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Multiple zeta functions: the double sine function and the signed double Poisson summation formula

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Abstract

We construct multiple zeta functions as absolute tensor products of usual zeta functions. The Euler product expression is established for the most basic case $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$ by using the signed double Poisson summation formula and the theory of the double sine function.

1. Introduction

A zeta function is usually considered to be a meromorphic function, satisfying a functional equation, with an Euler product expression over generalized primes. There should be an associated relation between the two sets of important objects $\{\text{zeros}, \text{poles}\}$ and $\{\text{generalized primes}\}$. Such a duality is usually written as an explicit formula or a trace formula.

In this paper we try to study a new kind of zeta function constructed from ordinary zeta functions using zeros and poles. We require that the zeros or poles of the new zeta function are sums of zeros or poles of the original zeta functions. We call such a new zeta function a *multiple zeta function*.

We are interested in the 'Euler product expression' for this new zeta function for two reasons. The first reason is the discovery itself of a new Euler product. The second reason is that, as in the case of the usual zeta functions, we would obtain some information on zeros and poles of the new zeta function via its Euler product. This might lead to a new result on zeros and poles of the original zeta functions.

The problem is non-trivial even in the simplest case, constructed from the Hasse zeta functions of two finite fields or from the Selberg zeta functions of two circles. In this paper, we use the theory of multiple sine functions to obtain a neat Euler product for this case.

In a forthcoming paper, we will establish the Euler product for the 'double Riemann zeta function', constructed from two copies of the Riemann zeta function. For that case, the Euler product can be obtained through an extension of our method based on the signed double explicit formula.

We now describe our results in more detail. Let

$$Z_j(s) = \prod_{\rho \in \mathbb{C}} (s - \rho)^{m_j(\rho)}$$

be 'zeta functions' expressed as a regularized product, where

$$m_j : \mathbb{C} \to \mathbb{Z}$$

denotes the multiplicity function for $j = 1, \ldots, r$. (Later, we will specify the 'zeta functions' to

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be treated.) We define the absolute tensor product $(Z_1 \otimes \cdots \otimes Z_r)(s)$ as

$$(Z_1 \otimes \cdots \otimes Z_r)(s) = \prod_{\rho_1,\ldots,\rho_r \in \mathbb{C}} (s - (\rho_1 + \cdots + \rho_r))^{m(\rho_1,\ldots,\rho_r)}$$

with

$$m(\rho_1,\ldots,\rho_r) = m_1(\rho_1)\cdots m_r(\rho_r) \times \begin{cases} 1 & \mathrm{Im}(\rho_j) \geqslant 0 \ (j = 1,\ldots,r), \\ (-1)^{r-1} & \mathrm{Im}(\rho_j) < 0 \ (j = 1,\ldots,r), \\ 0 & \text{otherwise.} \end{cases}$$

This definition originates from [Kur92a]. We also refer to the excellent survey of Manin [Man95]. The notation of the regularized product is due to Deninger [Den92]; see [HKW03] for the required regularized products. The absolute tensor product was studied by Schröter [Sch96] under the name of 'Kurokawa tensor product'.

We are especially interested in the case of Hasse zeta functions $Z_j(s) = \zeta(s, A_j)$ for rings $A_1,\ldots,A_r$. We recall that the Hasse zeta function $\zeta(s, A)$ of a ring A is defined to be

$$\zeta(s, A) = \prod_{\mathbf{m}} (1 - N(\mathbf{m})^{-s})^{-1}$$

where $\mathbf{m}$ runs over maximal left ideals of A up to the following equivalence:

$$\mathbf{m}_1 \sim \mathbf{m}_2 \Longleftrightarrow A/\mathbf{m}_1 \text{ and } A/\mathbf{m}_2 \text{ are isomorphic as left } A\text{-modules},$$

and $N(\mathbf{m}) = \#\,\mathrm{End}_{A\text{-mod}}(A/\mathbf{m})$ (see [Kur96] and [Fuk98]). For a commutative ring A, the above function $\zeta(s, A)$ coincides with the usual Hasse zeta function

$$\zeta(s, A) = \prod_{\mathbf{m}} (1 - N(\mathbf{m})^{-s})^{-1},$$

when $\mathbf{m}$ runs over maximal ideals of A and $N(\mathbf{m}) = \#(A/\mathbf{m})$.

For simplicity, we write

$$\zeta(s, A_1 \otimes \cdots \otimes A_r) = \zeta(s, A_1) \otimes \cdots \otimes \zeta(s, A_r).$$

Actually, as was explained by Manin [Man95], we expect that our multiple zeta function would be the zeta function of the 'absolute tensor product'

$$A_1 \otimes_{\mathbf{F}_1} \cdots \otimes_{\mathbf{F}_1} A_r,$$

that is the tensor product over the (virtual) 'one element field' $\mathbf{F}_1$ (see [KOW03] for the absolute mathematics over '$\mathbf{F}_1$'). In any event, we note that $\zeta(s, A_1 \otimes \cdots \otimes A_r)$ has the following additive structure on zeros and poles: if $\zeta(s, A_j) = 0$ or ∞ and $\mathrm{Im}(s_j)$ $(j = 1,\ldots,r)$ all have the same sign, then $\zeta(s_1 + \cdots + s_r, A_1 \otimes \cdots \otimes A_r) = 0$ or ∞.

Such an additive structure was crucial in the study of Hasse zeta functions of positive characteristic (congruence zeta functions) pursued by Grothendieck [Gro77] and Deligne [Del74], where Euler products were important for restricting the region of zeros and poles and reaching the analogue of the Riemann hypothesis.

