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Parametric Representation of Univalent Mappings in Several Complex Variables

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Abstract. Let B be the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm. We prove that the analog of the Carathéodory set, *i.e.* the set of normalized holomorphic mappings from B into $\mathbb{C}^n$ of "positive real part", is compact. This leads to improvements in the existence theorems for the Loewner differential equation in several complex variables. We investigate a subset of the normalized biholomorphic mappings of B which arises in the study of the Loewner equation, namely the set $S^0(B)$ of mappings which have parametric representation. For the case of the unit polydisc these mappings were studied by Poreda, and on the Euclidean unit ball they were studied by Kohr. As in Kohr's work, we consider subsets of $S^0(B)$ obtained by placing restrictions on the mapping from the Carathéodory set which occurs in the Loewner equation. We obtain growth and covering theorems for these subsets of $S^0(B)$ as well as coefficient estimates, and consider various examples. Also we shall see that in higher dimensions there exist mappings in $S(B)$ which can be imbedded in Loewner chains, but which do not have parametric representation.

1 Loewner Chains and Biholomorphic Mappings in $\mathbb{C}^n$

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)'$, equipped with an arbitrary norm $\|\cdot\|$. The symbol $'$ means the transpose of vectors and matrices. Let $B_r = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B = B_1$. In the case of one complex variable B_r is denoted by U_r and U_1 by U. If $G \subset \mathbb{C}^n$ is an open set, let $H(G)$ denote the set of holomorphic mappings from G into $\mathbb{C}^n$. If $f \in H(B_r)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. Let $S(B_r)$ be the set of normalized biholomorphic mappings in $H(B_r)$. The set of normalized convex (respectively starlike) mappings of B_r is denoted by $K(B_r)$ (respectively $S^*(B_r)$). When $n = 1$, the sets $S(U)$, $S^*(U)$ and $K(U)$ are denoted by S, S^* and K respectively.

By $L(\mathbb{C}^n, \mathbb{C}^m)$ we denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I denote the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. For each $z \in \mathbb{C}^n \setminus \{0\}$, let $T(z) = \{l_z \in L(\mathbb{C}^n, \mathbb{C}) : l_z(z) = \|z\|, \|l_z\| = 1\}$. This set is nonempty, by the Hahn-Banach theorem. It is used to define mappings of B of "positive real part". When $\|\cdot\|$ is the Euclidean norm, $T(z)$ reduces to only one element given by $l_z(w) = \langle w, z/\|z\|\rangle$, $w \in \mathbb{C}^n$, for any $z \in \mathbb{C}^n \setminus \{0\}$, where $\langle u, v\rangle = \sum_{j=1}^n u_j \bar{v}_j$ is the Euclidean inner product of $\mathbb{C}^n$.

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We recall that a mapping $F\colon B \times [0,\infty) \to \mathbb{C}^n$ is called a *Loewner chain* if $F(\cdot,t)$ is biholomorphic on B, $F(0,t) = 0$, $DF(0,t) = e^t I$ for $t \geq 0$ and

$$F(z,s) \prec F(z,t), \quad z \in B,\ 0 \leq s \leq t < \infty,$$

where the symbol $\prec$ means the usual subordination.

It is known that starlikeness can be characterized in terms of Loewner chains: $f \in S^*(B)$ iff $f(z,t) = e^t f(z)$, $z \in B$, $t \geq 0$, is a Loewner chain [Pf-Su1]. For the analytical characterization of starlikeness the reader may consult [Su1] and [Su2].

A locally biholomorphic mapping $f \in H(B)$, normalized by $f(0) = 0$ and $Df(0) = I$, is called *close-to-starlike* if there is a mapping $g \in S^*(B)$ such that

$$\operatorname{Re} l_z\big([Df(z)]^{-1} g(z)\big) > 0,$$

for all $z \in B \setminus \{0\}$ and $l_z \in T(z)$.

Let $C(B)$ denote the set of close-to-starlike mappings on B. Pfaltzgraff and Suffridge [Pf-Su1] showed that close-to-starlikeness can be also characterized in terms of Loewner chains: $f \in C(B)$ iff there is a mapping $g \in S^*(B)$ such that

$$f(z,t) = f(z) + (e^t - 1)g(z), \quad z \in B,\ t \geq 0,$$

is a Loewner chain.

Next we recall the notion of spirallikeness, due to Gurganus [Gu] on the unit Euclidean ball of $\mathbb{C}^n$, and Suffridge [Su3] on the unit ball of a complex Banach space. If $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ we define

$$m(A) = \inf\big\{ \operatorname{Re} l_z\big(A(z)\big) : z \in \mathbb{C}^n, \|z\| = 1, l_z \in T(z) \big\}.$$

Let A be such that $m(A) > 0$. A normalized biholomorphic mapping $f \in H(B)$ is called *spirallike relative to A* if $f(B)$ is a spirallike domain with respect to A, that is

$$e^{-sA} f(B) \subset f(B), \quad s \geq 0,$$

where

$$e^{-sA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} s^k A^k.$$

Suffridge [Su3] showed that if f is a normalized locally biholomorphic mapping on B, then f is spirallike relative to a linear operator A with $m(A) > 0$ if and only if

$$\operatorname{Re} l_z\big([Df(z)]^{-1} A f(z)\big) > 0, \quad z \in B \setminus \{0\},$$

for each $l_z \in T(z)$.

If $A = e^{-i\alpha} I$, where $\alpha \in \mathbb{R}$, $|\alpha| < \pi/2$, and f is spirallike relative to A, we say that f is spirallike of type α ([Ha-Ko1]).

Hamada and Kohr [Ha-Ko1] showed that spirallikeness of type α has the following characterization in terms of Loewner chains: f is spirallike of type α if and only if

$$f(z,t) = e^{(1-ia)t} f(e^{iat} z), \quad z \in B,\ t \geq 0,$$

is a Loewner chain, where $a = \tan\alpha$. However, it is known that in general a spirallike mapping relative to an operator A cannot be "nicely" imbedded in a Loewner chain on the unit ball B of $\mathbb{C}^n$ with $n \geq 2$ [Ha-Ko1]. Let $\hat{S}_\alpha(B)$ denote the set of spirallike mappings of type α on B.

The set $\mathcal{M}$ of normalized mappings of "positive real part" on B plays a fundamental role in the study of the Loewner differential equation. As in [Ko3], we shall introduce various subsets of $\mathcal{M}$, and consider properties of the corresponding solutions of the Loewner differential equation.

Definition 1.1 Let $g\colon U \to \mathbb{C}$ be a holomorphic univalent function such that $g(0) = 1$, $g(\bar{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$ (so, g has real coefficients in its power series expansion), $\operatorname{Re} g(\zeta) > 0$ on U, and assume g satisfies the following conditions

$$\begin{cases} \min_{|\zeta|=r} \operatorname{Re} g(\zeta) = \min\{g(r), g(-r)\} \\ \max_{|\zeta|=r} \operatorname{Re} g(\zeta) = \max\{g(r), g(-r)\}, \end{cases} \tag{1.1}$$

for $r \in (0,1)$. For example, the assumption (1.1) is satisfied by all functions which are convex in the direction of the imaginary axis and symmetric about the real axis (see [He-Sh]).

Let

$$\mathcal{M}_g = \Big\{ p \in H(B) : p(0) = 0, Dp(0) = I, \\ \frac{1}{\|z\|} l_z\big(p(z)\big) \in g(U), z \in B \setminus \{0\}, l_z \in T(z) \Big\}.$$

Note that, if $l_z \in T(z)$ then

$$\frac{1}{\|z\|} l_z\big(p(z)\big) = 1 + O(\|z\|) \to 1 \quad \text{as } z \to 0.$$

For $g(\zeta) = (1+\zeta)/(1-\zeta)$, $\zeta \in U$, we obtain the well known set $\mathcal{M}_g = \mathcal{M}$ of mappings with "positive real part on B", *i.e.*

$$\mathcal{M} = \big\{ p \in H(B) : p(0) = 0, Dp(0) = I, \operatorname{Re} l_z\big(p(z)\big) > 0, z \in B \setminus \{0\}, l_z \in T(z) \big\}.$$

We first establish coefficient estimates and upper bounds for the growth of mappings in $\mathcal{M}$ (compare with [Ko1]). These lead to the conclusion that $\mathcal{M}$ is compact.

Theorem 1.2 *Let $p \in \mathcal{M}$ and $P_m = \frac{1}{m!} D^m p(0)$ for $m \geq 1$. Then the following assertions hold:*

(i) $\big| l_z\big(P_m(z)\big)\big| \leq 2$ *for* $m \geq 2$, $\|z\| = 1$ *and* $l_z \in T(z)$. *These bounds are sharp when* B *is the unit ball of* $\mathbb{C}^n$ *with respect to a* p*-norm,* $1 \leq p \leq \infty$.

(ii) $\|P_m(z)\| \leq 2k_m$ *for* $m \geq 2$ *and* $\|z\| = 1$, *where* $k_m = m^{m/(m-1)}$.

