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Nilpotent signalizer functors on finite groups

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In a recent paper Martineau has shown that the study of groups with a fixed-point-free automorphism group can be applied to help in the investigation of signalizer functors. We prove further results about signalizer functors by this method.

Following the idea of Martineau applied in a recent paper [5] we obtain theorems which have as a consequence the following result.

THEOREM. *Let* r *be a prime,* G *a finite group and* A *an abelian* r*-subgroup of* G *with* $m(A) \geq 3$. *Suppose that* θ *is an* A*-signalizer functor on* G *such that* $\theta\left(C_G(a)\right)$ *is nilpotent for each* $a \in A^{\#}$. *Then* θ *is complete and* $\theta(G)$ *is nilpotent.*

We use the notation of [3] together with the following. Let Ψ denote a class of finite groups. We say that the A-signalizer functor θ on G (where G denotes a finite group and A is an abelian r-subgroup of G , r some prime) is a Ψ A-signalizer functor on G if $\theta\left(C_G(a)\right) \in \Psi$ for each $a \in A^{\#}$. If θ is a Ψ A-signalizer functor on G then we require, in addition to the usual requirements for a subgroup of G to belong to $M_\theta(A)$, that the elements of $M_\theta(A)$ belong to Ψ .

If we take Ψ to be the class of soluble groups then our notation agrees with that of Goldschmidt. The above theorem may now be stated as a theorem about nilpotent signalizer functors on finite groups:

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THEOREM 1. *Let* r *denote a prime,* G *a finite group and* A *an abelian* r*-subgroup of* G *with* $m(A) \geq 3$. *Suppose that* θ *is a nilpotent* A*-signalizer functor on* G . *Then* θ *is complete.*

Theorem 1 is an immediate consequence of the following theorem.

THEOREM 2. *Let* r *denote a prime,* G *a finite group and* A *an abelian* r*-subgroup of* G *with* $m(A) \geq 3$. *Let* p *denote some prime distinct from* r *and* Π *the class of finite groups with a normal Sylow* p*-subgroup. Suppose that* θ *is a* Π A*-signalizer functor on* G . *Then* θ *is complete.*

We also prove another theorem which includes as special cases both Theorem 1 and the theorem of Martineau [5].

THEOREM 3. *Let* r *denote a prime,* G *a finite group and* A *an elementary abelian* r*-subgroup of* G *with* $m(A) \geq 3$. *Let* θ *denote an* A*-signalizer functor on* G . *Assume that for each* $a \in A^{\#}$ *each element of* $\theta(C_G(A))$ *centralizes any element of* $\theta(C_G(a))$ *with relatively prime order. Then* θ *is complete.*

REMARK. Let r denote a prime, G a finite group and A an abelian r-subgroup of G . Suppose that θ is a supersoluble A-signalizer functor on G . We may then define a soluble A-signalizer functor on G - say $\bar{\theta}$ - by defining $\bar{\theta}(C_G(a)) = \theta(C_G(a))$ for each $a \in A^{\#}$. If $m(A) = 3$ then examples exist in which θ is complete but $\bar{\theta}$ is not complete. If $m(A) \geq 4$ then both θ and $\bar{\theta}$ must be complete.

The reason that this may happen is that $\mathcal{H}_{\theta}(A)$ may be a proper subset of $\mathcal{H}_{\bar{\theta}}(A)$ in the case $m(A) = 3$.

Proof of Theorem 2

We assume that G, A, r and θ satisfy the hypothesis of Theorem 2. Hence $O_p(\theta(C_G(a)))$ is a Sylow p-subgroup of $\theta(C_G(a))$ for each $a \in A^{\#}$. Hence if $P \in \mathcal{H}^*_{\theta}(A\ ;\ p)$, then by [2] Lemma 3.2 we may conclude that $P \geq O_p(\theta(C_G(a)))$ for each $a \in A^{\#}$. It follows that for each

$a \in A^{\#}$ we have

$$C_p(a) = O_p\big(\theta\big(C_G(a)\big)\big) .$$

Since A is noncyclic we find

$$P = \left\langle C_p(a) \mid a \in A^{\#} \right\rangle = \left\langle O_p\big(\theta\big(C_G(a)\big)\big) \mid a \in A^{\#} \right\rangle .$$

Thus P is normalized by $\theta\big(C_G(A)\big)$. By [2] Theorem 3.1 we conclude that $И^*_\theta(A ; p) = \{P\}$.

