. Now it is not hard to see that $g > 1$ and $\psi(Gp(3)) = 25$. Then the lower bound (5.8) gives $g \geq 16$.

Let Γ be an NEC group with signature $(0; +; [3, 3, 4, 4]; \{\})$. It is easy to define a homomorphism of Γ onto $Gp(3)$. Now $\mu(\Gamma)/2\pi = 5/6$ and $g \leq 16$. Therefore, $Gp(3)$ has symmetric genus 16.

8. The groups of order 40. There are 14 groups of order 40; 3 of these are abelian. Table II gives the symmetric genus of each non-abelian group. Let $\langle m, n, r\rangle$ denote the metacyclic group of order mn with presentation

$$A^m = B^n = 1, \qquad B^{-1}AB = A^r,$$

where $r^n \equiv 1 (\mathrm{mod}\ m)$ and $\gcd(r-1, m) = 1$; see [**18**, § 2].

The only group of order 40 which is not toroidal is the semidirect product G of Z_5 by Z_8 given by the presentation $\langle a, c \mid a^8 = c^5 = 1, a^{-1}ca = c^2\rangle$. Since a^4 is the unique element of order 2, if G were toroidal, then it would be in one of the Proulx classes (2.13), (2.14), or (2.18). But G has no abelian subgroup of index 2, and so it can not be in class (2.13) or (2.14). Since G is clearly not in class (2.18), we conclude $\sigma(G) > 1$.

Next, we note that all elements in G of orders 2, 4, and 5 are contained in a normal subgroup of index 2. Suppose that the triangle group $\Gamma = \Gamma(r, s, t)$ maps onto G. At least two of the ordinary periods r, s, or t must be different from 2, 4, or 5. In particular, since the element of order 2 is a non-generator, none of the ordinary periods may be 2. These restrictions eliminate all triangle groups that have non-euclidean area less than that obtained when $r = 4$, $s = 8$, and $t = 8$. Now let $\Gamma = \Gamma(4, 8, 8)$, and define $\phi: \Gamma \to G$ by

$$\begin{aligned} &x_1 \to aca, \\ \phi: &x_2 \to c^{-1}a^3, \\ &x_3 \to c^{-1}a^{-3}. \end{aligned}$$

This gives non-euclidean area $\mu(\Gamma)/2\pi = 1/2$ and genus 11.

TABLE III. THE SYMMETRIC GENUS OF GROUPS OF ORDER 42

	Group	Genus	Reason
1.	D_{21}	0	*
2.	$Z_3 \times D_7$	1	(3.10)
3.	$Z_7 \times D_3$	1	(3.10)
4.	$\langle 7, 6, 5\rangle$	1	(2.5)
5.	$Z_2 \times \langle 7, 3, 2\rangle$	8	$\Gamma(3, 6, 6)$

It is easily seen that no NEC group Γ which has a plus sign in its signature and maps onto G can have area smaller than 1/2. In fact, any reflection must map to a non-generator of G, so that we may assume that Γ has signature with $k = 0$. It is not difficult to see that the only remaining NEC group to consider has signature $(1; -; [r, s]; \{\})$. Since r and s are not 2, they must be at least 4 and the area is too large again.

9. The groups of order 42. There are 6 groups of order 42; only one is abelian. Table III gives the symmetric genus of each non-abelian group G.

Let G be the group $Z_2 \times \langle 7, 3, 2\rangle$ which has presentation $\langle a, b, c \mid a^7 = b^3 = c^2 = [a, c] = [b, c] = 1, b^{-1}ab = a^2\rangle$. Since G has a single element of order 2 and its subgroup of index 2 is non-abelian and contains all odd order elements, the group G is not toroidal.