(iii) *for each* $r \in (0, 1)$ *there is a constant* $M = M(r)$, *which is independent of* p, *such that* $\|p(z)\| \leq M(r)$ *for* $\|z\| \leq r$.

In fact $M(r) \leq \frac{4r}{(1-r)^2}$.

Proof Fix $z \in \mathbb{C}^n$ with $\|z\| = 1$ and $l_z \in T(z)$. Let $q(\zeta) = \frac{1}{\zeta} l_z\big(p(\zeta z)\big)$ for $\zeta \in U \setminus \{0\}$ and $q(0) = 1$. Then q is holomorphic on the unit disc and $\operatorname{Re} q(\zeta) > 0$ on U. Indeed, since there is a one-to-one correspondence between $T(\alpha z)$ and $T(z)$ given by $l_{\alpha z}(\cdot) = \frac{|\alpha|}{\alpha} l_z(\cdot)$, for each $\alpha \in \mathbb{C} \setminus \{0\}$, we obtain for $\zeta \in U \setminus \{0\}$ that

$$\operatorname{Re} q(\zeta) = \operatorname{Re} \frac{1}{\zeta} l_z\big(p(\zeta z)\big) = \frac{1}{|\zeta|} l_{\zeta z}\big(p(\zeta z)\big) > 0.$$

Thus q belongs to the Carathéodory class and hence

$$\Big|\frac{1}{m!} q^{(m)}(0)\Big| \leq 2, \quad m \geq 2$$

(see *e.g.* [Po]). On the other hand, using the Taylor expansion of q, we obtain

$$\begin{aligned} q(\zeta) &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} q^{(m)}(0)\zeta^m \\ &= 1 + \sum_{m=2}^{\infty} l_z\Big(\frac{1}{m!} D^m p(0)(z^m)\Big)\,\zeta^{m-1}. \end{aligned}$$

Consequently, identifying the coefficients in the power series expansions, we deduce that

$$q^{(m-1)}(0) = \frac{1}{m} l_z\big(D^m p(0)(z^m)\big), \quad m \geq 2.$$

The desired bounds in (i) now follow. In order to see that these bounds are sharp on the unit ball of $\mathbb{C}^n$ with respect to a p-norm, $1 \leq p \leq \infty$, let $|\lambda| = 1$ and

$$p(z) = \Big(z_1 \frac{1 + \lambda z_1}{1 - \lambda z_1}, \ldots, z_n \frac{1 + \lambda z_n}{1 - \lambda z_n}\Big)',$$

for $z = (z_1, \ldots, z_n)' \in B$. Then $p \in H(B)$, $p(0) = 0$, $Dp(0) = I$ and using the expression of $T(z)$ in [Su2, Section 2] we can show that $\operatorname{Re} l_z\big(p(z)\big) > 0$ for all $z \in B \setminus \{0\}$ and $l_z \in T(z)$.

We next prove the second statement. Our proof depends on an inequality given in [Har] (also see [Har-Re-Sh]) which uses the numerical radius of a homogeneous polynomial mapping to estimate the norm. Harris' result is that if $P_m \colon \mathbb{C}^n \to \mathbb{C}^n$ is a homogeneous polynomial mapping of degree m, then

$$\|P_m\| \leq k_m |V(P_m)|, \quad m \geq 1,$$

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where $k_m = m^{m/(m-1)}$ when $m > 1$ and $k_1 = e$ [Har, Theorem 1]. In this inequality the norm of P_m is

$$\|P_m\| = \sup\{\|P_m(z)\| : z \in \bar{B}\},$$

and $|V(P_m)|$ is the numerical radius of P_m, that is

$$|V(P_m)| = \lim_{s \to 1-0} \sup\{|\lambda| : \lambda \in V(P_{m,s})\},$$

where $P_{m,s}(z) = P_m(sz)$, $0 < s < 1$, and $V(P_{m,s})$ is the numerical range of $P_{m,s}$, *i.e.*,

$$V(P_{m,s}) = \left\{ l_z\big(P_{m,s}(z)\big) : l_z \in T(z), \|z\| = 1 \right\}.$$

Now, let $P_m = \frac{1}{m!} D^m f(0)$. Then P_m is a homogeneous polynomial of degree m and thus we obtain

$$\|P_m\| \le k_m |V(P_m)|.$$

Taking into account the first part of the proof and the above relations, we easily deduce that

$$|V(P_m)| \le 2, \quad m \ge 2,$$

and consequently

$$\|P_m\| \le 2k_m, \quad m \ge 2.$$

This completes the proof of (ii). Finally it suffices to apply (ii), to deduce that

$$\begin{aligned} \|p(z)\| &\le r + \sum_{m=2}^{\infty} \|P_m(z)\| \le r\Big(1 + 2\sum_{m=2}^{\infty} k_m r^{m-1}\Big) \\ &\le r\Big(1 + 4\sum_{m=2}^{\infty} m r^{m-1}\Big) = M(r) \le \frac{4r}{(1-r)^2}, \quad \|z\| \le r < 1, \end{aligned}$$

where we have used the fact that $k_m \le 2m$ for $m \ge 2$. This completes the proof.

Corollary 1.3 *The set $\mathcal{M}$ is compact.*

Proof It is clear that $\mathcal{M}$ is a normal family. Suppose that $\{p_k\}_{k \ge 1}$ is a sequence of mappings in $\mathcal{M}$ which converges locally uniformly to $p \in H(B)$. Then $p(0) = 0$, $Dp(0) = I$, and if z is fixed with $\|z\| = 1$ and $l_z \in T(z)$, the function

$$q(\zeta) = \begin{cases} \frac{1}{\zeta} l_z\big(p(\zeta z)\big), & \zeta \ne 0 \\ 1, & \zeta = 0 \end{cases}$$

is holomorphic on U and satisfies $\operatorname{Re} q(\zeta) \ge 0$. But $q(0) = 1$ and hence $\operatorname{Re} q(\zeta) > 0$ for $\zeta \in U$ by the minimum principle for harmonic functions. This implies that $p \in \mathcal{M}$.

The basic existence theorem for the Loewner differential equation on B (see [Pf, Theorem 2.1]) can now be improved by omitting the boundedness assumptions on

$h(z,t)$. This property follows automatically from the fact that $\mathcal{M}$ is compact. We note that for the case of the maximum norm $\|\cdot\|$, it is shown in [Por1, Corollary 1] that

$$\|h(z,t)\| \leq \|z\| \cdot \frac{1+\|z\|}{1-\|z\|}, \quad z \in B,\ t \geq 0.$$

We also explicitly state that the solution is locally Lipschitz in t locally uniformly with respect to $z \in B$ rather than just locally absolutely continuous in t. The Lipschitz property may be used together with a version of Vitali's theorem in several complex variables (Theorem 1.9) to show that the exceptional set of measure 0 in t for which the differential equation is not satisfied is independent of z.

Theorem 1.4 *Let $g\colon U \to \mathbb{C}$ satisfy the assumptions of Definition 1.1 and let $h_t(z) = h(z,t)\colon B \times [0,\infty) \to \mathbb{C}^n$ satisfy the following conditions:*

(i) *for each $t \geq 0$, $h_t(\cdot) \in \mathcal{M}_g$;*
(ii) *for each $z \in B$, $h(z,t)$ is a measurable function of $t \in [0,\infty)$.*

Then the limit

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s) \tag{1.2}$$

exists locally uniformly on B for each $s \geq 0$, where $v = v(z,s,t)$ is the unique solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \quad \text{a.e. } t \geq s,\ v(s) = z. \tag{1.3}$$

The mapping $v(z,s,t) = e^{s-t}z + \cdots$ is a univalent Schwarz mapping on B and it is a locally Lipschitz function of $t \geq s$ locally uniformly with respect to $z \in B$.

Moreover, $f(z,s) = f\big(v(z,s,t),t\big)$, $z \in B$, $0 \leq s \leq t < \infty$ (thus $f(z,s)$ is a Loewner chain), $f(z,\cdot)$ is a locally Lipschitz function on $[0,\infty)$ locally uniformly with respect to $z \in B$, and for a.e. $t \geq 0$,

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \forall z \in B.$$

Proof We have to show that $f(z,t)$ is a locally Lipschitz function of $t \in [0,\infty)$ locally uniformly with respect to $z \in B$, and for almost all $t \geq 0$,

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \forall z \in B.$$

The other conclusions can be obtained directly from [Por3, Theorems 2 and 3] and [Pf, Theorem 2.1]. (It is observed in the proof of [Pf, Theorem 2.1] that $v(z,s,t)$ is a locally Lipschitz function of $t \geq s$ locally uniformly with respect to $z \in B$.) We will use similar arguments to [Por1, Lemma 4].

