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ON THE ADDITIVE GROUPS OF SUBDIRECTLY IRREDUCIBLE RINGS

YASUYUKI HIRANO AND ISAO MOGAMI

In this paper we study the additive group structure of subdirectly irreducible rings and their hearts. We give an example of a torsion-free, non-reduced abelian group which is not the underlying additive group of any associative subdirectly irreducible ring. It is a counterexample to a theorem in Feigelstock's book "Additive Groups of Rings."

1. Introduction.

In [1] and [2], S. Feigelstock studied the additive group structure of subdirectly irreducible rings, and the results of these papers are collected in § 4, Chapter 4 of [3]. Moreover, in [4], Feigelstock proved that if R is a commutative subdirectly irreducible ring with heart S, then either R is a field or $S^2 = 0$. In this paper, first of all, we attempt to extend this result to certain classes of non-commutative rings. That is, we shall prove that a subdirectly irreducible PI-ring (respectively a one-sided duo ring) with square-nonzero heart is a simple Artinian ring (respectively a division ring). However, an example in

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Jacobson's book [6] shows that a subdirectly irreducible ring with square-nonzero heart need not be simple in general. As a corollary, we determine the additive group structure of those rings. Next we shall give a complete classification of the additive groups of the hearts of subdirectly irreducible rings. Using it we shall also give a necessary condition for an abelian group G to be the additive group of some subdirectly irreducible ring with square-nonzero heart. Finally, we shall give a counterexample to Feigelstock [3, Theorem 4.4.4] : A torsion-free abelian group G is the additive group of some associative subdirectly irreducible ring if and only if G is not reduced. But, in general, the if part of this assertion is not true. In fact, the group $Q^+ \oplus Z$ is a counterexample to the above assertion. Therefore the classification of the torsion-free additive groups of associative subdirectly irreducible rings is not complete.

2. Notation and terminology.

Throughout this paper, R will denote an associative ring (possibly without 1). The additive group of R is denoted by R^+, in particular, Q^+ means the full rational group. The quasi-cyclic (that is p-Prüfer) group will be denoted by $Z(p^\infty)$, and a cyclic group of order m, by $Z(m)$. A ring R is said to be subdirectly irreducible if the intersection S of all its nonzero ideals is not 0. In this case, the ideal S is called the heart of R. We say that R is a *PI*-ring if R satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible. A ring R is called right duo (respectively left duo) if every right (respectively left) ideal of R is two-sided. A ring R is called a finite-chain ring if the lattice of ideals of R forms a finite chain. Following Feigelstock [3], an abelian group G is said to be a subdirectly irreducible ring group if there exists an (associative) subdirectly irreducible ring R such that $R^+ = G$.

3. Main results.

We begin our study by proving the following theorem which is a generalization of Feigelstock [4, Theorem 4].

THEOREM 1. *Let R be a subdirectly irreducible ring with square-nonzero heart S. Then the following hold:*

(1) If R is a PI-ring, then R is a simple Artinian ring.

(2) If S is a right (or left) duo ring, then R is a division ring. In particular, if R is a right (or left) duo ring, then R is a division ring.

Proof. First we claim that in general, the square-nonzero heart S is a simple ring. If the left annihilator $\ell(S) = \{r \in R \mid rs = 0$ for all $s \in S\}$ of S is a nonzero ideal, then $S \subseteq \ell(S)$. This implies that $S^2 \subseteq \ell(S)S = 0$, which contradicts our assumption. Therefore we have $\ell(S) = 0$. Similarly we have the right annihilator $r(S) = 0$. Thus, for any nonzero $a \in S$, SaS is a nonzero ideal of R. Since R is subdirectly irreducible, we conclude $SaS = S$. This means that S is a simple ring.

(1) If R satisfies a polynomial identity, then by [8, Theorem 2 and Addendum], S has a non-trivial centre C. Since S is simple, C must be a field. Thus S has an identity element e. Let r be an arbitrary element of R. Then we have $S(r-er) = 0$. Since $r(S) = 0$, we have $r = er \in S$ for all $r \in R$. Therefore R equals S, and hence R is a simple ring with identity. By Kaplansky's theorem [7, Theorem 1], R is finite dimensional over its centre. In particular, R is Artinian.