Now let $\Gamma_1 = \Gamma(3, 6, 6)$. Define the homomorphism $\phi: \Gamma_1 \to G$ by

$$\phi: \begin{aligned} x_1 &\to ab, \\ x_2 &\to bc, \\ x_3 &\to bca^{-1}. \end{aligned}$$

The area associated with Γ_1 is $\mu(\Gamma_1)/2\pi = 1/3$ which corresponds to genus 8. Suppose that Γ is another NEC group which maps onto G and has minimal non-euclidean area. The unique element in G of order 2 is a power of any element of even order. Therefore, if there are any generators of G of even order, all generators of order 2, including reflections, are redundant. In any set of generators, the projection of at least two of them onto $\langle 7, 3, 2\rangle$ must have order at least 3, and the projection of at least one generator onto Z_2 must have order 2. These facts imply that if Γ has signature with $p = 0$, then the signature is either $(0; +; [3, 6, 6]; \{\})$ or $(0; +; [3, 3]; \{()\})$. Both of these groups satisfy $\mu(\Gamma)/2\pi = 1/3$ and map onto G. If $p > 0$, then a tedious calculation shows that Γ must have signature $(1; -; [3, 3]; \{\})$. But this group also has $\mu(\Gamma)/2\pi = 1/3$ and maps onto G. Therefore, the symmetric genus of G equals 8.

10. The remaining small groups. There are four groups of order less than 32 with symmetric genus greater than 1. Both $Z_2 \times Q$ and $(Z_3)^3$ have already been considered. We have seen that for each of these groups, the lower bound of Theorem 2 is attained. The group $SL(2, 3)$ has the following presentation as an image of the triangle group $\Gamma(3, 3, 4)$

$$X^3 = Y^3 = (XY)^4 = 1, \qquad [X, Y]^2 = (XY)^2. \tag{10.1}$$

Therefore, SL(2, 3) acts on a surface of genus 2, and we already know $\sigma(\mathrm{SL}(2, 3)) > 1$.

TABLE IV. THE GENUS OF GROUPS OF ORDER LESS THAN 32

	Group	Order	$\gamma(G)$	$\sigma(G)$
1.	$Z_2 \times Q$	16	3	5
2.	SL(2, 3)	24	1	2
3.	$Z_9 \times_\varphi Z_3$	27	4	7
4.	$(Z_3)^3$	27	7	10

The group $Z_9 \times_\varphi Z_3$ is an image of the triangle group $\Gamma(3, 9, 9)$. Since all elements of order 3 are in a proper normal subgroup, this NEC group must have minimal area. These groups are listed above with their graph theoretic genus and symmetric genus.

All other groups G whose orders satisfy $32 < |G| < 48$ are either abelian or dihedral with two exceptions. The first exception is the metacyclic group $\langle 13, 3, 3\rangle$ of order 39 which is toroidal because it is in Proulx class (2.18). Finally, there is the group $\langle a, c \mid a^{11} = c^4 = 1, c^{-1}ac = a^{-1}\rangle$ of order 44. This group is also toroidal because it is in Proulx class (2.13) with $p = 2$ and $s = 11$.

11. The groups of symmetric genus 2. Here we use the completed classification of the regular maps of genus 2 to classify the groups of symmetric genus 2.

Let H be a group of automorphisms of a regular map on a surface of genus 2. Then H acts on a Riemann surface of genus 2, and therefore $\sigma(H) \leq 2$. It may well be that $\sigma(H) < 2$, of course.

On the other hand, suppose $\sigma(G) = 2$. Then G acts on a Riemann surface X of genus 2. We apply Lemma A. Assume first that G preserves the orientation of X. In this case, if $|G| > 24$, then G is a rotation group of a regular map.

Assume next that G acts on X reversing orientation. Let G^+ be the subgroup of G consisting of the orientation-preserving automorphisms. If $|G| > 48$, then G is the full group of a regular map. If $48 \geq |G| > 24$, then $24 \geq |G^+| > 12$ and G is a group of a regular map unless G^+ is a quotient of $\Gamma(3, 3, 4)$ or $\Gamma(3, 3, 5)$; in these cases, G^+ would have order 24 or 15.