Fix $z_0 \in B$ and let $0 \le s \le t < \infty$. From (1.3) one deduces that

$$v(z_0, s, t) - z_0 = -\int_s^t h\big(v(z_0, s, \tau), \tau\big)\, d\tau.$$

Since [Pf, Lemma 2.2] holds on the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm, we conclude that

$$\frac{\|v(z,s,t)\|}{\big(1-\|v(z,s,t)\|\big)^2} \le e^{s-t}\frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B. \tag{1.4}$$

On the other hand, from the statement (iii) of Theorem 1.2, we obtain

$$\|v(z_0, s, t) - z_0\| \le M(\|z_0\|)(t-s). \tag{1.5}$$

From (1.2) and (1.4), we deduce that

$$\|f(z,t)\| \le e^t \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B,\ s \ge 0,$$

and using Cauchy's integral formula it is not difficult to prove that for all $T > 0$ and $r \in (0,1)$ there is an $L = L(r,T) > 0$ such that

$$\|f(z_1,t) - f(z_2,t)\| \le L\|z_1 - z_2\|, \quad \|z_1\| \le r,\ \|z_2\| \le r,\ t \in [0,T]. \tag{1.6}$$

Consequently, using the relations (1.5) and (1.6), we obtain

$$\begin{aligned}\|f(z_0,s) - f(z_0,t)\| &= \big\| f\big(v(z_0,s,t),t\big) - f(z_0,t)\big\| \\ &\le L\|v(z_0,s,t) - z_0\| \le LM(t-s) = K(t-s),\end{aligned}$$

for $0 \le s \le t \le T$. Therefore, $f(z_0,\cdot)$ is Lipschitz on $[0,T]$ locally uniformly with respect to z_0, for each $T \ge 0$.

It remains to prove that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \ge 0,\ \forall z \in B.$$

For this purpose it suffices to differentiate both sides of the equality

$$f(z,s) = f\big(v(z,s,t),t\big)$$

with respect to t and to use (1.3), to deduce that

$$\begin{aligned}0 &= Df(w,t)|_{w=v(z,s,t)}\frac{\partial v}{\partial t}(z,s,t) + \frac{\partial f}{\partial t}(w,t)|_{w=v(z,s,t)} \\ &= -Df(w,t)|_{w=v(z,s,t)}h\big(v(z,s,t),t\big) + \frac{\partial f}{\partial t}(w,t)|_{w=v(z,s,t)},\end{aligned}$$

for almost all $t \geq s \geq 0$ and all $z \in B$. This completes the proof.

Taking into account Theorem 1.4 we introduce the following definition (*cf.* [Por1], [Ko3]. Also see [Ha-Ko2]).

Definition 1.5 Let $g\colon U \to \mathbb{C}$ satisfy the assumptions of Definition 1.1. Also let $f \in H(B)$. We say $f \in S_g^0(B)$ if there is a mapping $h\colon B \times [0,\infty) \to \mathbb{C}^n$, satisfying the assumptions from Theorem 1.4, such that

$$\lim_{t\to\infty} e^t v(z,t) = f(z)$$

locally uniformly on B, where $v = v(z,t)$ is the unique solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \quad \text{a.e. } t \geq 0,\ v(z,0) = z,$$

for all $z \in B$.

The set $S_g^0(B)$ is called *the set of mappings which have g-parametric representation on B* (*cf.* [Ko3]). If $g(\zeta) = (1+\zeta)/(1-\zeta)$, we denote the set $S_g^0(B)$ by $S^0(B)$, and we call this latter set *the set of mappings which have parametric representation on B* (*cf.* [Por1], [Ko3], [Ha-Ko2]).

It is clear that $S_g^0(B) \subseteq S^0(B) \subseteq S(B)$ and in the case $n = 1$, $S^0(U) = S$, by [Po, Theorems 6.1 and 6.3]. However, if $n > 1$, $S(B)$ is a larger set than $S^0(B)$. On the other hand, we shall prove that in several complex variables there are mappings which can be imbedded in Loewner chains without having parametric representation (Example 2.12).

In the rest of this paper we shall study certain properties of the set $S_g^0(B)$, in particular growth and covering theorems and bounds of coefficients of mappings in $S_g^0(B)$. These results generalize to the case of the unit ball with respect to an arbitrary norm some results obtained by Poreda [Por1] in the case of the unit polydisc and by Kohr [Ko3] in the case of the Euclidean ball. Also Hamada and Kohr [Ha-Ko2] obtained certain results concerning parametric representation of univalent mappings on bounded balanced convex domains with the Minkowski function of class C^1 in $\mathbb{C}^n \setminus \{0\}$. We shall see that the most important subsets of $S(B)$ are also subsets of $S_g^0(B)$ for certain values of g.

We next give the following result which may be used to generate many examples of mappings in $S_g^0(B)$ (respectively $S^0(B)$). This result is a consequence of [Pf, Theorem 2.3] and [Por3, Theorem 6].

Lemma 1.6 *Let $f\colon B \times [0,\infty) \to \mathbb{C}^n$ be such that $f(\cdot,t) \in H(B)$, $f(0,t) = 0$, $Df(0,t) = e^t I$, for each $t \geq 0$ and $f(z,\cdot)$ is a locally Lipschitz continuous function of $t \in [0,\infty)$ locally uniformly with respect to $z \in B$. Let $g\colon U \to \mathbb{C}$ satisfy the conditions of Definition 1.1 and let $h\colon B \times [0,\infty) \to \mathbb{C}^n$ satisfy the requirements of Theorem 1.4. Suppose that*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geq 0,$$

for all $z \in B$. Further, assume there exists an increasing sequence $\{t_m\}$ such that $t_m > 0$, $t_m \to \infty$ and

$$\lim_{m\to\infty} e^{-t_m} f(z,t_m) = F(z)$$

locally uniformly on B. Then $f(z,t)$ is a Loewner chain and

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$$

locally uniformly on B for each $s \geq 0$, where $v = v(z,s,t)$ is the solution of the initial value problem (1.3). Hence $f(z) = f(z,0) \in S_g^0(B)$.

In connection with the above result we say that a mapping $f(z,t)$ is a g-*Loewner chain* (*cf.* [Ko3]) if f and g satisfy the assumptions of Lemma 1.6. In the case of one variable, if $g(\zeta) = (1+\zeta)/(1-\zeta)$, $\zeta \in U$, a g-Loewner chain is a Loewner chain in the usual sense, by [Po, Theorem 6.2].

Let $S_g^1(B)$ denote the set of those mappings in $S(B)$ which can be imbedded in g-Loewner chains. That is, $f \in S_g^1(B)$ if and only if there is a g-Loewner chain $f(z,t)$ such that $f(z) = f(z,0)$, $z \in B$. When $g(\zeta) = (1+\zeta)/(1-\zeta)$, let $S_g^1(B) = \tilde{S}(B)$. Combining the results of Theorem 1.4 and Lemma 1.6, one concludes that $S_g^0(B) = S_g^1(B)$ and therefore,

$$S^0(B) = \tilde{S}(B) \subseteq S(B).$$

Example 1.7 Let $0 < c \leq 1$ and let $f\colon B \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping on B such that

$$(1+\|z\|)\|[Df(z)]^{-1}D^2 f(z)(z,\cdot)\| \leq 2c, \quad z \in B.$$

Then $f \in S_g^0(B)$, where $g(\zeta) = (1+c\zeta)/(1-c\zeta)$, $\zeta \in U$.

Proof In [Ha-Ko-Li, Theorem 4.2] and [Ha-Ko4, Theorem 3.2] it is shown that if f satisfies the above assumption, then f is starlike on B. Therefore $f(z,t) = e^t f(z)$ is a Loewner chain. Moreover, if $p(z) = [Df(z)]^{-1} f(z) - z$, $z \in B$, then in the proof of [Ha-Ko4, Theorem 3.2] it is proved that $\|p(z)\| \leq c$ for $z \in B$. Since $p \in H(B)$, $p(0) = 0$, it follows in view of the Schwarz lemma that

$$\|p(z)\| \leq c\|z\|, \quad z \in B.$$

Hence, if $h(z,t) = [Df(z)]^{-1} f(z)$ for $z \in B$ and $t \geq 0$, then it is obvious to deduce that

$$\Big|(1-c^2)l_z\big(h(z,t)-z\big) - 2c^2\|z\|\Big| \leq 2c\|z\|, \quad z \in B\setminus\{0\},\ l_z \in T(z),\ t \geq 0.$$

This relation is equivalent to

$$\left|\frac{1}{\|z\|}l_z\big(h(z,t)\big) - \frac{1+c^2}{1-c^2}\right| \leq \frac{2c}{1-c^2}, \quad z \in B\setminus\{0\},\ l_z \in T(z),\ t \geq 0.$$

Therefore we have shown that $h(z,t) \in \mathcal{M}_g$, $z \in B$, $t \geq 0$, where $g(\zeta) = (1+c\zeta)/(1-c\zeta)$. Moreover, since

$$\lim_{t\to\infty} e^{-t} f(z,t) = f(z)$$

locally uniformly on B and since $f(z,t)$ satisfies all assumptions from Lemma 1.6, we conclude that $f(z) = f(z,0) \in S_g^0(B)$, as desired. This completes the proof.