(2) Since the heart S is a simple right (or left) duo ring, it is easy to see that S is a division ring. Now, the rest of the proof proceeds in the same way as in the proof of (1).

As a direct consequence of Theorem 1, we obtain the following

COROLLARY 2. *Let R be a PI-ring or a one-sided duo ring. If R is a finite-chain ring, then every proper ideal of R is nilpotent.*

Proof. Let $R \supsetneq J_0 \supsetneq J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_n \supsetneq 0$ be the lattice of ideals of R. It suffices to show that J_0 is nilpotent. If $J_n^2 \neq 0$, then by Theorem 1, R is a simple ring. This is a contradiction. Hence we have $J_n^2 = 0$. Similarly we have $J_i^2 \subseteq J_{i+1}$ for all $0 \leq i \leq n-1$. Therefore we conclude that J_0 is nilpotent.

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The following example shows that if R is not a PI-ring or a one-sided duo ring, then the assertions of Theorem 1 and Corollary 2 need not be true in general.

EXAMPLE 3. Let M be an $\aleph_n$-dimensional vector space over some field D. We consider the complete ring R of linear transformations in M. Let e be an infinite cardinal number not exceeding $\aleph_n$ and let L_e denote the totality of linear transformations in M of rank less than e. Then, by Jacobson [6, Theorem 1, p. 93], the ideals $R \supsetneqq L_{\aleph_n} \supsetneqq L_{\aleph_{n-1}} \supsetneqq \cdots \supsetneqq L_{\aleph_0} \supsetneqq 0$ are the only ideals. As is well known, R is a von Neumann regular ring, and so every ideal of R is idempotent.

The following is a generalization of [4, Corollary 5].

COROLLARY 4. *Let R be a subdirectly irreducible ring with square-nonzero heart S. If R is a PI-ring or a one-sided duo ring, then either $R^+ = \bigoplus_\alpha Q^+$ or $R^+ = \bigoplus_\beta Z(p)$, where p is some prime, α and β are some cardinal numbers.*

Proof. By Theorem 1 R is a simple Artinian ring. Hence the centre C of R is a field. Thus our assertion follows from the fact that R is a vector space over the prime field of C.

Next we shall determine the additive groups of the hearts of subdirectly irreducible rings.

THEOREM 5. *Let G be an abelian group. Then the following are equivalent:*

(1) G is the additive group of the heart of some subdirectly irreducible ring.

(2) Either $G = \bigoplus_\alpha Q^+$ or $G = \bigoplus_\beta Z(p)$, where p is some prime, α and β are some cardinal numbers.

Proof. (1) $\Rightarrow$ (2). Let S be the heart of a subdirectly irreducible ring. Let p be a prime. Then either $pS = 0$ or $pS = S$. If $pS = 0$ for some prime p, then S is a vector space over the field $GF(p)$, and hence $S^+ = \bigoplus_\beta Z(p)$ for some cardinal β. On the other hand if $pS = S$ for all primes p, then S^+ is divisible. Hence, by [5, Theorem 23.1] we have the decomposition $S^+ = \bigoplus_\alpha Q^+ \oplus_{p \text{ a prime}} \bigoplus [\bigoplus_{\alpha_p} Z(p^\infty)]$ for some cardinal numbers α, α_p. Let n be an arbitrary nonzero integer. If the ideal $T_n = \{a \in R \mid na = 0\} \neq 0$, then $S \subseteq T_n$, and so $nS = 0$. But this contradicts the divisibility of S^+. Thus, S^+ is torsion-free in this case. Therefore we conclude that $S^+ = \bigoplus_\alpha Q^+$ for some cardinal α.

(2) ==> (1). If $G = \bigoplus_\alpha Q^+$, then G is the additive group of a field extension of degree α of Q. On the other hand, if $G = \bigoplus_\beta Z(p)$, then G is the additive group of a field extension of degree β of the field $GF(p)$. This completes the proof.