Thus, if G is a group of symmetric genus 2, then either G is a group of a regular map (on a surface of genus 2), a small group (of order at most 24), or G^+ is a quotient of one of these two exceptional triangle groups.

First we consider the groups of the regular maps of genus 2. Table V is from [**10**, p. 140]. Only one of each pair of dual maps is listed.

The small automorphism groups have symmetric genus zero. For each of the last

TABLE V. THE REGULAR MAPS OF GENUS 2

Map	Vertices	Edges	Faces	Rotation group	Group	Order
$\{8, 8\}_{1,0}$	1	4	1	Z_8	D_8	16
$\{5, 10\}_2$	1	5	2	Z_{10}	D_{10}	20
$\{6, 6\}_2$	2	6	2	$Z_2 \times Z_6$	$Z_2 \times D_6$	24
$\{4, 8\}_{1,1}$	2	8	4	$\langle -2, 4 \mid 2\rangle$		32
$\{4, 6 \mid 2\}$	4	12	6	$(4, 6 \mid 2, 2)$		48
$\{3, 4+4\}$	6	24	16	$\langle -3, 4 \mid 2\rangle$		96

https://doi.org/10.1017/S0017089500030457 Published online by Cambridge University Press

three maps, it is necessary to construct the full automorphism group. We use the approach in [**24**, Th. 2].

Let $M_2 = \{4, 6 \mid 2\}$ [**10**, p. 110]. The rotation group G_{24} (using the notation of [**3**]) of M_2 has presentation

$$R^4 = S^6 = (RS)^2 = 1, \qquad R^2S = SR^2.$$

Adjoin an element W of order 2 that transforms the elements of G_{24} according to the automorphism $\beta(R) = R^{-1}$, $\beta(S) = S^{-1}$. We obtain a larger group G_{24}^* of order 48 with presentation

$$W^2 = R^4 = S^6 = (RS)^2 = (WR)^2 = (WS)^2 = 1, \qquad R^2S = SR^2.$$

However, both G_{24}^* and its subgroup G_{24} have symmetric genus one; the larger group is in Proulx class (3.4).

Next let $M_1 = \{4, 8\}_{1,1}$ [**10**, p. 115]. The full group G of the map M_1 has order 32 and presentation

$$T^2 = Y^2 = Z^4 = (TY)^2 = (TZ)^2 = (YZ)^8 = 1, \qquad ZY = (YZ)^3.$$

Again $\sigma(G) = 1$; G is also in Proulx class (3.4).

Finally let $M_3 = \{3, 4 + 4\}$ [**10**, pp. 115, 116]. The rotation group G_{48} [**3**] of M_3 has presentation

$$R^8 = S^3 = (RS)^2 = 1, \qquad R^4S = SR^4.$$

The full group G_{48}^* has presentation

$$W^2 = R^8 = S^3 = (RS)^2 = (WR)^2 = 1, \qquad R^4S = SR^4.$$

It is not difficult to establish that $G_{48} \cong \mathrm{GL}(2, 3)$ Each of these two groups acts on a surface of genus 2 and has symmetric genus at most 2. But we know $\sigma(\mathrm{GL}(2, 3)) > 1$. Therefore, $\sigma(\mathrm{GL}(2, 3)) = \sigma(G_{48}^*) = 2$.

Now we consider the quotients of the two exceptional triangle groups. Let $\Gamma = \Gamma(3, 3, k)$. The triangle group Γ has presentation

$$x^3 = y^3 = (xy)^k = 1. \tag{11.1}$$

Suppose H is a quotient of Γ by a surface group K. Since K contains no elements of finite order, the relations (11.1) must also be satisfied in H. In particular, H is generated by elements of order 3.

The only group of order 15 is cyclic, of course, and is not generated by elements of order 3. Hence Z_{15} is not a quotient of $\Gamma(3, 3, 5)$.