Example 1.8 Let $f\colon B \to \mathbb{C}^n$ be a normalized locally biholomorphic mapping on the unit Euclidean ball B of $\mathbb{C}^n$ such that

$$(1 - \|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\| \le 1, \quad z \in B.$$

Then $f \in S^0(B)$.

Proof Let

$$f(z,t) = f(ze^{-t}) + (e^t - e^{-t})Df(ze^{-t})z, \quad z \in B,\ t \ge 0.$$

In [Pf, Theorem 2.4] it is shown that if f satisfies the above assumption, then $f(z,t)$ is a Loewner chain and thus f is biholomorphic on B. Also if we let

$$E(z,t) = -(1 - e^{-2t})[Df(ze^{-t})]^{-1}D^2f(ze^{-t})(ze^{-t},\cdot), \quad z \in B,\ t \ge 0,$$

then in the proof of [Pf, Theorem 2.4] it is shown that $\|E(z,t)\| \le \|z\|$. Moreover, if

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z), \quad z \in B,\ t \ge 0,$$

then $h(\cdot\,,t) \in H(B)$, $h(0,t) = 0$, $Dh(0,t) = I$ and $\operatorname{Re}\langle h(z,t), z\rangle > 0$ for $z \in B \setminus \{0\}$ and $t \ge 0$. Therefore $h(z,t) \in \mathcal{M}$. Further, since $\lim_{t\to\infty} e^{-t}f(z,t) = z$ locally uniformly on B, one concludes in view of Lemma 1.6 that $f(z) = f(z,0) \in S^0(B)$. This completes the proof.

Pfaltzgraff remarks that the constant 1 in the hypothesis of Example 1.8 must be strengthened to $1/3$ if instead of the Euclidean norm we use an arbitrary norm of $\mathbb{C}^n$ (see [Pf, p. 67]).

We conclude this section with a result which gives a sufficient condition for a Loewner chain to satisfy the generalized Loewner differential equation. The proof makes use of a version of Vitali's theorem in several complex variables (see *e.g.* [Na, p. 9]).

We recall that *a set of uniqueness* for the holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ is a subset Q of Ω with the property that if Φ is a holomorphic function on Ω and $\Phi|_Q = 0$ then $\Phi \equiv 0$. We note that there exist countable sets of uniqueness, for example any countable dense subset of Ω is a set of uniqueness.

Theorem 1.9 **(Vitali's Theorem in Several Complex Variables)** *Let Ω be a domain in $\mathbb{C}^n$ and let $Q \subset \Omega$ be a set of uniqueness for the holomorphic functions on Ω. Suppose that $\{\Phi_k\}_{k\ge 1}$ is a sequence of holomorphic functions on Ω which is locally bounded and which has the property that $\{\Phi_k\}_{k\ge 1}$ converges for all $z \in Q$. Then there exists a holomorphic function Φ on Ω such that $\Phi_k \to \Phi$ locally uniformly on Ω.*

We note that this theorem also can be applied to holomorphic mappings whose target space is $\mathbb{C}^n$.

Theorem 1.10 *Let $f(z,t)$ be a Loewner chain which is locally Lipschitz in t locally uniformly with respect to z. Then there is a mapping $h = h(z,t)$ such that $h(z,t) \in \mathcal{M}$ for each $t \geq 0$, $h(z,t)$ is measurable in t for each $z \in B$, and for a.e. $t \geq 0$,*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \forall z \in B. \tag{1.7}$$

Moreover, if there is a sequence $\{t_m\}$ such that $t_m > 0$, $t_m \to \infty$ and

$$\lim_{m\to\infty} e^{-t_m} f(z,t_m) = F(z)$$

locally uniformly on B, then $f(z) = f(z,0) \in S^0(B)$.

Proof First we prove (1.7). To this end, let $v = v(z,s,t)$ be the transition mapping defined by the chain $f(z,t)$, *i.e.*

$$f(z,s) = f\big(v(z,s,t),t\big), \quad z \in B,\ 0 \leq s \leq t < \infty.$$

Taking into account the normalization of $f(z,t)$, we deduce that $Dv(0,s,t) = e^{s-t}I$ for $t \geq s \geq 0$. By using the subordination property and applying the mean value theorem to the real and imaginary parts of the components of f, we obtain

(1.8)
$$\begin{aligned}\frac{1}{r}[f(z,t+r) - f(z,t)] &= \frac{1}{r}\big[f(z,t+r) - f\big(v(z,t,t+r),t+r\big)\big] \\ &= A(z,t,r)\Big(\frac{1}{r}[z - v(z,t,t+r)]\Big), \quad z \in B,\ t \geq 0,\ r > 0,\end{aligned}$$

where $A(z,t,r)$ is a real-linear operator which tends to the invertible complex linear operator $Df(z,t)$ as $r \to +0$. In view of this we deduce that the difference quotient in the first member of (1.8) has a limit as $r \to +0$ if and only if the same is true of the difference quotient in the last member of (1.8). Since $f(z,t)$ is locally Lipschitz in t locally uniformly with respect to z, the difference quotients on the left-hand side of (1.8) are locally bounded holomorphic functions of z. Let Q be a countable set of uniqueness for the holomorphic functions on B. For each $z \in Q$ the limit as $r \to +0$ of these difference quotients exists except when $t \in E_z$, where E_z is a subset of $[0,\infty)$ of measure 0. The set $E = \bigcup\{E_z : z \in Q\}$ also has measure 0, and Vitali's theorem implies that for $t \notin E$, the difference quotient on the left-hand side of (1.8) has a limit as $r \to +0$ which is holomorphic in z. Moreover, since $v(z,s,t)$ is a Schwarz mapping and $Dv(0,s,t) = e^{s-t}I$, the difference quotient on the right has a limit $h(z,t)$ in $\mathcal{M}$, by [Su2, Lemmas 1 and 3]. The mapping $h(z,t)$ is measurable in $t \in [0,\infty)$ for each $z \in B$, since $\frac{\partial f}{\partial t}(z,t)$ and $[Df(z,t)]^{-1}$ are measurable in t.

Finally, it suffices to apply Lemma 1.6 with $g(\zeta) = (1+\zeta)/(1-\zeta)$, to deduce that $f(z,0) \in S^0(B)$. This completes the proof.

Remark 1.11 In higher dimensions, univalent solutions of the generalized Loewner equation (1.7) need not be unique (*cf.* [Bec2]). For if $f(z,t)$ is a Loewner chain which satisfies the differential equation (1.7), and if $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ is a normalized entire biholomorphic mapping, not the identity, then $g(z,t) = \Phi\big(f(z,t)\big)$ is another Loewner chain satisfying (1.7).

2 Main Results

2.1 Growth Results for Mappings in $S_g^0(B)$

One of the main results of this section is a growth theorem for mappings in $S_g^0(B)$. To this end, we need to use the following lemma (*cf.* [Ko3]).

Lemma 2.1 *Let g and h satisfy the assumptions of Theorem 1.4. If $v = v(z,s,t)$ is the solution of the initial value problem (1.3), then*

$$(2.1)\qquad e^s\|z\| \exp \int_{\|v(z,s,t)\|}^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right]\frac{dx}{x} \\ \leq e^t\|v(z,s,t)\| \leq e^s\|z\| \exp \int_{\|v(z,s,t)\|}^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right]\frac{dx}{x},$$

for $z \in B$ and $t \geq s \geq 0$.

Proof Fix $s \geq 0$ and $z \in B \setminus \{0\}$ and let $v(t) = v(z,s,t)$. Also let $l_z \in T(z)$. Then for all t, t' with $s \leq t < t'$, we have

$$\big|\, \|v(t)\| - \|v(t')\| \,\big| \leq \|v(t) - v(t')\| \leq \left\| \int_t^{t'} \frac{dv(\tau)}{d\tau}\, d\tau \right\| \\ \leq \int_t^{t'} \left\| \frac{dv(\tau)}{d\tau} \right\| d\tau = \int_t^{t'} \big\| -h\big(v(\tau),\tau\big) \big\|\, d\tau \leq M(t'-t),$$

by the statement (iii) of Theorem 1.2. Hence $\|v(t)\|$ is absolutely continuous for $t \in [s,\infty)$ and thus $\|v(t)\|$ is differentiable a.e. on $[s,\infty)$. Moreover,

$$\frac{\partial \|v\|}{\partial t} = \mathrm{Re}\left[l_v\Big(\frac{\partial v}{\partial t}\Big)\right]$$

for $l_v \in T\big(v(t)\big)$ a.e. on $[s,\infty)$, by [Ka, Lemma 1.3]. Equivalently,

$$(2.2)\qquad \frac{\partial \|v\|}{\partial t} = -\,\mathrm{Re}\big[l_v\big(h(v,t)\big)\big], \quad \text{a.e. on } [s,\infty).$$