Next we shall describe the additive group structure of a subdirectly irreducible ring with square-nonzero heart.

THEOREM 6. *Let R be a subdirectly irreducible ring with square-nonzero heart S. Then either R^+ is torsion-free, non-reduced or $R^+ = \bigoplus_\alpha Z(p)$ for some prime p and some cardinal number α.*

Proof. First we consider the case that R^+ is torsion-free. In this case, by Theorem 5, we have $S^+ = \bigoplus_\beta Q^+$ for some cardinal β. Therefore R^+ is not reduced.

Next, assume that R^+ is not torsion-free. Then, clearly, the heart is contained in the ideal consisting of all torsion elements in R. Again by Theorem 5 we obtain $S^+ = \bigoplus_\gamma Z(p)$ for some prime p and some cardinal γ. Now we claim that $pR = 0$. If $pR \neq 0$, then $S \subseteq pR$, and so $S^2 \subseteq S(pR) = 0$. This is a contradiction. Thus R can be considered as a vector space over the field $GF(p)$, and hence $R^+ = \bigoplus_\alpha Z(p)$ for some cardinal α.

If an abelian additive group G has the form $\bigoplus_\alpha Z(p)$ for some prime p and some cardinal number α, then G is the additive group of a field extension of degree α of the field $GF(p)$, and so G is an associative aubdirectly irreducible ring group. In [3, Theorem 4.4.4], Feigelstock asserted that every torsion-free, non-reduced abelian group is an associative subdirectly irreducible ring group. But the following example shows that Feigelstock's assertion is not valid in general.

EXAMPLE 7. The additive group $Q^+ \oplus Z$ is not an associative subdirectly irreducible ring group.

Proof. Suppose, to the contrary, that there exists an associative subdirectly irreducible ring R such that $R^+ = Q^+ \oplus Z$. Let S be the heart of R . Then we conclude that $S^+ = Q^+(1,0)$, by Theorem 5. Now put $e = (1,0)$ and $f = (0,1)$. Since $S = Qe$ is an ideal of R , we have $e^2 = q_1 e$, $ef = q_2 e$ and $fe = q_3 e$ for some $q_1, q_2, q_3 \in Q$. By the associativity of R , we have $q_1 q_2 e = (ef)e = e(fe) = q_1 q_3 e$. If $q_1 \neq 0$, then we obtain $q_2 = q_3$. In this case, R is a commutative subdirectly irreducible ring with square-nonzero heart S . But this is impossible by Corollary 4. Hence, we have $e^2 = 0$, and so $Z(q_2,q_3)e$ is a nonzero ideal of R , where $Z(q_2,q_3)$ denotes the subring of Q , generated by q_2 and q_3 . Evidently, the ideal $Z(q_2,q_3)e$ does not contain S . This contradicts the fact that S is the heart of R . This completes the proof.

In view of Example 7, it seems that there is not a torsion-free, non-divisible, subdirectly irreducible ring group. However we have the following example. Therefore the following remains as an open problem: Characterize the torsion-free abelian groups which occur as the additive groups of associative subdirectly irreducible rings.

EXAMPLE 8. Again we consider the ring R in Example 3, and we put $D = Q$, $L_{\aleph_0} = L$. Then clearly L is a vector space over Q . Hence, $L^+ = \oplus_\alpha Q^+$ for some cardinal α . Let T be the subring of R generated by L and 1. We claim that T is a subdirectly irreducible ring such that $T^+ = [\oplus_\alpha Q^+] \oplus Z$ for some cardinal α . Let x be an arbitrary nonzero element of T. By the first part of the proof of Theorem 1, we see that L is a simple ring with $\ell(L) = r(L) = 0$. Hence $LxL \neq 0$, and so $LxL = L$. This shows that L is the heart of T. It is easy to see that $T^+ = [\bigoplus_\alpha Q^+] \oplus Z$ for some cardinal α.

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Department of Mathematics
Okayama University
Okayama 700
JAPAN

Tsuyama College of Technology
Tsuyama 708
and
Department of Mathematics
Okayama University
Okayama 700
JAPAN

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