We expect that our multiple zeta functions also have Euler products of the following form:

$$\zeta(s, A_1) \otimes \cdots \otimes \zeta(s, A_r) = \prod_{(\mathbf{m}_1,\ldots,\mathbf{m}_r)} H_{(\mathbf{m}_1,\ldots,\mathbf{m}_r)}(N(\mathbf{m}_1)^{-s},\ldots,N(\mathbf{m}_1)^{-s}).$$

Here the $\mathbf{m}_i$ run over the maximal (left) ideals of A_i and $H_{(\mathbf{m}_1,\ldots,\mathbf{m}_r)}(T_1,\ldots,T_r)$ is a power series in $T_1,\ldots,T_r$ of the constant term 1 with a possible degeneration at $(\mathbf{m}_1,\ldots,\mathbf{m}_r)$ when $N(\mathbf{m}_i) = N(\mathbf{m}_j)$ for some $i \neq j$. More generally, we expect that the multiple zeta function $Z_1(s) \otimes \cdots \otimes Z_r(s)$

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has an Euler product

$$Z_1(s) \otimes \cdots \otimes Z_r(s) = \prod_{(p_1,\ldots,p_r) \in P_1 \times \cdots \times P_r} H_{(p_1,\ldots,p_r)}(N(p_1)^{-s}, \ldots, N(p_r)^{-s})$$

when each zeta function $Z_j(s)$ has an Euler product

$$Z_j(s) = \prod_{p \in P_j} H_p^j(N(p)^{-s})$$

and satisfies a functional equation; here $H_p^j(T)$ is a power series in T and $H_{(p_1,\ldots,p_r)}(T_1, \ldots, T_r)$ is a power series in $(T_1, \ldots, T_r)$ with a possible degeneration at $(p_1, \ldots, p_r)$ when $N(p_i) = N(p_j)$ for some $i \neq j$.

In this paper we investigate

$$\zeta(s, \mathbf{F}_p \otimes \mathbf{F}_q) = \zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$$

for primes p and q. We prove that it has a kind of Euler product expression in terms of the polylogarithm

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

and its variations. Note that the original 'Euler product' is

$$\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1} = H_p(p^{-s}) = \exp\left(\sum_{n=1}^{\infty} \frac{p^{-ns}}{n}\right) = \exp(\mathrm{Li}_1(p^{-s})),$$

where

$$H_p(T) = (1 - T)^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n}\right) = \exp(\mathrm{Li}_1(T)).$$

We use the following notation for two functions $F(s)$ and $G(s)$: $F(s) \cong G(s)$ if there exists a polynomial $Q(s)$ satisfying $F(s) = e^{Q(s)} G(s)$.

Theorem 1. *For distinct primes p and q, the following expression holds in* $\mathrm{Re}(s) > 0$*:*

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \cong (1 - p^{-s})^{1/2} (1 - q^{-s})^{1/2} \times \exp\left(\frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot(\pi n (\log p / \log q))}{n} p^{-ns} + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot(\pi n (\log q / \log p))}{n} q^{-ns}\right).$$

Remark 1.1. We see that $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q)$ is essentially $H_{p,q}(p^{-s}, q^{-s})$ with

$$H_{(p,q)}(T_1, T_2) = (1 - T_1)^{1/2} (1 - T_2)^{1/2} \times \exp\left(\frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot(\pi n (\log p / \log q))}{n} T_1^n + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot(\pi n (\log q / \log p))}{n} T_2^n\right),$$

which is indeed a power series in T_1 and T_2 as we expected.

Remark 1.2. Recently, Akatsuka [Aka03] calculated $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \otimes \zeta(s, \mathbf{F}_r)$ for distinct primes p, q and r. For $\zeta(s, \mathbf{F}_{p_1}) \otimes \cdots \otimes \zeta(s, \mathbf{F}_{p_r})$ see [KW04].

We prove Theorem 1 by showing a corresponding expression for the double sine function. We recall the multiple sine functions studied in [Kur91, Kur92a, Kur92b, KK03a], and we use the multiple Hurwitz zeta function due to Barnes [Bar04]

$$\zeta_r(s, z, \underline{\omega}) = \sum_{n_1,\ldots,n_r=0}^{\infty} (n_1 \omega_1 + \cdots + n_r \omega_r + z)^{-s}$$

for $\underline{\omega} = (\omega_1, \ldots, \omega_r)$. The definitions of the multiple gamma function and the multiple sine function are as follows:

$$\Gamma_r(z, \underline{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, z, \underline{\omega})\bigg|_{s=0}\right) = \left(\prod_{\mathbf{n}\geqslant\mathbf{0}}(\mathbf{n}\cdot\underline{\omega} + z)\right)^{-1},$$

$$S_r(z, \underline{\omega}) = \Gamma_r(z, \underline{\omega})^{-1}\Gamma_r(\omega_1 + \cdots + \omega_r - z, \underline{\omega})^{(-1)^r} = \left(\prod_{\mathbf{n}\geqslant\mathbf{0}}(\mathbf{n}\cdot\underline{\omega} + z)\right)\left(\prod_{\mathbf{n}\geqslant\mathbf{1}}(\mathbf{n}\cdot\underline{\omega} - z)\right)^{(-1)^{r-1}}.$$

When $r = 2$, we have $\underline{\omega} = (\omega_1, \omega_2)$ and

$$S_2(z, (\omega_1, \omega_2)) = \Gamma_2(z, (\omega_1, \omega_2))^{-1}\Gamma_2(\omega_1 + \omega_2 - z, (\omega_1, \omega_2)).$$

We say that a real number α is *generic* if and only if

$$\lim_{m\to\infty}\|m\alpha\|^{1/m} = 1,$$

where we put $\|x\| := \min\{|x - n| : n \in \mathbb{Z}\}$ for $x \in \mathbb{R}$. For example:

1) if $\alpha \in (\overline{\mathbb{Q}} \cap \mathbb{R}) \setminus \mathbb{Q}$, then α is generic;
2) let $x, y \in \overline{\mathbb{Q}} \cap \mathbb{R}$; if $\alpha = \log x/\log y \notin \mathbb{Q}$, then α is transcendental and generic [Bak75, Theorem 3.1].

Theorem 2. *Let ω_1 and ω_2 be positive real numbers. Assume that ω_1/ω_2 and ω_2/ω_1 are both generic. Then the double sine function has the following expression in* $\mathrm{Im}(z) > 0$*:*

$$\begin{aligned} S_2(z, (\omega_1, \omega_2)) = \exp\Bigg(&\frac{1}{2i}\sum_{k=1}^{\infty}\frac{1}{k}\cot\left(\pi k\frac{\omega_2}{\omega_1}\right)e^{2\pi ik(z/\omega_1)} + \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi n\frac{\omega_1}{\omega_2}\right)e^{2\pi in(z/\omega_2)} \\ &+ \frac{1}{2}\log(1 - e^{2\pi i(z/\omega_1)}) + \frac{1}{2}\log(1 - e^{2\pi i(z/\omega_2)}) \\ &+ \frac{\pi iz^2}{2\omega_1\omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)z + \frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} + 3\right)\Bigg). \end{aligned}$$

Since

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \cong S_2\left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right)$$

by Proposition 4.1 below, Theorem 2 directly implies Theorem 1. Now the main step in proving Theorem 2 (and hence Theorem 1) is to establish the following signed double Poisson summation formula, which is a special case of signed multiple explicit formulae.