On the other hand, let $p\colon U \to \mathbb{C}$ be given by

$$p(\zeta) = \begin{cases} \frac{1}{\zeta} l_z\big(h(\zeta\frac{z}{\|z\|}, t)\big), & \zeta \neq 0 \\ 1, & \zeta = 0. \end{cases}$$

Then $p \in H(U)$, $p(0) = g(0) = 1$, and since $h(z,t) \in \mathcal{M}_g$ we deduce that $p(\zeta) \in g(U)$ for $\zeta \in U$. Indeed, since there is a one-to-one correspondence between $T(\alpha z)$ and $T(z)$ given by $l_{\alpha z}(\cdot) = \frac{|\alpha|}{\alpha} l_z(\cdot)$, for each $\alpha \in \mathbb{C} \setminus \{0\}$, one deduces that $l_{z/\|z\|}(\cdot) = l_z(\cdot)$, and thus for $\zeta \in U \setminus \{0\}$ we have

$$\begin{aligned} p(\zeta) &= \frac{1}{\zeta} l_z\Big(h_t\Big(\zeta \frac{z}{\|z\|}\Big)\Big) = \frac{1}{\zeta} l_{z/\|z\|}\Big(h_t\Big(\zeta \frac{z}{\|z\|}\Big)\Big) \\ &= \frac{1}{\| \zeta \frac{z}{\|z\|} \|} l_{\zeta z/\|z\|}\Big(h_t\Big(\zeta \frac{z}{\|z\|}\Big)\Big) \in g(U). \end{aligned}$$

Therefore $p \prec g$, and by the subordination principle we deduce that $p(U_r) \subseteq g(U_r)$ for each r, $0 < r < 1$. Next, in view of the maximum and respectively the minimum principle for harmonic functions, we conclude that

$$\min\{g(|\zeta|), g(-|\zeta|)\} \le \operatorname{Re} p(\zeta) \le \max\{g(|\zeta|), g(-|\zeta|)\}, \quad \zeta \in U.$$

For $\zeta = \|z\|$ in the above relations, we obtain

$$\|z\| \min\{g(\|z\|), g(-\|z\|)\} \le \operatorname{Re} l_z\big(h(z,t)\big) \le \|z\| \max\{g(\|z\|), g(-\|z\|)\}. \tag{2.3}$$

We now integrate in both sides of (2.2) with respect to t and use (2.3), to obtain

$$\begin{aligned} -\int_{\|z\|}^{\|v\|} \frac{dx}{x \min\{g(x), g(-x)\}} &= -\int_s^t \frac{1}{\|v(\tau)\| \min\big\{ g\big(\|v(\tau)\|\big), g\big(-\|v(\tau)\|\big)\big\}} \\ &\quad \cdot \frac{d\|v(\tau)\|}{d\tau}\, d\tau \ge \int_s^t d\tau = t - s \end{aligned}$$

and

$$\begin{aligned} -\int_{\|z\|}^{\|v\|} \frac{dx}{x \max\{g(x), g(-x)\}} &= -\int_s^t \frac{1}{\|v(\tau)\| \max\big\{ g\big(\|v(\tau)\|\big), g\big(-\|v(\tau)\|\big)\big\}} \\ &\quad \cdot \frac{d\|v(\tau)\|}{d\tau}\, d\tau \le \int_s^t d\tau = t - s. \end{aligned}$$

Finally straightforward computations in the above relations yield (2.1), as desired. This completes the proof.

We now are able to obtain the following growth result for the set $S_g^0(B)$. This result generalizes [Ko3, Theorem 2.3].

Theorem 2.2 *Let $g\colon U \to \mathbb{C}$ satisfy the assumptions of Definition 1.1 and $f \in S_g^0(B)$. Then*

$$\begin{aligned} &\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1 \right] \frac{dx}{x} \\ &\qquad \le \|f(z)\| \le \|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1 \right] \frac{dx}{x}, \qquad z \in B. \end{aligned} \tag{2.4}$$

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Proof Since $f \in S_g^0(B)$ we have

$$f(z) = \lim_{t\to\infty} e^t v(z,t) \tag{2.5}$$

locally uniformly on B, where $v = v(z,t)$ is the solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \quad \text{a.e. } t \geq 0,\ v(z,0) = z,$$

for all $z \in B$. Taking into account the relations (2.1), one deduces that

(2.6)
$$\|z\| \exp \int_{\|v(z,t)\|}^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right] \frac{dx}{x}$$
$$\leq e^t \|v(z,t)\| \leq \|z\| \exp \int_{\|v(z,t)\|}^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{dx}{x}, \quad z \in B,\ t \geq 0.$$

Since
$$\lim_{t\to\infty} e^t \|v(z,t)\| = \|f(z)\| < \infty,$$
we must have
$$\lim_{t\to\infty} \|v(z,t)\| = \lim_{t\to\infty} e^{-t} \|e^t v(z,t)\| = 0.$$

Letting $t \to \infty$ in (2.6) and using (2.5), we obtain the estimate (2.4), as desired. This completes the proof.

We remark that if $f(z,t)$ is a g-Loewner chain, then using similar reasoning as in the above result, we obtain the following growth theorem.

Corollary 2.3 *Let $g\colon U \to \mathbb{C}$ satisfy the assumptions of Definition 1.1 and $f(z,t)$ be a g-Loewner chain. Then*

$$\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right] \frac{dx}{x}$$
$$\leq \|e^{-t} f(z,t)\| \leq \|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{dx}{x}, \quad z \in B,\ t \geq 0.$$

A case of particular interest in Theorem 2.2 is the case $g(\zeta) = (1+\zeta)/(1-\zeta)$, $\zeta \in U$. We remark that in view of Lemma 1.6, all mappings in $S^*(B)$, $\hat{S}_\alpha(B)$ and $C(B)$ belong to $S^0(B)$. To see this, it suffices to observe that the Loewner chains which characterize the above sets satisfy the assumptions of Lemma 1.6. Therefore, we have the following inclusion relations

- $S^*(B) \subset C(B) \subset S^0(B)$ and $\hat{S}_\alpha(B) \subset S^0(B), \quad |\alpha| < \dfrac{\pi}{2}.$

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We have the following growth result for the set $S^0(B)$ (*cf.* [Por1], [Ko3]). In particular, the growth result for normalized starlike ([Ba-Fi-Go], [Chu], [Ha2]) and close-to-starlike mappings [Che-Re] can be deduced from Theorem 2.2.

Corollary 2.4 *If $f \in S^0(B)$ then*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in B.$$

Consequently, $f(B) \supseteq B_{1/4}$.

Remark 2.5 We remark that if $B(p)$ denotes the unit ball with respect to a p-norm $\|\cdot\|$, $1 \le p \le \infty$, where

$$\|z\| = \begin{cases} [\sum_{j=1}^n |z_j|^p]^{1/p}, & 1 \le p < \infty \\ \max_{1\le j\le n} |z_j|, & p = \infty, \end{cases}$$

then the result of Corollary 2.4 is sharp. To see this, let

$$f(z) = \left(\frac{z_1}{(1-z_1)^2}, \dots, \frac{z_n}{(1-z_n)^2} \right)', \quad z = (z_1, \dots, z_n)' \in B(p).$$

Then f is normalized starlike on $B(p)$ (see *e.g.* [Ha-Ko2]) and for $z = (r, 0, \dots, 0)' \in B(p)$, with $r \in (0,1)$, $\|z\| = r$ and $\|f(z)\| = \frac{r}{(1-r)^2}$. Moreover, for $z = (-r, 0, \dots, 0)' \in B(p)$, we have $\|z\| = r$ and $\|f(z)\| = \frac{r}{(1+r)^2}$.

On the other hand, from Corollary 2.4 we obtain the following important consequence in higher dimensions.

Corollary 2.6 *$S^0(B)$ is a normal family. Thus in $\mathbb{C}^n$, $n \ge 2$, $S(B)$ is a larger set than $S^0(B)$.*

Actually we believe that $S^0(B)$ is a compact set.

2.2 Examples of Mappings in $S_g^0(B)$

The following result, obtained recently in [Gr-Ha-Ko-Su], gives many examples of mappings in $S^0(B)$ when B is the unit Euclidean ball of $\mathbb{C}^n$. Properties of the operator $\Psi_{n,0,\beta}$ have been recently studied in [Gr-Ko-Ko].

Theorem 2.7 *Let $\alpha \in [0,1]$ and $\beta \in [0,1/2]$ be such that $\alpha + \beta \le 1$. If $f \in S$ then $\Psi_{n,\alpha,\beta}(f) \in S^0(B)$, where*

$$\Psi_{n,\alpha,\beta}(f)(z) = \left(f(z_1), z_2\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\left(f'(z_1)\right)^{\beta}, \dots, z_n\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\left(f'(z_1)\right)^{\beta} \right)',$$

for $z = (z_1, \dots, z_n)' \in B$. The branches of the power functions are chosen such that

$$\left(\frac{f(z_1)}{z_1}\right)^{\alpha}\Big|_{z_1=0} = 1 \quad \text{and} \quad \left(f'(z_1)\right)^{\beta}\Big|_{z_1=0} = 1.$$

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In the case $\alpha = 0$ and $\beta = 1/2$ we obtain the Roper-Suffridge extension operator [Ro-Su1]. Properties of this operator have been investigated in [Ro-Su1], [Gr-Ko].