In the course of the proof we will need the following result.

LEMMA. *Let H denote a finite r'-group and B an elementary abelian r-group of operators on H with order r^3 where r is some prime. Assume that for each $b \in B^{\#}$ the subgroup $C_H(b)$ contains a normal Sylow p-subgroup where p is some fixed prime. Then H contains a normal Sylow p-subgroup.*

Proof. We assume that H is a minimal counterexample to the lemma. Thus B is an elementary abelian r-group of operators on H with order r^3. It is immediate from our choice of H that $O_{p}(H) = 1$.

The argument given above shows that H possesses a unique B-invariant Sylow p-subgroup T. By [3] Theorem 6.2.2 (iii) and our hypothesis, $C_T(b)$ is the normal Sylow p-subgroup of $C_H(b)$ for each $b \in B^{\#}$.

Now [2] Lemma 2.1 shows that $T = \langle C_T(B_1) \mid B/B_1$ is cyclic$\rangle$. By our choice of H we have $T \neq 1$ so that $C_T(B_1) \neq 1$ for some subgroup B_1 of index r in B.

Let K denote a maximal B-invariant subgroup of H which contains $C_H(b)$ for some $b \in B_1^{\#}$. Then by our choice of H and B_1, we have $1 \neq C_T(B_1) \leq T \cap K$. Since $O_p(H) = 1$ we conclude that $K = N_H(T\cap K)$. Since T is a p-group and $T \cap K = N_T(T\cap K)$ we must have $T \leq K$. Hence $K = N_H(T)$. We conclude that $C_H(b) \leq N_H(T)$ for each $b \in B_1^{\#}$.

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Since B_1 is a noncyclic abelian group of operators on H it follows that $H = \left\langle C_H(b) \mid b \in B_1^{\#} \right\rangle \leq N_H(T)$. This is a contradiction and completes the proof of the lemma.

We now complete the proof of Theorem 2.

By the lemma just proved if H is an A-invariant r'-subgroup of G such that $C_H(a) \leq \theta\big(C_G(a)\big)$ for each $a \in A$ then $P \cap H \triangleleft H$. Hence $H \in \mathcal{U}_\theta(A)$. This implies that if $H \in \mathcal{U}_\theta^*(A)$ and $P \cap H \neq 1$ then $H = N_H(P)$.

The same argument as is given in the proof of the lemma shows that there exists some subgroup B of index r in A such that $C_P(B) \neq 1$. Again by an argument given in the proof of the lemma we can conclude that $N_G(P)$ contains $\theta\big(C_G(b)\big)$ for each $b \in B^{\#}$.

We now complete the proof by making two applications of Lemma 2.6 of [2]. We suppose that the theorem is true for groups of order less than $|G|$. If P is normal in G then we may apply part 2 of the lemma mentioned, taking $X = P$, and easily conclude that θ is complete. Otherwise $N = N_G(P)$ is a proper subgroup of G and part 1 of the lemma is applicable. This completes the proof.

Proof of Theorem 3

Let G, A, r and θ be as given in the hypothesis of the theorem. We prove the theorem by means of a sequence of lemmas.

LEMMA 1. *θ is soluble.*

Proof. Let $H \in \mathcal{U}_\theta(A)$ so that $C_H(A) \leq \theta\big(C_G(A)\big)$ and, for each $a \in A^{\#}$, we have $C_H(a) \leq \theta\big(C_G(a)\big)$. It follows that the group H and the operator group induced on H by conjugation by elements of A satisfy the hypothesis of the main theorem of [6]. Hence H is soluble. Thus θ is soluble.

LEMMA 2. *If p denotes a prime then $\mathcal{U}_\theta^*(A\ ;\ p)$ is a singleton.*

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Proof. Since $\theta(C_G(A)) \leq \theta(C_G(a))$ for each $a \in A^\#$, we conclude from our hypothesis that $\theta(C_G(A))$ is nilpotent. Let p denote some prime. Write $\theta(C_G(A)) = P_1 \times D$ where P_1 is the Sylow p-subgroup of $\theta(C_G(A))$ and D is a p'-group. Choose $P \in \mathcal{H}^*_\theta(A\ ;\ p)$ such that $P \geq P_1$. By [2] Theorem 3.1, if $\overline{P} \in \mathcal{H}^*_\theta(A\ ;\ p)$ then $\overline{P} = P^x$ for some $x \in \theta(C_G(A))$. Since $P_1 \leq P$, we may assume that $x \in D$. But

$$P = \left\langle C_P(a) \mid a \in A^\# \right\rangle = \left\langle P \cap \theta(C_G(a)) \mid a \in A^\# \right\rangle ,$$

so by our hypothesis P is centralized by D. In particular P is centralized by x so that $\overline{P} = P$. This proves the lemma.