Theorem 3. *Let $H(t)$ be an odd function in $L^1(\mathbb{R})$ with $H(t) = O(t^{-2})$ as $|t| \to \infty$, and put*

$$\widetilde{H}(u) = \int_{-\infty}^{\infty}H(t)e^{itu}\,dt.$$

Assume that a/b and b/a are both generic and that the test function $H(t)$ satisfies

$$\widetilde{H}(x) = O(\mu^x) \tag{1.1}$$

as $x \to \infty$ for some $0 < \mu < 1$. Then we have

$$\begin{aligned} &\sum_{k,n>0}H\left(2\pi\left(\frac{k}{a} + \frac{n}{b}\right)\right) + \frac{1}{2}\left(\sum_{k>0}H\left(2\pi\frac{k}{a}\right) + \sum_{n>0}H\left(2\pi\frac{n}{b}\right)\right) \\ &\qquad = -\frac{ia}{4\pi}\sum_{k>0}\cot\left(\pi\frac{ka}{b}\right)\widetilde{H}(ka) - \frac{ib}{4\pi}\sum_{n>0}\cot\left(\pi\frac{nb}{a}\right)\widetilde{H}(nb) - \frac{iab}{8\pi^2}\widetilde{H}'(0). \end{aligned} \tag{1.2}$$

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We prove Theorem 3 in § 2 using a method that goes back to Cramér [Cra19] (see also [Den98, Vor03]). Then we prove Theorem 2 by applying Theorem 3 in § 3. We show Theorem 1 from Theorem 2 in § 4. Finally, we treat $\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_p)$ in § 5. Our method of using the signed multiple explicit formula is applicable to general multiple zeta functions. In fact, in a forthcoming paper we will calculate the double Riemann zeta function $\zeta(s)\otimes\zeta(s)$ using this method.

2. Signed double Poisson summation formula

In this section we prove Theorem 3.

Lemma 2.1. *Assume that α is generic. Then the power series*

$$\sum_{n=1}^{\infty}\cot(\pi n\alpha)x^n \tag{2.1}$$

converges absolutely in $|x|<1$.

Proof. As α is generic, we have $\|n\alpha\|^{-1}=O(e^{\varepsilon n})$ as $n\to\infty$ for any $\varepsilon>0$. Since $\cot(\pi x)\sim 1/(\pi x)$ as $x\to 0$, we have $\cot(\pi n\alpha)=O(e^{\varepsilon n})$ for any $\varepsilon>0$. □

Proof of Theorem 3. We use Cramér's method from [Cra19]. Put $Z_a(s)=\sinh(as/2)$ and $Z_b(s)=\sinh(bs/2)$. Let D_T be the region defined by

$$D_T=\Big\{s\in\mathbb{C}\ \Big|\ |s|>\alpha, |\mathrm{Re}(s)|<\alpha, 0<\mathrm{Im}(s)<T\Big\}$$

with $0<\alpha<\min\{2\pi/a,2\pi/b\}$. By Cauchy's theorem for an odd function h which is regular in D_T, we have

$$\sum_{\substack{0<\mathrm{Im}(\rho_a)\\ \mathrm{Im}(\rho_b)<T}} h(\rho_a+\rho_b)=\frac{1}{(2\pi i)^2}\int_{\partial D_T}\int_{\partial D_T} h(s_1+s_2)\frac{Z_a'}{Z_a}(s_1)\frac{Z_b'}{Z_b}(s_2)\,ds_1\,ds_2, \tag{2.2}$$

where ρ_a and ρ_b denote the zeros of $Z_a(s)$ and $Z_b(s)$, respectively, and the integrals along ∂D_T are taken counter-clockwise. Considering the limits as $T\to\infty$ of both sides of (2.2), we have

$$\sum_{\mathrm{Im}(\rho_a),\mathrm{Im}(\rho_b)>0} h(\rho_a+\rho_b)=\frac{1}{(2\pi i)^2}\int_{\partial D}\int_{\partial D} h(s_1+s_2)\frac{Z_a'}{Z_a}(s_1)\frac{Z_b'}{Z_b}(s_2)\,ds_1\,ds_2, \tag{2.3}$$

where

$$D=\Big\{s\in\mathbb{C}\ \Big|\ |\mathrm{Re}(s)|<\alpha, |s|>\alpha, \mathrm{Im}(s)>0\Big\}.$$

We decompose $\partial D=C_1\cup C_2\cup C_3$ with

$$C_1=\Big\{s\in\partial D\ \Big|\ \mathrm{Re}(s)=-\alpha\Big\},\quad C_2=\Big\{s\in\partial D\ \Big|\ |s|=\alpha\Big\},\quad C_3=\Big\{s\in\partial D\ \Big|\ \mathrm{Re}(s)=\alpha\Big\}.$$

We compute each double integral $I_{ij}=(1/(2\pi i)^2)\int_{C_i}\int_{C_j}$ in (2.3).