Next, we consider conditions under which the mapping $F\colon B \to \mathbb{C}^n$ given by $F(z) = P(z)z$ belongs to $S_g^*(B)$, where P is a complex valued holomorphic function with $P(0) = 1$ and $S_g^*(B)$ denotes the subset of $S_g^0(B)$ consisting of those normalized starlike mappings f of B such that

$$\frac{1}{\|z\|} l_z\big(w(z)\big) \in g(U), \quad z \in B \setminus \{0\},\ l_z \in T(z),$$

where $w(z) = [Df(z)]^{-1} f(z)$, $z \in B$.

We have the following result:

Theorem 2.8 *Let $P\colon B \to \mathbb{C}$ be a holomorphic function on B such that $P(0) = 1$ and let $F(z) = P(z)z$, $z \in B$. Also let $g\colon U \to \mathbb{C}$ satisfy the requirements of Definition 1.1. Then $F \in S_g^*(B)$ if and only if*

$$1 + \frac{DP(z)z}{P(z)} \in \frac{1}{g}(U), \quad z \in B.$$

Proof We will use similar arguments to those in [Pf-Su2, Theorem 2]. Let $L(z) = \frac{DP(z)(\cdot)}{P(z)}$. Then

$$[DF(z)]^{-1} = \frac{1}{P(z)}\left(I - \frac{zL(z)(\cdot)}{1 + L(z)z}\right), \quad z \in B.$$

Let $l_z \in T(z)$, for $z \in B \setminus \{0\}$. A short computation yields that

$$w(z) = [DF(z)]^{-1}F(z) = \frac{z}{1 + L(z)z},$$

and since $l_z \in T(z)$, we obtain

$$\frac{1}{\|z\|} l_z\big(w(z)\big) = \frac{1}{1 + L(z)z}.$$

Therefore we deduce that $F \in S_g^*(B)$ if and only if $\frac{1}{1+L(z)z} \in g(U)$. This completes the proof.

We remark that if $g(\zeta) = (1+\zeta)/(1-\zeta)$, this result has recently been obtained in [Pf-Su2].

We shall also give some applications of the above result. We remark that in the case $g(\zeta) = (1+\zeta)/(1-\zeta)$, the result below was obtained by Pfaltzgraff and Suffridge [Pf-Su2]. In this case Corollary 2.9 gives the following extension result: if each $f_j \in S^*$, $j = 1, \dots, n$, then $F \in S^*(B)$.

Corollary 2.9 *For each $j = 1, 2, \dots, n$, let $f_j(\zeta)$ be a normalized holomorphic function on U such that $\frac{f_j(\zeta)}{\zeta f_j'(\zeta)} \prec g(\zeta)$, for $\zeta \in U$, where $g\colon U \to \mathbb{C}$ satisfy the requirements*

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from Definition 1.1. Moreover, assume $1/g$ *is a convex function on* U. *If* $\lambda_j \geq 0$ *and* $\sum_{j=1}^{n} \lambda_j = 1$, *then*

$$F(z) = z \prod_{j=1}^{n} \left(\frac{f_j(z_j)}{z_j} \right)^{\lambda_j}, \quad z \in B, \tag{2.7}$$

is a mapping in $S_g^*(B)$.

Proof Note that since each f_j satisfies the assumptions in the hypothesis, we have $f_j \in S_g^*$. Thus the mapping F given by (2.7) is a holomorphic mapping on B, and also $F(0) = 0$ and $DF(0) = I$. Let P denote the product in the statement and let $L(z) = DP(z)(\cdot)/P(z)$. Then

$$1 + L(z)(z) = \sum_{j=1}^{n} \lambda_j \frac{z_j f_j'(z_j)}{f_j(z_j)}$$

and since $\frac{z_j f_j'(z_j)}{f_j(z_j)} \prec \frac{1}{g(z_j)}$ and $1/g$ is a convex function, we have

$$\frac{1}{1 + L(z)z} = \frac{1}{\sum_{j=1}^{n} \lambda_j \frac{z_j f_j'(z_j)}{f_j(z_j)}} \in g(U).$$

Finally, it suffices to apply the result of Theorem 2.8 to deduce the desired conclusion. This completes the proof.

Corollary 2.10 *For each* $j = 1, 2, \ldots, n$, *let* f_j *be a normalized starlike function of order* $1/2$ *on* U *and let* F *be defined by (2.7). Then* $F \in S_g^*(B)$, *where* $g(\zeta) = 1 + \zeta$.

We have seen that the class of spirallike mappings of type α, $|\alpha| < \pi/2$, is a subclass of $S^0(B)$. However, in general a spirallike mapping relative to a linear operator need not belong to $S^0(B)$. In other words, in higher dimensions there exist mappings in $S(B) \setminus S^0(B)$, which do not have parametric representation. We have the following example on the unit Euclidean ball of $\mathbb{C}^n$:

Example 2.11 Let $n = 2$ and $f \colon B \subset \mathbb{C}^2 \to \mathbb{C}^2$,

$$f(z) = (z_1 + a z_2^2, z_2)', \quad z = (z_1, z_2)' \in B.$$

Let $A \colon \mathbb{C}^2 \to \mathbb{C}^2$, $A(z) = (2z_1, z_2)'$, $z = (z_1, z_2)' \in B$. Then $m(A) > 0$ and

$$[Df(z)]^{-1} A f(z) = (2z_1, z_2)', \quad z = (z_1, z_2)' \in B,$$

hence f is a normalized spirallike mapping relative to A, for all $a \in \mathbb{C}$. In particular $f \in S(B)$. Let $a \in \mathbb{R}$ with $a > 2\sqrt{15}$. Let $z_0 = (0, 1/2)'$. Then $f(z_0) = (a/4, 1/2)'$ and

$$\|f(z_0)\| > 2 = \frac{\|z_0\|}{(1 - \|z_0\|)^2}.$$

Taking into account Corollary 2.4, one deduces that $f \notin S^0(B)$.

The above observations suggest that one should consider another subset of $S(B)$, namely the set $S^1(B)$ consisting of those *normalized biholomorphic mappings of B which can be imbedded in Loewner chains*. Thus

$f \in S^1(B)$ if and only if there is a Loewner chain $f(z,t)$ such that $f(z,0) = f(z)$, for $z \in B$.

Combining Theorem 1.4 and Definition 1.5, we have the following inclusions:

- $$S^0(B) = \tilde{S}(B) \subseteq S^1(B) \subseteq S(B).$$

In the case of one complex variable, $S^0(U) = S^1(U) = S$, by [Po, Theorems 6.1, 6.2 and 6.3].

The following example shows that $S^0(B)$ is a proper subset of $S^1(B)$ in higher dimensions.

Example 2.12 (i) As noted in Remark 1.11, if $f(z,t)$ is a (normalized) Loewner chain and $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ is an entire normalized biholomorphic mapping, not the identity, then $\Phi\big(f(z,t)\big)$ is also a (normalized) Loewner chain.

We first remark that for any such Φ there exists a point $z_0 \in \mathbb{C}^n$ such that $\|\Phi(z_0)\| > \|z_0\|$. For otherwise, Φ maps B to itself and Cartan's theorem on fixed points (*cf.* [Ru]) implies that Φ must be the identity. After conjugation with a unitary transformation, we may assume that there exists $\rho > 0$ such that $\|\Phi(\rho, 0, \dots, 0)\| > \rho$.

Now let B be the unit Euclidean ball of $\mathbb{C}^n$ and consider the Loewner chain

$$f(z,t) = \left(\frac{e^t z_1}{(1-z_1)^2}, \dots, \frac{e^t z_n}{(1-z_n)^2}\right)', \quad z = (z_1, \dots, z_n)' \in B,$$

whose initial element $f(z) = f(z,0)$ satisfies

$$\|f(r, 0, \dots, 0)\| = \frac{r}{(1-r)^2}, \quad 0 \le r < 1.$$

Choose r such that $\frac{r}{(1-r)^2} = \rho$, where ρ is as above. It is obvious that $\Phi \circ f$ is the first element of a Loewner chain, thus $\Phi \circ f \in S^1(B)$, and

$$\big\| \Phi\big(f(r, 0, \dots, 0)\big) \big\| > \frac{r}{(1-r)^2}.$$

In view of Corollary 2.4 we conclude that $\Phi \circ f \notin S^0(B)$.