There are 15 groups of order 24; 3 of these are abelian. Almost all of these groups have a normal Sylow 3-subgroup and consequently are not generated by elements of order 3. The exceptions are S_4 and $\mathrm{SL}(2, 3)$. But S_4 is clearly not generated by elements of order 3 either, since 3-cycles are even. We have already seen that $\mathrm{SL}(2, 3)$ is a quotient of $\Gamma(3, 3, 4)$. Thus we have the following result.

Lemma 5. *Let H be a group of order* 24. *If H is a quotient of* $\Gamma(3, 3, 4)$ *by a surface group, then* $H \cong \mathrm{SL}(2, 3)$.

Now suppose G is a group of order 48 containing $\mathrm{SL}(2, 3)$ that acts on X reversing orientation. Represent X as U/K where K is a surface group. Then there is a proper

https://doi.org/10.1017/S0017089500030457 Published online by Cambridge University Press

NEC group Γ such that $\Gamma/K \cong G$. The canonical Fuchsian subgroup of Γ must be the triangle group $\Gamma(3,3,4)$, and $\Gamma^+/K \cong \mathrm{SL}(2,3)$.

Let $H = \mathrm{SL}(2,3)$ have presentation (10.1), which is essentially unique. A basic result of Singerman shows that there are only two possibilities for the NEC group Γ and two corresponding ways to construct the extension of H by Z_2 [**24**, p. 22].

To obtain G in the first case, adjoin an element T of order 2 that transforms the elements of H according to the automorphism

$$\alpha(X) = X^{-1}, \qquad \alpha(Y) = Y^{-1}.$$

Then G has presentation

$$T^2 = X^3 = Y^3 = (XY)^4 = (TX)^2 = (TY)^2 = 1, \qquad [X, Y]^2 = (XY)^2.$$

In this case G is a quotient of the extended triangle group with signature $(0; +; [\,]; \{(3,3,4)\})$. But it is not hard to see that $G \cong \mathrm{GL}(2,3)$, and we already know $\sigma(G) = 2$.

In the other case, adjoin an element S of order 2 that transforms the elements of H according to the automorphism

$$\beta(X) = Y^{-1}, \qquad \beta(Y) = X^{-1}.$$

Then G has presentation

$$S^2 = X^3 = Y^3 = (XY)^4 = 1, \qquad SXS = Y^{-1}, \qquad [X, Y]^2 = (XY)^2.$$

In this case G is a quotient of the NEC group with signature $(0; +; [3]; \{(4)\})$ which has presentation

$$c^2 = x^3 = [c, x]^4 = 1.$$

Now G acts on a surface of genus 2, so that $\sigma(G) \leq 2$. It turns out that $G \cong P_{48}$, the group with presentation (3.1). Hence P_{48} is another group of symmetric genus 2.

This completes the classification of the groups with $\sigma = 2$.

THEOREM 4. *There are exactly* 4 *groups with symmetric genus* 2; *these groups are* SL(2, 3), GL(2, 3), P_{48}, *and* G^*_{48}.

We must mention here that regular hypermaps could have been employed in the proof of Theorem 4. Hypermaps are more general than maps; a good introduction to hypermaps is given in [**9**]. A finite group G that is a quotient of a triangle group $\Gamma(m, n, k)$ is a group of a regular hypermap [**9**, Th. 9, p. 346]. The regular hypermaps of genus 2 have been classified [**9**], and SL(2, 3) can be obtained as the group of a regular hypermap [**9**, p. 350]. We have chosen to use only the more familiar regular maps in our classification, however.

Theorem 4 is closely related to the following result of Tucker.

THEOREM A [**26**]. *The unique group with graph theoretic genus* 2 *is the group* G^*_{48}.

Tucker's result could have been used to shorten the proof of Theorem 4, of course. We have chosen an approach that shows the relevance of the work on regular maps. Our approach could be used (together with [**22**]) to classify the groups of symmetric genus 3.