First we treat the integral along the vertical lines:

$$I_{33}=\frac{1}{(2\pi)^2}\int_0^{\infty}\int_0^{\infty} h(2\alpha+i(t_1+t_2))\frac{Z_a'}{Z_a}(\alpha+it_1)\frac{Z_b'}{Z_b}(\alpha+it_2)\,dt_1\,dt_2. \tag{2.4}$$

Note that

$$\frac{Z_a'}{Z_a}(\alpha+it_1)=\frac{a}{2}+a\sum_{k=1}^{\infty}e^{-ka(\alpha+it_1)}$$

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and

$$\frac{Z_b'}{Z_b}(\alpha + it_2) = \frac{b}{2} + b\sum_{n=1}^{\infty} e^{-nb(\alpha+it_2)}.$$

We define

$$H_\alpha(t) = h(2\alpha + it).$$

Then (2.4) can be written as

$$I_{33} = \frac{1}{4\pi^2}\sum_{k,n\geqslant 0} \varepsilon_{k,n} ab \int_0^\infty \int_0^t H_\alpha(t) e^{-ka(\alpha+it_1)} e^{-nb(\alpha+i(t-t_1))}\, dt_1\, dt,$$

where we put

$$\varepsilon_{k,n} = \begin{cases} \frac{1}{4} & k = n = 0, \\ \frac{1}{2} & k = 0, n \neq 0 \text{ or } k \neq 0, n = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} I_{33} &= \frac{ab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka\neq nb}} \varepsilon_{k,n} e^{-(ka+nb)\alpha} \int_0^\infty \frac{H_\alpha(t)(e^{-ikat} - e^{-inbt})}{-i(ka-nb)}\, dt \\ &\quad + \frac{ab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \int_0^\infty tH_\alpha(t) e^{-ikat}\, dt. \end{aligned} \tag{2.5}$$

Similarly, we compute

$$\begin{aligned} I_{11} &= -\frac{ab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka\neq nb}} \varepsilon_{k,n} e^{-(ka+nb)\alpha} \int_0^\infty \frac{H_\alpha(-t)(e^{ikat} - e^{inbt})}{i(ka-nb)}\, dt \\ &\quad - \frac{ab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \int_0^\infty tH_\alpha(-t) e^{ikat}\, dt. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6) we have

$$\begin{aligned} I_{11} + I_{33} &= \frac{ab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka\neq nb}} \varepsilon_{k,n} e^{-(ka+nb)\alpha} \int_{-\infty}^\infty \frac{H_\alpha(t)(e^{-ikat} - e^{-inbt})}{-i(ka-nb)}\, dt \\ &\quad + \frac{ab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \int_{-\infty}^\infty tH_\alpha(t) e^{-ikat}\, dt. \\ &= \frac{iab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka\neq nb}} \varepsilon_{k,n} \frac{e^{-(ka+nb)\alpha}}{ka-nb}(\widetilde{H_\alpha}(-ka) - \widetilde{H_\alpha}(-nb)) \\ &\quad + \frac{ab}{4\pi^2}\sum_{\substack{k,n\geqslant 0\\ ka=nb}} \varepsilon_{k,n} e^{-2ka\alpha} \widetilde{tH_\alpha(t)}(-ka). \end{aligned}$$

The assumption that a/b is generic implies that the second sum consists of only one term with $k = n = 0$.

Next we calculate I_{13}. Since $h(i(t_1+t_2)) = H_0(t_1+t_2)$ and Z_a'/Z_a is an odd function, we have

$$\begin{aligned} I_{13} &= \frac{-1}{(2\pi i)^2}\int_\infty^0\int_0^\infty h(i(t_1+t_2))\frac{Z_a'}{Z_a}(-\alpha+it_1)\frac{Z_b'}{Z_b}(\alpha+it_2)\,dt_1\,dt_2 \\ &= \frac{ab}{4\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}e^{-(ka+nb)\alpha}\int_0^\infty \frac{H_0(t)(e^{ikat}-e^{-inbt})}{i(ka+nb)}\,dt + \frac{ab}{16\pi^2}\int_0^\infty tH_0(t)\,dt. \end{aligned} \tag{2.7}$$

Similarly,

$$I_{31} = \frac{ab}{4\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}e^{-(ka+nb)\alpha}\int_0^\infty \frac{H_0(-t)(e^{-ikat}-e^{inbt})}{i(ka+nb)}\,dt + \frac{ab}{16\pi^2}\int_0^\infty tH_0(t)\,dt. \tag{2.8}$$

Therefore, (2.7) and (2.8) lead to

$$I_{13}+I_{31} = -\frac{iab}{4\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}\frac{e^{-(ka+nb)\alpha}}{(ka+nb)}(\widetilde{H_0}(ka)-\widetilde{H_0}(-nb)) - \frac{iab}{16\pi^2}\widetilde{H_0}'(0).$$

Letting $\alpha\to 0$ gives

$$\lim_{\alpha\to 0}(I_{11}+I_{33}+I_{13}+I_{31}) = -\frac{iab}{2\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}\frac{\widetilde{H_0}(ka)ka-\widetilde{H_0}(nb)nb}{k^2a^2-n^2b^2} - \frac{iab}{8\pi^2}\widetilde{H_0}'(0), \tag{2.9}$$

since $\widetilde{H_0}' = i\widetilde{tH_0(t)}$.

Next we treat $I_2 := I_{21}+I_{22}+I_{23}$. We compute

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{C_2}\left(\frac{1}{2\pi i}\int_{\partial D} h(s_1+s_2)\frac{Z_a'}{Z_a}(s_1)\,ds_1\right)\frac{Z_b'}{Z_b}(s_2)\,ds_2 \\ &= \frac{1}{2\pi i}\int_{C_2}\sum_{\rho_a} h(\rho_a+s_2)\frac{Z_b'}{Z_b}(s_2)\,ds_2, \end{aligned}$$

where ρ_a runs through the zeros of $Z_a(s)$ with $\mathrm{Im}(\rho)>0$. Putting $s_2=\alpha e^{i\theta}$, we reach

$$\lim_{\alpha\to 0} I_2 = \frac{1}{2\pi}\int_\pi^0\sum_{\rho_a} h(\rho_a)\,d\theta = -\frac{1}{2}\sum_{\rho_a} h(\rho_a). \tag{2.10}$$

We similarly deal with $I_2' := I_{12}+I_{22}+I_{32}$ to obtain

$$\lim_{\alpha\to 0} I_2' = -\frac{1}{2}\sum_{\rho_b} h(\rho_b). \tag{2.11}$$

The integral I_{22}, which appears in both (2.10) and (2.11), tends to 0 as $\alpha\to 0$. Thus, taking (2.9), (2.10) and (2.11) into account, (2.3) equals

$$-\frac{iab}{2\pi^2}\sum_{(k,n)\neq(0,0)}\varepsilon_{k,n}\frac{\widetilde{H_0}(ka)ka-\widetilde{H_0}(nb)nb}{k^2a^2-n^2b^2} - \frac{iab}{8\pi^2}\widetilde{H_0}'(0) - \frac{1}{2}\sum_{k>0}H_0\left(2\pi\frac{k}{a}\right) - \frac{1}{2}\sum_{n>0}H_0\left(2\pi\frac{n}{b}\right).$$

Theorem 3 now follows from the formulae

$$\sum_{n>0}\frac{2ka}{k^2a^2-n^2b^2}+\frac{1}{ka} = \frac{\pi}{b}\cot\left(\pi\frac{ka}{b}\right), \quad \sum_{k>0}\frac{2nb}{n^2b^2-k^2a^2}+\frac{1}{nb} = \frac{\pi}{a}\cot\left(\pi\frac{nb}{a}\right). \qquad \square$$

Remark 2.2. When we apply calculations of the above type, in the style of Cramér [Cra19], to zeta functions $Z_1(s)$ and $Z_2(s)$ that have Euler products with functional equations instead of $\sinh(as/2)$ and $\sinh(bs/2)$, we obtain the signed double explicit formula.