COROLLARY. *If* $H \in \mathcal{H}_\theta(A)$ *then* H *contains a unique* A*-invariant Hall* π*-subgroup for any set of primes* π.

We may now suppose, in addition to the existing assumptions, that G is a counterexample of minimum order to Theorem 3. In the end we will derive a contradiction and the theorem will be proved.

LEMMA 3. θ *is locally complete.*

Proof. This is Lemma 5.1 of [2].

Now let $p, q \in \pi(\theta)$, let $\mathcal{H}^*_\theta(A\ ;\ p) = \{P\}$ and let $\mathcal{H}^*_\theta(A\ ;\ q) = \{Q\}$.

Suppose that $H \in \mathcal{H}^*_\theta(A\ ;\ p, q)$. We will prove that $H = PQ$. Suppose that $H \neq PQ$. If $H \geq P$ then we may write $H = PX$ where X is characterized as the largest subgroup of Q which is normalized by A and is permutable (setwise) with P. Similarly if $H \geq Q$ then $H = QY$ where Y is the largest subgroup of P which is normalized by A and is permutable with Q.

For the remainder of the proof we will reserve the symbols p, q, P, Q, X and Y for the situation just described. It will be some time before we will use the assumption that $X \neq Q$ (or equivalently $Y \neq P$).

LEMMA 4. *Suppose* $H \in \mathcal{H}^*_\theta(A\ ;\ p, q)$ *and that* M *is an* A*-invariant*

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subgroup of $F(H)$ *with* $O_p(M) \neq 1 \neq O_q(M)$. *Then* H *is the only element of* $\mathcal{H}_\theta^*(A\ ;\ p,\ q)$ *which contains* M .

Proof. (The following proof was obtained by adapting Lemma 4 of [4].)

Let $Z(F(H)) = Z$. We will first prove the lemma in the special case that $M = Z$. To this end let $K \in \mathcal{H}_\theta^*(A\ ;\ p,\ q)$ satisfy $K \geq Z$. Denote by Z_p (respectively Z_q) the Sylow p-subgroup (respectively q-subgroup) of Z . Then $Z_p \in \mathcal{H}_\theta(A\ ;\ p)$ so by Lemma 3 we may form $\theta(N_G(Z_p))$. Now $Z_p \triangleleft H \in \mathcal{H}_\theta^*(A\ ;\ p,\ q)$ so H is the unique A-invariant Hall $\{p,\ q\}$-subgroup of $\theta(N_G(Z_p))$. Since $Z_q \leq O_q(H) \cap K$ and $N_K(Z_p) \leq H$ we obtain by Lemma 2.3 of [2] that $Z_q \leq O_q(K)$. Hence $O_p(K) \leq \theta(C_G(Z_q))$ so that $O_p(K) \leq H$. By symmetry in p and q , $O_q(K) \leq H$. We conclude then that $F(K) \leq H$.

Since $Z_q \leq O_q(K)$ and $Z_q \neq 1$ we have $O_q(K) \neq 1$. Similarly $O_p(K) \neq 1$. We also know that $F(K) \leq H$ so in particular $Z(F(K)) \leq H$. Hence we may reverse the roles of H and K above to obtain $F(H) \leq K$.

We now work with $F(H)$ instead of Z . Since $O_p(H) \leq \theta(N_G(O_q(H)))$ we obtain $O_p(H) \leq O_p(N_K(O_q(H)))$. Hence $O_p(H) \leq O_p(K)$. Similarly $O_q(H) \leq O_q(K)$ and therefore $F(H) \leq F(K)$. Interchanging the roles of H and K we get the reverse inclusion and so deduce that $F(H) = F(K)$.

Now we see that $K = H$ since each is the unique A-invariant subgroup of $\theta(N_G(F(H)))$. Hence H is the unique element of $\mathcal{H}_\theta^*(A\ ;\ p,\ q)$ to contain $Z(F(H))$.

