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MULTIPLICATION OF OPERATORS BY C^∞ FUNCTIONS

BY

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Let $\mathscr{D}_+$ (respectively, $\mathscr{D}_-$) denote the set of all complex-valued infinitely differentiable functions defined on the reals whose supports are bounded on the left (respectively, right). Under addition and convolution $\mathscr{D}_+$ is a commutative algebra. We define P to be the family of all the operators A mapping $\mathscr{D}_+$ into itself such that

$$A(\phi*\psi) = \phi*A(\psi) \quad (\text{all } \phi, \psi \in \mathscr{D}_+).$$

We may make P into a vector space by defining addition and scalar multiplication in the usual way. In [3] it is shown that P is algebraically isomorphic to the space $\mathscr{D}'_+$ of distributions having left-bounded support; the operator $\{F\}$ corresponding to any $F \in \mathscr{D}'_+$ is given by

$$\{F\}(\phi) = F*\phi \quad (\text{all } \phi \in \mathscr{D}_+).$$

Consequently, each $A \in P$ is linear.

The space P is sequentially complete when endowed with a locally convex topology defined simply in terms of the ordinary pointwise convergence of functions. We define the product $\alpha \cdot A$ of an infinitely differentiable function α and an element A of P, and, using the sequential completeness, show that $\alpha \cdot A$ belongs to P. If A is the operator defined by a locally integrable function f then $\alpha \cdot A$ is simply the operator of the pointwise product αf of the functions α and f. The differentiation operator D belongs to P. We show that multiplication by $-t$, where t is the real variable, corresponds to the algebraic derivative, that is, to differentiation with respect to D.

We shall use the following notation: If f is any function defined on the reals and t is any real number, we denote by f_t the function defined by

$$f_t(\tau) = f(t-\tau) \quad (-\infty < \tau < \infty).$$

Then $F*\phi(t) = \langle F, \phi_t \rangle$ for all $F \in \mathscr{D}'_+$, all real t and all $\phi \in \mathscr{D}_+$. For any $\phi \in \mathscr{D}_+$ and any real number t the equation

$$\rho_{\phi,t}(A) = |A(\phi)(t)| \quad (A \in P)$$

defines a seminorm $\rho_{\phi,t}$ on the space P. We endow P with the locally convex topology defined by the family of seminorms $\{\rho_{\phi,t} : \phi \in \mathscr{D}_+, -\infty < t < \infty\}$. It is clear that if $A_n (n=0, 1, 2, \ldots)$ is a sequence in P then $A_0 = \lim A_n$ (as $n \to \infty$) if and only if $A_0(\phi)(t) = \lim A_n(\phi)(t)$ for all $\phi \in \mathscr{D}_+$ and all real t.

THEOREM 1. *The space P is sequentially complete.*

If α is any infinitely differentiable function and if $A \in P$ we denote by $\alpha \cdot A$ the rule which assigns to any $\phi \in \mathscr{D}_+$ the function $(\alpha \cdot A)(\phi)$ defined as follows:

$$(\alpha \cdot A)(\phi)(t) = A(\alpha_t \phi)(t) \quad (-\infty < t < \infty).$$

THEOREM 2. *If f is a locally integrable function then* $\alpha \cdot \{f\} = \{\alpha f\}$ *for all infinitely differentiable functions* α.

Proof. For $\phi \in \mathscr{D}_+$ and $-\infty < t < \infty$,

$$\begin{aligned}(\alpha \cdot \{f\})(\phi)(t) &= \{f\}(\alpha_t \phi)(t) \\ &= f^*(\alpha_t \phi)(t) \\ &= \int_{-\infty}^{\infty} f(t-u)\alpha_t(u)\phi(u)\, du \\ &= \int_{-\infty}^{\infty} f(t-u)\alpha(t-u)\phi(u)\, du \\ &= (\alpha f)^*\phi(t) \\ &= \{\alpha f\}(\phi)(t)\end{aligned}$$

THEOREM 3. *If A belongs to P then* $\alpha \cdot A$ *belongs to P for all infinitely differentiable functions* α.

Proof. Let $\delta_n (n=1, 2, \ldots)$ be a "δ-sequence" in $\mathscr{D}_+$. Then

$$A(\psi)(t) = \lim_{n \to \infty} \delta_n^* A(\psi)(t) = \lim_{n \to \infty} A(\delta_n)^* \psi(t)$$

for all $\psi \in \mathscr{D}_+$ and all real t. Therefore the equation

$$A(\alpha_t \phi)(t) = \lim_{n \to \infty} \{A(\delta_n)\}(\alpha_t \phi)(t) = \lim_{n \to \infty} (\alpha \cdot \{A(\delta_n)\})(\phi)(t)$$

holds for all $\phi \in \mathscr{D}_+$ and all real t. But $\alpha \cdot \{A(\delta_n)\} \in P$ by Theorem 2; the conclusion $\alpha \cdot A \in P$ is then a consequence of Theorem 1.

DEFINITION. Let γ be the function defined by $\gamma(\tau) = -\tau$. For each A in P we define $A' = \gamma \cdot A$.

LEMMA. *The equation*

$$T'(\phi)(t) = \gamma(t)T(\phi)(t) - T(\gamma\phi)(t) \quad (-\infty < t < \infty, \phi \in \mathscr{D}_+)$$

holds for all $T \in P$.

Proof. Since

$$\gamma_t(\tau)\phi(\tau) = -(t-\tau)\phi(\tau) = \gamma(t)\phi(\tau) - \gamma(\tau)\phi(\tau) \quad (\text{all } \tau)$$

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we may write $\gamma_t\phi=\gamma(t)\phi-\gamma\phi$. Consequently,

$$\begin{aligned} T'(\phi)(t) &= T(\gamma_t\phi)(t) \\ &= T(\gamma(t)\phi-\gamma\phi)(t) \\ &= \gamma(t)T(\phi)(t)-T(\gamma\phi)(t) \end{aligned}$$

for all $T\in P$.

We may make P into a commutative algebra by defining AB to be the composition of the operator A with the operator B.

THEOREM 4. *The equation* $(AB)'=A'B+AB'$ *holds for all* A *and* B *in* P.

Proof. By the lemma,

$$\begin{aligned} (AB)'(\phi) &= \gamma(A(B(\phi)))-A(B(\gamma\phi)) \\ &= \gamma(A(B(\phi)))-A(\gamma(B(\phi)))+A(\gamma(B(\phi))-B(\gamma\phi)) \\ &= A'(B(\phi))+A(B'(\phi)) \\ &= (A'B)(\phi)+(AB')(\phi) \end{aligned}$$

for all $\phi\in\mathscr{D}_+$.

COROLLARY. *The equation* $(D^n)'=nD^{n-1}$ *holds for all positive integers* n.

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