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3. Expression of the double sine function

We prove Theorem 2 from Theorem 3.

Lemma 3.1. *We have*

$$\frac{d^2}{dz^2}\log(1-e^{iaz}) = -\sum_{n=-\infty}^{\infty}\frac{1}{(z-2\pi n/a)^2}.$$

Proof. Since

$$\log(1-e^{iaz}) = -\frac{\pi i}{2}+\frac{iaz}{2}+\log\left(2\sin\frac{az}{2}\right)$$

and

$$2\sin\frac{az}{2} = az\prod_{n=1}^{\infty}\left(1-\left(\frac{az}{2\pi n}\right)^2\right),$$

we have

$$\frac{d^2}{dz^2}\log(1-e^{iaz}) = -\frac{1}{z^2}-\sum_{n=1}^{\infty}\left(\frac{1}{(z-2\pi n/a)^2}+\frac{1}{(z+2\pi n/a)^2}\right) = -\sum_{n=-\infty}^{\infty}\frac{1}{(z-2\pi n/a)^2}. \qquad \square$$

Proof of Theorem 2. Apply the odd function

$$H(t) = \frac{1}{(z-t)^2}-\frac{1}{(z+t)^2}$$

with $z\in\mathbb{C}$, $\mathrm{Im}(z)>0$, to our summation formula (1.2). As we have

$$\widetilde{H}(x) = \int_{-\infty}^{\infty}H(t)e^{ixt}\,dt = 2\pi i\,\mathrm{Res}_{t=z}(H(t)e^{ixt}) = -2\pi x e^{ixz},$$

condition (1.1) is satisfied. Since $\widetilde{H}'(0) = -2\pi$, if we put

$$F(z) = \sum_{k,n\geqslant 1}\left(\frac{1}{(z-2\pi(k/a+n/b))^2}-\frac{1}{(z+2\pi(k/a+n/b))^2}\right)$$
$$+\frac{1}{2}\sum_{k>0}\left(\frac{1}{(z-2\pi(k/a))^2}-\frac{1}{(z+2\pi(k/a))^2}\right)+\frac{1}{2}\sum_{n>0}\left(\frac{1}{(z-2\pi(n/b))^2}-\frac{1}{(z+2\pi(n/b))^2}\right),$$

the summation formula (1.2) gives

$$\begin{aligned} F(z) &= \frac{i}{2}\sum_{k>0}\cot\left(\pi\frac{ka}{b}\right)ka^2e^{ikaz}+\frac{i}{2}\sum_{n>0}\cot\left(\pi\frac{nb}{a}\right)nb^2e^{inbz}+\frac{iab}{4\pi}\\ &= \frac{d^2}{dz^2}\left(\frac{1}{2i}\sum_{k>0}\frac{1}{k}\cot\left(\pi\frac{ka}{b}\right)e^{ikaz}+\frac{1}{2i}\sum_{n>0}\frac{1}{n}\cot\left(\pi\frac{nb}{a}\right)e^{inbz}\right)+\frac{iab}{4\pi}. \end{aligned} \tag{3.1}$$

Now we have (see [KK03a])

$$S_2(z,\omega_1,\omega_2) = e^{c_0+c_1z+c_2z^2}\frac{z\prod'_{n_1,n_2\geqslant 0}P_2(-z/(n_1\omega_1+n_2\omega_2))}{\prod_{n_1,n_2\geqslant 1}P_2(z/(n_1\omega_1+n_2\omega_2))} \tag{3.2}$$

for $P_2(u) := (1-u)\exp(u+u^2/2)$. Hence, setting $n_1 = k$, $n_2 = n$, $\omega_1 = 2\pi/a$ and $\omega_2 = 2\pi/b$, we

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have

$$\begin{aligned}\frac{d^2}{dz^2}\log S_2(z,(\omega_1,\omega_2)) &= -\frac{1}{z^2} - \sum_{n_1,n_2\geqslant 1}\left(\frac{1}{(z+n_1\omega_1+n_2\omega_2)^2} - \frac{1}{(z-(n_1\omega_1+n_2\omega_2))^2}\right)\\ &\quad - \sum_{n_1\geqslant 1}\frac{1}{(z+n_1\omega_1)^2} - \sum_{n_2\geqslant 1}\frac{1}{(z+n_2\omega_2)^2} + 2c_2\\ &= F(z) - \frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{1}{(z-2\pi(k/a))^2} - \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{(z-2\pi(n/b))^2} + 2c_2\\ &= \frac{d^2}{dz^2}\left(\frac{1}{2i}\sum_{k>0}\frac{1}{k}\cot\left(\pi\frac{ka}{b}\right)e^{ikaz} + \frac{1}{2i}\sum_{n>0}\frac{1}{n}\cot\left(\pi\frac{nb}{a}\right)e^{inbz}\right.\\ &\quad \left. + \frac{1}{2}\log(1-e^{iaz}) + \frac{1}{2}\log(1-e^{ibz})\right) + 2c_2 + \frac{iab}{4\pi}, \end{aligned} \tag{3.3}$$

where we have used (3.1) and Lemma 3.1. So, if we put

$$\begin{aligned} E(z) &:= \log S_2(z,(\omega_1,\omega_2)) - \left(\frac{1}{2i}\sum_{k>0}\frac{1}{k}\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)e^{2\pi ikz/\omega_1}\right.\\ &\quad \left. + \frac{1}{2i}\sum_{n>0}\frac{1}{n}\cot\left(\pi\frac{n\omega_1}{\omega_2}\right)e^{2\pi inz/\omega_2} + \frac{1}{2}\log(1-e^{(2\pi i/\omega_1)z}) + \frac{1}{2}\log(1-e^{(2\pi i/\omega_2)z})\right), \end{aligned} \tag{3.4}$$

it follows that $d^2E(z)/dz^2$ is constant and that $E(z)$ is a polynomial of degree 2.