(ii) For example, let $n = 2$ and $\Phi(z) = (z_1, z_2 + z_1^2)'$, $z = (z_1, z_2)' \in \mathbb{C}^2$. Then Φ is an entire normalized biholomorphic mapping on $\mathbb{C}^2$. Also if

$$f(z,t) = \left(\frac{e^t z_1}{(1-z_1)^2}, \frac{e^t z_2}{(1-z_2)^2}\right)', \quad z \in B,\ t \ge 0,$$

then

$$\Phi\big(f(z,t)\big) = \left(\frac{e^t z_1}{(1-z_1)^2}, \frac{e^t z_2}{(1-z_2)^2} + \frac{e^{2t} z_1^2}{(1-z_1)^4}\right)', \quad z \in B,\ t \ge 0,$$

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is a Loewner chain. Let $f(z) = f(z,0)$, $z \in B$. It is clear that for each $r \in (0,1)$,

$$\left\| \Phi\big(f(r,0)\big) \right\| = \frac{r}{(1-r)^2}\sqrt{1+\frac{r^2}{(1-r)^4}} > \frac{r}{(1-r)^2}.$$

Therefore $\Phi \circ f \notin S^0(B)$. We remark that $\left\{ e^{-t}\Phi\big(f(z,t)\big) \right\}_{t\geq 0}$ is not a normal family.

Note that if $f(z,t)$ is a Loewner chain which is locally Lipschitz in $t \geq 0$ locally uniformly with respect to $z \in B$, and such that $\{e^{-t}f(z,t)\}_{t\geq 0}$ is a normal family, then in view of Theorem 1.10, $f(z) = f(z,0) \in S^0(B)$. Thus f satisfies the growth estimate from Corollary 2.4.

2.3 Subsets of Mappings in $S_g^0(B)$ With $g(\zeta) = 1 + \zeta$

We next study the set $S_g^0(B)$ when $g(\zeta) = 1+\zeta$, $\zeta \in U$.

In this case $g(U)$ is the open disc centered at 1 and of radius 1. Therefore,

$$\mathcal{M}_g = \Big\{ p \in H(B) : p(0) = 0, Dp(0) = I, \\ \left| \frac{1}{\|z\|} l_z\big(p(z)\big) - 1 \right| < 1, z \in B \setminus \{0\}, l_z \in T(z) \Big\}.$$

Let $u \in \mathbb{C}^n$ with $\|u\| = 1$ and $l_u \in T(u)$. For a normalized locally biholomorphic mapping f on B, let

$$G_f(\alpha,\beta) = \frac{2\alpha}{l_u\Big([Df(\alpha u)]^{-1}\big(f(\alpha u) - f(\beta u)\big)\Big)} - \frac{\alpha+\beta}{\alpha-\beta}, \quad \alpha,\beta \in U.$$

Let $\mathcal{G}$ denote the set of all normalized locally biholomorphic mappings f on B that satisfy the condition $\operatorname{Re} G_f(\alpha,\beta) > 0$, for all $\alpha,\beta \in U$, $|\beta| \leq |\alpha|$, $u \in \mathbb{C}^n$ with $\|u\| = 1$ and $l_u \in T(u)$. This set, called the set of *quasi-convex mappings of type A*, has been recently introduced by Roper and Suffridge [Ro-Su2], as a natural generalization to higher dimensions of the convex functions in the plane. They proved the inclusion relation

- $$K(B) \subset \mathcal{G} \subset S^*(B),$$

and obtained a number of interesting properties of the mappings in $\mathcal{G}$. In particular they showed that in the case of the Euclidean norm the 1/2 growth result for the set $K(B)$ is also valid for the set $\mathcal{G}$. We shall prove that this result remains true in the case of an arbitrary norm. We shall refer to the set $\mathcal{G}$ as the set of quasi-convex mappings.

Now let $f \in \mathcal{G}$. In [Ro-Su2] it is shown that f satisfies the relation

$$\operatorname{Re}\left\{ \frac{\|z\|}{l_z\big([Df(z)]^{-1}f(z)\big)} \right\} > \frac{1}{2}, \quad z \in B \setminus \{0\},\ l_z \in T(z),$$

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which is equivalent to

$$\left|\frac{1}{\|z\|}l_z\big([Df(z)]^{-1}f(z)\big) - 1\right| < 1, \quad z \in B\setminus\{0\},\ l_z \in T(z).$$

If we set $p(z) = [Df(z)]^{-1}f(z)$ and use the above inequality, one deduces that $p \in \mathcal{M}_g$. Moreover, let $f(z,t) = e^t f(z)$. Since $f \in \mathcal{G}$ it follows that $f \in S^*(B)$, and thus $f(z,t)$ is a Loewner chain. If $h(z,t) = p(z)$, $z \in B$, $t \geq 0$, then obviously the differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad z \in B,\ t \geq 0,$$

holds. Since $\lim_{t\to\infty} e^{-t}f(z,t) = f(z)$ locally uniformly on B and $h(z,t) \in \mathcal{M}_g$, we see that $f(z,t)$ satisfies all assumptions of Lemma 1.6. Consequently, $f(z,t)$ is a g-Loewner chain and $f(z,0) = f(z) \in S_g^0(B)$. Therefore we have proved that $\mathcal{G} \subset S_g^0(B)$, and thus we have the following inclusions:

- $$K(B) \subset \mathcal{G} \subset S_g^0(B) \quad \text{with } g(\zeta) = 1+\zeta,\ \zeta \in U.$$

These inclusions between $K(B)$, $\mathcal{G}$ and $S_g^0(B)$ with $g(\zeta) = 1 + \zeta$, were one of the motivations for considering the set $S_g^0(B)$. In particular the growth results for normalized convex mappings on B and quasi-convex mappings can be deduced from Theorem 2.2 (see [Su4], [Fi-Th], [Ro-Su2], [Ha-Ko3], [Ko2], [Ha1]).

Corollary 2.13 *Let $g(\zeta) = 1 + \zeta$, $\zeta \in U$, and $f \in S_g^0(B)$. Then*

$$\frac{\|z\|}{1+\|z\|} \leq \|f(z)\| \leq \frac{\|z\|}{1-\|z\|}, \quad z \in B.$$

Consequently, $f(B) \supseteq B_{1/2}$.

Note that the above growth result is sharp in the case of the unit ball $B(p)$ with respect to a p-norm, $1 \leq p \leq \infty$. To see this, let

$$f(z) = \left(\frac{z_1}{1-z_1}, \dots, \frac{z_n}{1-z_n}\right)', \quad z = (z_1,\dots,z_n)' \in B(p).$$

Then f is normalized biholomorphic on $B(p)$ and using the expression of $T(z)$ in [Su2, Section 2] we can prove that

$$\left|\frac{1}{\|z\|}l_z\big([Df(z)]^{-1}f(z)\big) - 1\right| < 1, \quad 0 < \|z\| < 1,\ l_z \in T(z).$$

Thus $f \in S_g^0(B)$ with $g(\zeta) = 1 + \zeta$. Moreover, for $z = (r, 0, \dots, 0)'$ with $r \in [0, 1)$, we have $\|z\| = r$ and $\|f(z)\| = \frac{r}{1-r}$. Also for $z = (-r, 0, \dots, 0)'$, $\|z\| = r$ and $\|f(z)\| = \frac{r}{1+r}$.

2.4 Bounds of Coefficients of Mappings in $S_g^0(B)$

We now prove the following estimate for the second order coefficients of mappings in the set $S_g^0(B)$ (compare with [Por1, Theorem 3] and [Ko3, Theorem 2.4]). Note that if $f(z,t)$ is a g-Loewner chain, then $f(z,0)$ satisfies (2.8).

Theorem 2.14 *Let $g\colon U \to \mathbb{C}$ satisfy the assumptions of Definition 1.1 and let $f \in S_g^0(B)$. Then*

$$\Big|\frac{1}{2!}l_w\big(D^2 f(0)(w,w)\big)\Big| \le |g'(0)|, \quad w \in \mathbb{C}^n,\ \|w\| = 1,\ l_w \in T(w). \tag{2.8}$$

Proof Since $f \in S_g^0(B)$, there is a mapping $h = h(z,t) \in \mathcal{M}_g$ such that

$$f(z) = \lim_{t\to\infty} e^t v(z,t)$$

locally uniformly on B, where $v(t) = v(z,t)$ is the solution of the initial value problem

$$\frac{\partial v}{\partial t}(z,t) = -h(v,t), \quad \text{a.e. } t \ge 0,\ v(z,0) = z.$$

Let $f\colon B \times [0,\infty) \to \mathbb{C}^n$ be given by

$$f(z,s) = \lim_{t\to\infty} e^t w(z,s,t)$$

locally uniformly on B, where $w(t) = w(z,s,t)$ is the unique solution of the initial value problem

$$\frac{\partial w}{\partial t} = -h(w,t), \quad \text{a.e. } t \ge s,\ w(s) = z, \tag{2.9}$$

for $z \in B$ and $s \ge 0$. Then it is obvious to see that $w(z,0,t) = v(z,t)$, for all $z \in B$ and $t \ge 0$, hence $f(z,0) = f(z)$, $z \in B$.