We now turn to the general case. Let M_p (respectively M_q) denote the Sylow p-subgroup (respectively q-subgroup) of M . By hypothesis $M \leq F(H)$. Suppose $M \leq K \in \mathcal{H}_\theta^*(A\ ;\ p,\ q)$. Then $\theta(C_G(M_q))$ contains Z so $C_H(M_q) = H \cap \theta(C_G(M_q))$ is the unique A-invariant Hall $\{p,\ q\}$-subgroup of $\theta(C_G(M_q))$. Since $K \in \mathcal{H}_\theta(A\ ;\ p,\ q)$ we deduce that $M_p \leq C_K(M_q) \leq H$ and hence $M_p \leq O_p(C_K(M_q))$. By Lemma 2.3 of [2] we find that $M_p \le O_p(K)$. Now the same argument yeilds $O_p(K) \le C_K(M_p) \le H$. By symmetry in p and q we also have $O_p(K) \le H$ and hence $F(K) \le H$. We have also shown that $M \le F(K)$.

Interchanging the roles of H and K in the last argument we find that $F(H) \le K$. But then $Z(F(H)) \le K$ so we may finally conclude that $H = K$. This completes the proof of Lemma 4.

LEMMA 5. *Let* $H \in \mathcal{N}_\theta(A\ ;\ p,\ q)$ *and assume that* $Z(Q) \le H$. *Then* $H \cap P = O_p(H)(H \cap Y)$.

Proof. We have already observed that $\theta(C_G(A))$ is nilpotent. In particular if $H \in \mathcal{N}_\theta(A\ ;\ p,\ q)$ then since $C_H(A) = H \cap C_G(A) \le \theta(C_G(A))$ we deduce that $C_H(A)$ is nilpotent. Thus $SL(2,\ q)$ is not involved in $C_H(A)$.

By Corollary 1 of [1] applied to the group $H/O_p(H)$, the prime q and the group of operators induced by A on $H/O_p(H)$ we have $H \cap P = O_p(H)C_{H \cap P}(Z(Q \cap H))N_{H \cap P}(J(Q \cap H))$. Now $Z(Q) \le H \cap Q$ so $Z(Q) \le Z(H \cap Q)$. Thus $C_{H \cap P}(Z(Q \cap H)) \le C_{H \cap P}(Z(Q))$. But $Z(Q)$ is an A-invariant subgroup of Q and $Q \in \mathcal{N}_\theta(A)$ so $Z(Q) \in \mathcal{N}_\theta(A)$. Therefore $\theta(C_G(Z(Q)))$ is soluble and Q is its unique A-invariant Sylow q-subgroup. Thus $P \cap \theta(C_G(Z(Q)))$ is contained in Y . Since $C_{H \cap P}(Z(Q)) \le P \cap \theta(C_G(Z(Q)))$ we have $C_{H \cap P}(Z(H \cap Q)) \le Y$.

Suppose that the lemma is false. Then we deduce that $N_{H \cap P}(J(H \cap Q)) \nleq Y$. Choose $Q^* \le Q$ maximal subject to: Q^* is A-invariant, $Z(Q) \le Q^*$ and $N_P(J(Q^*)) \nleq Y$. Since θ is locally complete, $N_P(J(Q)) \le Y$ so $Q^* \ne Q$.

Let P^* and $\overline{Q}$ be respectively the A-invariant Sylow p-subgroup and Sylow q-subgroup of $\theta(N(J(Q^*)))$. Let $K = P^*\overline{Q}$. Then $Z(Q) \le \overline{Q}$ and $\overline{Q}$ is A-invariant. Another application of Glauberman's factorization theorem, this time to $K/O_p(K)$, yields $P^* = O_p(K)C_{P^*}(Z(\overline{Q}))N_{P^*}(J(\overline{Q}))$.

Now $Z(Q) \le Q^* \le Q$ so that $Z(Q) \le Z(Q^*) \le J(Q^*) \le O_q(K)$. Hence $O_p(K)$ and $Z(Q)$ commute elementwise so that $O_p(K) \le Y$. From $Z(Q) \le \overline{Q} \le Q$ we have $Z(Q) \le Z(\overline{Q})$ which in turn yields $C_{P^*}(Z(\overline{Q})) \le Y$. Finally since $Q^* \le Q$ we have $Q^* < \overline{Q}$, so that maximality of Q^* yields $N_{P^*}(J(\overline{Q})) \le N_P(J(\overline{Q})) \le Y$. Hence $P^* \le Y$, contrary to the assumption that $N_P(J(Q^*)) \nleq Y$. This lemma is proved.