Thus, we put $E(z) = \alpha + \beta z + \gamma z^2$ and compute α, β and γ. We first calculate β and γ by considering

$$E(z+\omega_1) - E(z) = (\beta\omega_1 + \gamma\omega_1^2) + 2\gamma\omega_1 z. \tag{3.5}$$

It follows from (3.4) that (3.5) equals

$$\log\frac{S_2(z+\omega_1,(\omega_1,\omega_2))}{S_2(z,(\omega_1,\omega_2))} - \frac{1}{2i}\sum_{n>0}\frac{1}{n}\cot\left(\pi\frac{n\omega_1}{\omega_2}\right)(e^{2\pi in\omega_1/\omega_2}-1)e^{2\pi inz/\omega_2}$$
$$-\frac{1}{2}\log(1-e^{(2\pi i/\omega_2)(z+\omega_1)}) + \frac{1}{2}\log(1-e^{(2\pi i/\omega_2)z}).$$

The sum over n is computed as

$$\begin{aligned}-\frac{1}{2i}\sum_{n>0}\frac{1}{n}\cot\left(\pi\frac{n\omega_1}{\omega_2}\right)(e^{2\pi in\omega_1/\omega_2}-1)e^{2\pi inz/\omega_2} &= -\frac{1}{2}\sum_{n>0}\frac{1}{n}(1+e^{2\pi in\omega_1/\omega_2})e^{2\pi inz/\omega_2}\\ &= \frac{1}{2}\log(1-e^{(2\pi i/\omega_2)(z+\omega_1)}) + \frac{1}{2}\log(1-e^{(2\pi i/\omega_2)z}).\end{aligned}$$

We use formula (2.4) from [KK03a] to obtain

$$\frac{S_2(z+\omega_1,(\omega_1,\omega_2))}{S_2(z,(\omega_1,\omega_2))} = S_1(z,\omega_2)^{-1} = \left(2\sin\frac{\pi z}{\omega_2}\right)^{-1}.$$

Hence, (3.5) is equal to

$$\begin{aligned}-\log\left(2\sin\frac{\pi z}{\omega_2}\right) + \log(1-e^{(2\pi i/\omega_2)z}) &= -\log\left(2\sin\frac{\pi z}{\omega_2}\right) + \log\left(-2ie^{(\pi i/\omega_2)z}\sin\frac{\pi z}{\omega_2}\right)\\ &= -\frac{\pi i}{2} + \frac{\pi i}{\omega_2}z.\end{aligned}$$

Therefore, we have

$$\beta\omega_1 + \gamma\omega_1^2 = -\frac{\pi i}{2} \quad \text{and} \quad 2\gamma\omega_1 = \frac{\pi i}{\omega_2}.$$

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We thus obtain

$$\beta = -\frac{\pi i}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right) \quad \text{and} \quad \gamma = \frac{\pi i}{2\omega_1\omega_2}.$$

Next we deal with α by considering

$$E(z)+E\left(z+\frac{\omega_1}{2}\right)+E\left(z+\frac{\omega_2}{2}\right)+E\left(z+\frac{\omega_1+\omega_2}{2}\right)-E(2z). \tag{3.6}$$

The constant term of (3.6) is

$$3\alpha+\beta(\omega_1+\omega_2)+\gamma\left(\frac{\omega_1}{4}+\frac{\omega_2}{4}+\frac{(\omega_1+\omega_2)^2}{4}\right) = 3\alpha-\frac{\pi i}{4}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}+3\right). \tag{3.7}$$

On the other hand, we compute (3.6) by using (3.4). We write (3.6) as $\sum_{j=0}^{4} A_j$, where

$$A_0 = \log\frac{S_2(z,(\omega_1,\omega_2))S_2(z+\omega_1/2,(\omega_1,\omega_2))S_2(z+\omega_2/2,(\omega_1,\omega_2))S_2(z+(\omega_1+\omega_2)/2,(\omega_1,\omega_2))}{S_2(2z,(\omega_1,\omega_2))},$$

$$A_1 = -\frac{1}{2i}\sum_{k>0}\frac{\cot(\pi(k\omega_2/\omega_1))}{k}\left(e^{\frac{2\pi ikz}{\omega_1}}+e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_1}{2})}+e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_2}{2})}+e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_1+\omega_2}{2})}-e^{\frac{4\pi ikz}{\omega_1}}\right),$$

$$A_2 = -\frac{1}{2i}\sum_{n>0}\frac{\cot(\pi(n\omega_1/\omega_2))}{n}\left(e^{\frac{2\pi inz}{\omega_2}}+e^{\frac{2\pi in}{\omega_2}(z+\frac{\omega_2}{2})}+e^{\frac{2\pi in}{\omega_2}(z+\frac{\omega_1}{2})}+e^{\frac{2\pi in}{\omega_2}(z+\frac{\omega_2+\omega_1}{2})}-e^{\frac{4\pi inz}{\omega_2}}\right),$$

$$A_3 = -\frac{1}{2}\log\frac{\left(1-e^{\frac{2\pi i}{\omega_1}z}\right)\left(1-e^{\frac{2\pi i}{\omega_1}(z+\frac{\omega_1}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_1}(z+\frac{\omega_2}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_1}(z+\frac{\omega_1+\omega_2}{2})}\right)}{1-e^{\frac{4\pi i}{\omega_1}z}},$$

$$A_4 = -\frac{1}{2}\log\frac{\left(1-e^{\frac{2\pi i}{\omega_2}z}\right)\left(1-e^{\frac{2\pi i}{\omega_2}(z+\frac{\omega_1}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_2}(z+\frac{\omega_2}{2})}\right)\left(1-e^{\frac{2\pi i}{\omega_2}(z+\frac{\omega_1+\omega_2}{2})}\right)}{1-e^{\frac{4\pi i}{\omega_2}z}}.$$