Finally, we would like to thank the referee for several helpful suggestions.

REFERENCES

1. M. G. Brin and C. C. Squier, On the genus of $Z_3 \times Z_3 \times Z_3$, *European J. Combinatorics* **5** (1988), 431–443.

2. M. G. Brin, D. E. Rauschenberg and C. C. Squier, On the genus of the semidirect product of Z_9 by Z_3, *J. Graph Theory* **13** (1989), 49–61.

3. E. Bujalance and D. Singerman, The symmetry type of a Riemann surface, *Proc. London Math. Soc.* (3) **51** (1985), 501–519.

4. W. Burnside, *Theory of Groups of Finite Order*, (Cambridge University Press, 1911).

5. J. J. Cannon, *An introduction to the group theory language CAYLEY*, Computational Group Theory (M. Atkinson, ed.), (Academic Press, 1984), 145–183.

6. M. D. E. Conder, The symmetric genus of alternating and symmetric groups, *J. Combin. Theory Ser. B* **39** (1985), 179–186.

7. M. D. E. Conder, Hurwitz groups: a brief survey, *Bull. Amer. Math. Soc.* (N.S.) **23** (1990), 359–370.

8. M. D. E. Conder, R. A. Wilson and A. J. Woldar, The symmetric genus of sporadic groups, *Proc. Amer. Math. Soc.* **116** (1992), 653–663.

9. D. Corn and D. Singerman, Regular Hypermaps, *European J. Combinatorics* **9** (1988), 337–351.

10. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, Fourth Edition, (Springer-Verlag, 1957).

11. D. Garbe, Uber die regularen Zerlegungen geschlossener orientierbarer Flachen, *J. Reine Angew. Math.* **237** (1969), 39–55.

12. H. Glover and D. Sjerve, The genus of $PSL_2(q)$, *J. Reine Angew. Math.* **380** (1987), 59–86.

13. J. L. Gross and T. W. Tucker, *Topological Graph Theory*, (John Wiley and Sons, 1987).

14. A. Hurwitz, Uber algebraische gebilde mit eindeutigen transformationen in sich, *Math. Ann.* **41** (1893), 403–442.

15. C. L. May, Complex doubles of bordered Klein surfaces with maximal symmetry, *Glasgow Math. J.* **33** (1991), 61–71.

16. C. L. May, A lower bound for the real genus of a finite group, *Canad J. Math.* (to appear).

17. C. L. May and J. Zimmerman, The symmetric genus of finite abelian groups, *Illinois J. Math.* **37** (1993) 400–423.

18. C. L. May and J. Zimmerman, The symmetric genus of metacyclic groups, (to appear).

19. C. Maclachlan, Abelian groups of automorphisms of compact Riemann surfaces, *Proc. London Math. Soc.* **15** (1965), 699–712.

20. T. Pisanski and A. T. White, Nonorientable embeddings of groups, *European J. Combinatorics* **9** (1988), 445–461.

21. V. K. Proulx, Classification of the toroidal groups, *J. Graph Theory* **2** (1978), 269–273.

22. F. A. Sherk, The regular maps on a surface of genus three, *Canad. J. Math.* **11** (1959), 452–480.

23. D. Singerman, On the structure of non-Euclidean crystallographic groups, *Proc. Cambridge Philos. Soc.* **76** (1974), 233–240.

24. D. Singerman, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* **210** (1974), 17–32.

25. T. W. Tucker, Finite groups acting on surfaces and the genus of a group, *J. Combin. Theory Ser. B* **34** (1983), 82–98.

26. T. W. Tucker, There is one group of genus two, *J. Combin. Theory Ser. B* **36** (1984), 269–275.

27. A. T. White, *Graphs, Groups and Surfaces*, Revised Edition, (North-Holland, 1984).

Department of Mathematics
Towson Stage University
Baltimore
Maryland 21204
U.S.A.

https://doi.org/10.1017/S0017089500030457 Published online by Cambridge University Press