We now let $\Psi = \{H \in \mathcal{H}_\theta^*(A\ ;\ p,\ q)\ |\ H \ne PX$ or $QY\}$.

LEMMA 6. *If* $H \in \Psi$ *then*

(i) $O_p(H) \ne 1 \ne O_q(H)$,

(ii) $Z(P) \le H$ *and* $Z(Q) \le H$, *and*

(iii) $X \cap O_q(H) = 1 = Y \cap O_p(H)$.

Proof. We first prove *(i)*. Suppose by way of contradiction that $H \in \Psi$ and $O_p(H) = 1$. Since H is soluble, $O_q(H) \ne 1$. Now $H \in \mathcal{H}_\theta^*(A\ ;\ p,\ q)$, so it follows that H contains the unique A-invariant subgroup of $\theta(N_G(O_q(H)))$. Hence H contains $N_Q(O_q(H))$ and in particular H contains $Z(Q)$. Now by Lemma 5,

$$H \cap P = O_p(H)(H \cap Y) = H \cap Y \le Y .$$

Since $H = (H \cap Q)(H \cap P) \le QY$ and $H \in \mathcal{H}_\theta^*(A\ ;\ p,\ q)$ we conclude that $H = QY$. But this contradicts the choice of H . Hence $O_p(H) \ne 1$. Similarly $O_q(H) \ne 1$.

Let $H \in \Psi$. Then H is the unique A-invariant Hall $\{p,\ q\}$-subgroup of $\theta(N_G(O_p(H)))$. Hence $Z(P) \le H$. Similarly $Z(Q) \le H$. This proves *(ii)*.

To prove *(iii)* suppose that $H \in \Psi$ and, by way of contradiction, that $N = Y \cap O_p(H) \ne 1$. Then $M = NO_q(H)$ is contained in both YQ and H . On the other hand M and H satisfy the hypotheses of Lemma 4, so $H = YQ$. But this is contrary to the choice of H . Hence $N = 1$.

Similarly $X \cap O_q(H) = 1$.

LEMMA 7. *If* $H, K \in \Psi$ *and* $M \in \mathcal{H}_\theta(A\ ;\ p,\ q)$ *satisfies* $M \leq F(H) \cap K$ *then* $H = K$.

Proof. By Lemma 4 we may assume that M is a q-group. Let $Q_1 = [Z(P), M]$. Since $Z(P) \leq H$ by Lemma 6, it follows that Q_1 is a q-group. Again by Lemma 6 we have $Z(P) \leq K$ so that $Z(P) \leq Z(P\ K)$. Hence $Z(P) \leq O_{q,p}(K)$. Since $M \leq K$ we must have $Q_1 \leq O_{q,p}(K)$ and hence $Q_1 \leq O_q(K)$.

If $Q_1 \neq 1$ then we may apply Lemma 4 with $M = C_{F(H)}(Q_1)$. This yields $O_p(K) \leq H$. Hence $Q_1 O_p(K) \leq H$ and another application of Lemma 4, this time taking $M = Q_1 O_p(K)$, yields $H = K$.

If $Q_1 = 1$ then $[Z(P), M] = 1$ so that $M \leq X$. But then $X \cap O_q(H) \neq 1$ contradicting Lemma 6. This completes the proof.

LEMMA 8. *Suppose* $H \in \Psi$ *and* $y \in A$ *are such that*

$$O_p(H) \cap C_G(y) \neq 1 \neq O_q(H) \cap C_G(y) .$$

Then $C_P(y) \leq H$ *and* $C_Q(y) \leq H$. *Furthermore if* $K \in \Psi$ *and* $K \neq H$ *then* $C_{F(K)}(y) = 1$.

Proof. By Lemma 6 *(iii)*, $C_P(y) \nleq Y$ and $C_Q(y) \nleq X$. Hence the A-invariant Hall $\{p, q\}$-subgroup, S , of $\theta(C_G(y))$ is contained in neither PX nor QY . Hence there exists $K \in \Psi$ such that $S \leq K$. Now we may apply Lemma 4, taking $M = C_{F(H)}(y)$, to deduce that $K = H$.