Formula (2.5) from [KK03a] gives $A_0 = 0$. Next, A_1 is computed as follows:

$$\begin{aligned} A_1 &= -\frac{1}{2i}\sum_{\substack{k>0\\ \text{even}}}\frac{1}{k}\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)\left(2e^{\frac{2\pi ikz}{\omega_1}}+2e^{\frac{2\pi ik}{\omega_1}(z+\frac{\omega_2}{2})}\right)+\frac{1}{2i}\sum_{k>0}\frac{1}{k}\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)e^{\frac{4\pi ikz}{\omega_1}} \\ &= -\frac{1}{2i}\sum_{k>0}\frac{1}{k}\left(\cot\left(\pi\frac{2k\omega_2}{\omega_1}\right)\left(1+e^{\frac{2\pi ik\omega_2}{\omega_1}}\right)-\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)\right)e^{\frac{4\pi ikz}{\omega_1}} \\ &= -\frac{1}{2}\sum_{k>0}\frac{1}{k}e^{\frac{2\pi ik\omega_2}{\omega_1}}e^{\frac{4\pi ikz}{\omega_1}} \\ &= \frac{1}{2}\log\left(1-e^{\frac{4\pi i}{\omega_1}(z+\frac{\omega_2}{2})}\right). \end{aligned}$$

Here we have used the identity

$$\cot 2\theta(1+e^{2i\theta})-\cot\theta = ie^{2i\theta}$$

with $\theta = \pi(k\omega_2/\omega_1)$. Similarly, A_2 is calculated as

$$A_2 = \frac{1}{2}\log\left(1-e^{\frac{4\pi i}{\omega_2}(z+\frac{\omega_1}{2})}\right).$$

The remaining terms are easily computed as

$$A_3 = -\frac{1}{2}\log\left(1-e^{\frac{4\pi i}{\omega_1}(z+\frac{\omega_2}{2})}\right), \quad A_4 = -\frac{1}{2}\log\left(1-e^{\frac{4\pi i}{\omega_2}(z+\frac{\omega_1}{2})}\right).$$

Hence we have deduced that (3.6) is equal to $\sum_{j=0}^{4} A_j = 0$. Therefore, its constant term (3.7) vanishes, which leads to

$$\alpha = \frac{\pi i}{12}\biggl(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} + 3\biggr). \qquad \square$$

Remark 3.2. The referee indicated that the method used above to prove Theorem 2 may be modified as follows. Let $R(z)$ be the quotient of the left-hand side by the right-hand side in Theorem 2. Using the formulae

$$\frac{S_2(z,(\omega_1,\omega_2))}{S_2(z+\omega_1,(\omega_1,\omega_2))} = 2\sin\frac{\pi z}{\omega_2}, \quad \frac{S_2(z,(\omega_1,\omega_2))}{S_2(z+\omega_2,(\omega_1,\omega_2))} = 2\sin\frac{\pi z}{\omega_1}$$

and exponential expressions for trigonometric functions, we see that $R(z)$ is both ω_1-periodic and ω_2-periodic. Hence $R(z)$ is constant, since ω_1/ω_2 is an irrational real number. To calculate the constant function $R(z)$ one uses the natural branch of $\log R(z)$ and evaluates

$$3\log R(z) = \sum_{\omega\in T} \log R(z+\omega) - \log R(2z),$$

where $T = \{0, \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2\}$. For this, one applies the following functional equation for the natural branch of $\log S_2(z,(\omega_1,\omega_2))$ in the strip $0 < \mathrm{Re}(z) < \omega_1 + \omega_2$:

$$\sum_{\omega\in T} \log S_2(z+\omega,(\omega_1,\omega_2)) = \log S_2(2z,(\omega_1,\omega_2)),$$

which is immediate from the corresponding functional equation for the double Hurwitz zeta function.

Remark 3.3. The condition that ω_2/ω_1 and ω_1/ω_2 are generic is similar to the 'diophantine condition' used by Connes [Con85] to calculate cohomologies for (irrational rotational) non-commutative tori. The problem belongs to the category of so-called 'small denominator problems'.

Remark 3.4. We easily see that

$$\frac{\pi i z^2}{2\omega_1\omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) z + \frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} + 3\right) = \pi i \zeta_2(0, z, (\omega_1,\omega_2)),$$

which is a multiple Bernoulli polynomial of Barnes [Bar04].

Remark 3.5. The formula of Theorem 2 is valid for more general situations. For example, it holds for $\arg(\omega_2) < \arg(z) < \arg(\omega_1) + \pi$ when $0 \leqslant \arg(\omega_1) \leqslant \arg(\omega_2) < \pi$. We refer to [KW04] where $S_r(z;(\omega_1,\ldots,\omega_r))$ is also treated.

4. The product $\zeta(s,\mathbf{F}_p) \otimes \zeta(s,\mathbf{F}_q)$ for distinct primes p and q

We first describe a convenient form of the absolute tensor product of some zeta functions. Let Z_j $(j = 1,2)$ be meromorphic functions of order μ_j. We put the Hadamard product as

$$Z_j(s) = s^{k_j} e^{Q_j(s)} {\prod_{\rho\in\mathbb{C}}}' P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)}, \tag{4.1}$$

where $P_r(u) := (1-u)\exp(u + u^2/2 + \cdots + u^r/r)$, m_j denotes the multiplicity function with $k_j := m_j(0)$, and Q_j is a polynomial with $\deg Q_j \leqslant \mu_j$. Here the product over $\rho \in \mathbb{C}$ means

$$\lim_{R\to\infty} \prod_{0<|\rho|<R} P_{\mu_j}\left(\frac{s}{\rho}\right)^{m_j(\rho)}.$$

Assume, moreover, that Z_j is one of the following: the Riemann zeta function, the Hasse zeta function of a finite field, the Selberg zeta function of a circle and the Selberg zeta function of

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a Riemann surface. Then, by [HKW03] and [KKSW03], the absolute tensor product originally constructed as a regularized product turns out to be the following Hadamard product:

$$(Z_1 \otimes Z_2)(s) = s^{k_1 k_2} e^{Q(s)} \prod_{\rho_1, \rho_2 \in \mathbb{C}}{}' P_{\mu_1+\mu_2}\left(\frac{s}{\rho_1+\rho_2}\right)^{m(\rho_1,\rho_2)}, \tag{4.2}$$

where $Q(s)$ is a polynomial with $\deg Q \leqslant \mu_1 + \mu_2$ and

$$m(\rho_1, \rho_2) := m_1(\rho_1) m_2(\rho_2) \times \begin{cases} 1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) \geqslant 0, \\ -1 & \text{if } \operatorname{Im}(\rho_1), \operatorname{Im}(\rho_2) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

See [HKW03] regarding the existence of the (re-)regularized product for multiple zeta functions and [KKSW03] for its relation to the Hadamard product. Our restriction to four specific zeta functions can be weakened considerably by the method of [HKW03]. It is worthwhile consulting [Ill01, Ill02] for a study of regularized products.