If $C_{F(K)}(y) \neq 1$ for some $K \in \Psi$ then, since $C_{F(K)}(y) \leq H$, we may apply Lemma 7 with $M = C_{F(K)}(y)$ to deduce that $K = H$. This completes the proof of Lemma 8.

LEMMA 9. $|\Psi| \leq 1$.

Proof. Suppose that this is false and choose two distinct elements,

H_1 and H_2 of Ψ. We may choose noncyclic subgroups, B and C, each of index r in A such that $C_P(B) \cap O_p(H_1) \neq 1$ and $C_Q(C) \cap O_q(H_2) \neq 1$ by Lemma 2.1 of [2]. Let $z \in B \cap C^{\#}$. From our choice of B and Lemma 6 *(iii)* we deduce that $C_P(B) \nsubseteq Y$. In particular $C_P(z) \nsubseteq Y$. Similarly $C_Q(z) \nsubseteq X$. Hence the unique A-invariant Hall $\{p, q\}$-subgroup, S, of $\theta(C_G(z))$ is contained in neither PX nor QY. Thus we have $S \leq K$ for some $K \in \Psi$.

Now by Lemma 6 *(ii)* and Lemma 5 we have $P \cap K = O_p(K)(Y \cap K)$. Hence $|C_P(z)| = |C_{Y \cap K}(z)| \left|C_{O_p(K)}(z)\right|$, where we have used Lemma 6 *(iii)* in addition to the information already mentioned. Now $C_P(z) \nsubseteq Y$, so we must have $O_p(K) \cap C_P(z) \neq 1$. Similarly $O_q(K) \cap C_Q(z) \neq 1$.

But Lemma 8 now implies that z operates without fixed-points on $F(H)$ for any $H \in \Psi$ which is distinct from K. But this is contrary to our choice of z. The lemma is proved.

LEMMA 10. $PQ = QP$.

Proof. Assume that $PQ \neq QP$. Then there exists a noncyclic subgroup C of A such that $C_Q(C) \nsubseteq X$. Then for each $y \in C^{\#}$ the unique A-invariant Hall $\{p, q\}$-subgroup, $C_P(y)C_Q(y)$, of $\theta(C_G(y))$ is not contained in PX. If Ψ is non-empty let $H \in \Psi$. Then for each $y \in C$ either $C_P(y)C_Q(y) \leq H$ or $C_P(y) \leq Y$. Let $y_1, y_2, \ldots, y_r$ denote generators for the $r + 1$ nonidentity proper subgroups of C and let $P_i = C_P(y_i)$ and $Q_i = C_Q(y_i)$ for $1 \leq i \leq r$. We assume that the y_i are arranged so that for $i \leq s$ we have $P_i \leq H \cap P$ whilst for $i > s$ we have $P_i \leq Y$ (where s is some integer). Now by Theorem 5.3.16 of [3] it follows that $P = P_1P_2 \ldots P_r \subseteq (H \cap P)Y$. But $(H \cap P)Y \subseteq P$ so $P = (H \cap P)Y$. By Lemma 5, $P = O_p(H)Y = YO_p(H)$. Similarly $Q = O_q(H)X = XO_q(H)$.

Now $O_p(H)$ and $O_q(H)$ commute elementwise, $PX = XP$ and $QY = YQ$. Hence

$$PQ = O_p(H)YQ = O_p(H)QY = O_p(H)O_q(H)XY = O_q(H)O_p(H)XY$$
$$\subseteq O_q(H)PXY = O_q(H)XPY = O_q(H)XP = QP \ .$$

We obtain in the same way the reverse inequality and hence conclude that $PQ = QP$. This completes the proof of Lemma 10.

We now complete the proof of Theorem 3 by explicitly exhibiting $\theta(G)$. Let $p_1, p_2, \ldots, p_t$ denote the distinct prime divisors of $|G|$. Let P_i denote the unique element of $\mathcal{N}_\theta(A\ ;\ p_i)$ for $1 \le i \le t$. Then by Lemma 10 we have $P_iP_j = P_jP_i$ for all i and j . Hence $P_1P_2 \ldots P_t$ is a group and is clearly the unique maximal element of $\mathcal{N}_\theta(A)$. Hence θ is complete and so is the proof.

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