Here we compute this absolute tensor product for Hasse zeta functions of finite fields,

$$Z_1(s) = \zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1}, \quad Z_2(s) = \zeta(s, \mathbf{F}_q) = (1 - q^{-s})^{-1},$$

with p, q primes.

Proposition 4.1. *The absolute tensor product of Hasse zeta functions for finite prime fields is given as*

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \cong S_2\left(is, \left(\frac{2\pi}{\log p}, \frac{2\pi}{\log q}\right)\right).$$

Proof. We easily compute the Hadamard product (4.1) for the Hasse zeta function, which is given by

$$\zeta(s, \mathbf{F}_p) \cong s^{-1} \prod_{n=-\infty}^{\infty}{}' P_1\left(\frac{s}{(2\pi i/\log p)n}\right)^{-1}.$$

Thus, by (4.2),

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \cong s \prod_{k,n \in \mathbb{Z}}{}' P_2\left(\frac{s}{(2\pi i/\log p)k + (2\pi i/\log q)n}\right)^{m_{k,n}}$$

with

$$m_{k,n} := m\left(\frac{2\pi i}{\log p}k, \frac{2\pi i}{\log q}n\right) = \begin{cases} 1 & \text{if } k, n \geqslant 0, \\ -1 & \text{if } k, n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \cong s \frac{\displaystyle\prod_{k,n=0}^{\infty}{}' P_2\left(\frac{s}{(2\pi i/\log p)k + (2\pi i/\log q)n}\right)}{\displaystyle\prod_{k,n=1}^{\infty} P_2\left(-\frac{s}{(2\pi i/\log p)k + (2\pi i/\log q)n}\right)}.$$

We use the $r = 2$ case of the formula of Proposition 2.4 from [KK03a]:

$$S_2(z, (\omega_1, \omega_2)) \cong z \frac{\prod_{k,n=0}^{\infty}{}' P_2(-z/(\omega_1 k + \omega_2 n))}{\prod_{k,n=1}^{\infty} P_2(z/(\omega_1 k + \omega_2 n))}.$$

Putting $z = is$ and $\underline{\omega} = (\omega_1, \omega_2) = (2\pi/\log p, 2\pi/\log q)$, we reach the proposition. □

Proof of Theorem 1. We take $(\omega_1, \omega_2) = (2\pi/\log p, 2\pi/\log q)$ in Theorem 2. As remarked in § 1, $\omega_1/\omega_2 = (\log q)/(\log p)$ is generic, since p and q are distinct primes. Thus Proposition 4.1 gives the assertion of Theorem 1. □

Remark 4.2. By the same proof, Theorem 1 is also valid when p and q are powers of distinct primes.

5. The product $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p)$

In this section we treat $\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p)$ for a prime number p.

Theorem 4. *Let p be a prime number. Then for* $\mathrm{Re}(s) > 0$ *we have*

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p) \cong (1 - p^{-s})^{1 - is \log p/2\pi} \exp\left(-\frac{\mathrm{Li}_2(p^{-s})}{2\pi i}\right). \tag{5.1}$$

We prove Theorem 4 from the following result.

Theorem 5. *Let $\omega > 0$. Then for* $\mathrm{Im}(z) > 0$ *we have*

$$S_2(z, (\omega, \omega)) = \exp\left(-\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i z/\omega}) + \pi i\left(\frac{z^2}{2\omega^2} - \frac{z}{\omega} + \frac{5}{12}\right)\right)(1 - e^{2\pi i z/\omega})^{1 - z/\omega}.$$

Proof. We recall the formulae of the double sine function:

$$\begin{aligned} S_2(z, (\omega, \omega)) &= S_2\left(\frac{z}{\omega}, (1, 1)\right) && \text{[KK03a, Theorem 2.1(c)]} \\ &= \mathcal{S}_2\left(\frac{z}{\omega}\right)^{-1} \mathcal{S}_1\left(\frac{z}{\omega}\right) && \text{[KK03a, Example 3.6].} \end{aligned}$$

Here $\mathcal{S}_r(z)$ $(r = 1, 2)$ are the primitive multiple sine functions [KK03a]. We have by definition

$$\mathcal{S}_1(z) = 2 \sin \pi z$$

and the expression [KK03a, Theorem 2.8, (2.12)]

$$\mathcal{S}_2(z) = \exp\left(\frac{1}{2\pi i}\mathrm{Li}_2(e^{2\pi i z}) + z \log(1 - e^{2\pi i z}) - \frac{\pi i}{2} z^2 + \frac{\pi i}{12}\right)$$

for $\mathrm{Im}(z) > 0$. Then an easy calculation gives Theorem 5. □

Remark 5.1. As in Remark 3.4, we see that

$$\pi i\left(\frac{z^2}{2\omega^2} - \frac{z}{\omega} + \frac{5}{12}\right) = \pi i \zeta_2(0, z, (\omega, \omega))$$

is again a multiple Bernoulli polynomial.

Proof of Theorem 4. Let $z = is$ and $\omega = 2\pi/\log p$ in Theorem 5. □

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Remark 5.2. Theorems 4 and 5 can alternatively be proved in the same manner as Theorems 1 and 2. The Euler product (5.1) is deeply related to the generalized Kummer's formula for the double gamma function obtained in [KK03c].

Remark 5.3. For the case of powers of the same prime p, the calculation of the tensor product $\zeta(s, \mathbf{F}_{p^k}) \otimes \zeta(s, \mathbf{F}_{p^l})$, a generalization of Theorem 4, is described in [KK03b].

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