Fix $z \in B \setminus \{0\}$, $l_z \in T(z)$ and $t_0 \ge 0$. Let

$$p_{t_0}(\zeta) = \begin{cases} \frac{1}{\zeta} l_z\big(h_{t_0}(\zeta\frac{z}{\|z\|})\big), & \zeta \in U \setminus \{0\}, \\ 1, & \zeta = 0. \end{cases}$$

Then p_{t_0} is a holomorphic function on U, and as in the proof of Lemma 2.1 we have $p_{t_0}(\zeta) \in g(U)$ for $\zeta \in U$. Hence $p_{t_0} \prec g$, and thus $|p_t'(0)| \le |g'(0)|$ by the subordination principle.

Since

$$h_{t_0}\Big(\zeta\frac{z}{\|z\|}\Big) = \frac{z}{\|z\|}\zeta + \frac{1}{2!}D^2 h_{t_0}(0)\Big(\frac{z}{\|z\|}, \frac{z}{\|z\|}\Big)\zeta^2 + \cdots, \quad \zeta \in U,$$

we obtain by identifying the coefficients in the power series expansions that

$$p_{t_0}'(0) = \frac{1}{2!} l_z\left(D^2 h_{t_0}(0)\Big(\frac{z}{\|z\|}, \frac{z}{\|z\|}\Big)\right).$$

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Consequently, we deduce the following relation

$$\left| \frac{1}{2!} l_z \left(D^2 h_{t_0}(0) \left(\frac{z}{\|z\|}, \frac{z}{\|z\|} \right) \right) \right| \le |g'(0)|. \tag{2.10}$$

On the other hand, since $f(z,\cdot)$ is locally Lipschitz on $[0,\infty)$ locally uniformly with respect to $z \in B$ by Theorem 1.4, we deduce that $f(z,\cdot)$ is differentiable with respect to t for almost all $t \in [0,\infty)$. Moreover, since $f(z,s)$ satisfies the relation

$$f(z,s) = f\big(w(z,s,t),t\big),$$

we obtain by differentiating the above equality with respect to t and using (2.9) that for almost all $t \ge 0$,

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad z \in B.$$

Fix $T > 0$ and integrate both sides of the above equality, to obtain

$$f(z,T) - f(z,0) = \int_0^T Df(z,t)h(z,t)\,dt.$$

For fixed z, let $G_z \colon U \to \mathbb{C}$ be given by

$$G_z(\zeta) = f(\zeta z, T) - f(\zeta z, 0)$$

and

$$H_z(\zeta) = \int_0^T Df(\zeta z, t)h(\zeta z, t)\,dt.$$

Then $G_z(\zeta) = H_z(\zeta)$, $\zeta \in U$ and both mappings G_z and H_z are holomorphic on U. After simple computations, using the fact that $Df(0,t) = e^t I$, we deduce that

$$\frac{d^2 H_z}{d\zeta^2}(0) = \int_0^T [2D^2 f(0,t)(z,z) + e^t D^2 h(0,t)(z,z)]\,dt,$$

and hence

$$D^2 f(0,T)(z,z) - D^2 f(0,0)(z,z) = \int_0^T [2D^2 f(0,t)(z,z) + e^t D^2 h(0,t)(z,z)]\,dt.$$

By simple transformations this equality is equivalent to the following

$$e^{-2T} D^2 f(0,T)(z,z) - D^2 f(0,0)(z,z) = \int_0^T e^{-t} D^2 h(0,t)(z,z)\,dt,$$

hence

$$e^{-2T} l_z\big(D^2 f(0,T)(z,z)\big) - l_z\big(D^2 f(0,0)(z,z)\big) = \int_0^T l_z\big(e^{-t} D^2 h(0,t)(z,z)\big)\, dt. \tag{2.11}$$

As in Corollary 2.3, we have the following estimate

$$\|f(z,T)\| \le e^T \|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{dx}{x}. \tag{2.12}$$

Next, using the Cauchy formula

$$\frac{1}{2!} D^2 f(0,T)(u,u) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta u, T)}{\zeta^3}\, d\zeta, \quad r < 1,$$

for $u \in \mathbb{C}^n$, $\|u\| = 1$, and taking into account (2.12), we easily obtain that

$$\lim_{T\to\infty} e^{-2T} D^2 f(0,T)(z,z) = 0.$$

If we now let $T \to \infty$ in (2.11) and use the above equality and (2.10), we deduce that

$$\left| \frac{1}{2!} l_z\left(D^2 f(0,0)\left(\frac{z}{\|z\|}, \frac{z}{\|z\|}\right)\right)\right| \le |g'(0)|.$$

Obviously, the above relation is equivalent to

$$\left| \frac{1}{2!} l_w\big(D^2 f(0,0)(w,w)\big)\right| \le |g'(0)|, \quad \|w\| = 1,\ l_w \in T(w).$$

Since $f(z,0) = f(z)$, $z \in B$, the proof is complete.

For the norm of the second order Fréchet derivative of a mapping in $S_g^0(B)$ we have the following estimate.

Corollary 2.15 *Let $g\colon U \to \mathbb{C}$ satisfy the assumptions of Definition 1.1 and $f \in S_g^0(B)$. Then*

$$\left\| \frac{1}{2!} D^2 f(0)(z,z)\right\| \le 4|g'(0)|, \quad \|z\| = 1.$$

Proof It suffices to use similar arguments as in the second part of the proof of Theorem 1.2. For this purpose, let $P_2 = \frac{1}{2!} D^2 f(0)$. Then P_2 is a homogeneous polynomial of degree 2 and thus we obtain

$$\|P_2\| \le 4|V(P_2)|.$$

Taking into account (2.8) and the above relations, we easily deduce that

$$|V(P_2)| \le |g'(0)|$$

and the result now follows. This completes the proof.

For $g(\zeta) = (1+\zeta)/(1-\zeta)$, $\zeta \in U$, we obtain the following consequence. In particular, this result is satisfied by all mappings in $S^*(B)$. The bound (2.13) was obtained by Kohr [Ko1] in the case of mappings in $S^*(B)$ when B is the unit Euclidean ball of $\mathbb{C}^n$. We also note that this bound is sharp in the case of the p-norm, with $1 \le p \le \infty$.

Corollary 2.16 *If $f \in S^0(B)$ then*

$$\left| \frac{1}{2!} l_w\big(D^2 f(0)(w,w)\big) \right| \le 2, \quad \|w\| = 1,\ l_w \in T(w). \tag{2.13}$$

Moreover

$$\left\| \frac{1}{2!} D^2 f(0)(z,z) \right\| \le 8, \quad \|z\| = 1.$$

It would be interesting to see if the mappings in $S^1(B)$ satisfy the above bound. Note that using the growth result in Corollary 2.4, one may prove that if $f \in S^0(B)$ then

$$\left\| \frac{1}{k!} D^k f(0)(w^k) \right\| \le \left[\frac{e(k+1)}{2} \right]^2, \quad \|w\| = 1,\ k \ge 2.$$

We leave this bound as an exercise for the reader.

It would be interesting to study the following conjecture in several complex variables (this could be considered the n-dimensional version of the Bieberbach conjecture for the set S).

Conjecture 2.17 *If $f \in S^0(B)$ then*

$$\left| \frac{1}{k!} l_w\big(D^k f(0)(w^k)\big) \right| \le k, \quad \|w\| = 1,\ l_w \in T(w),\ k \ge 2.$$

Remark 2.18 In higher dimensions if $f \in S^0(B)$ it need not be true that

$$\left\| \frac{1}{k!} D^k f(0)(w^k) \right\| \le k, \quad \|w\| = 1,\ k \ge 2.$$

(However, in the case of the maximum norm the above inequalities are true for $k = 2$, as shown by Poreda [Por1], and are open for $k \ge 3$. On the other hand, Gong [Go2, Theorem 5.3.1] has recently proved that if f is normalized starlike on the unit polydisc of $\mathbb{C}^n$, then the above bounds hold for $k = 2, 3$, and are open for $k \ge 4$.)

To see this, let $n = 2$ and consider the space $\mathbb{C}^2$ with the Euclidean structure. Also let $f(z) = (z_1 + a z_2^2, z_2)'$ for $z = (z_1, z_2)' \in B$. If $|a| = 3\sqrt{3}/2$ then f is starlike (see [Ro-Su2, Example 5]). Thus $f \in S^0(B)$. However, for $w = (0,1)'$, we have

$$\left\| \frac{1}{2} D^2 f(0)(w^2) \right\| = |a| = \frac{3\sqrt{3}}{2} > 2.$$

For $g(\zeta) = 1 + \zeta$, $\zeta \in U$, we obtain the following bound for the second order coefficients of mappings in $S_g^0(B)$. In particular, this result is satisfied by all mappings in $K(B)$ and $\mathcal{G}$ respectively (also see [Ko1]). Note that (2.14) is sharp in the case of the unit ball $B(p)$ with respect to a p-norm, $1 \le p \le \infty$.

Corollary 2.19 *If $f \in S_g^0(B)$ with $g(\zeta) = 1 + \zeta$, $\zeta \in U$, then*

$$\Big| \frac{1}{2!} l_w\big(D^2 f(0)(w, w)\big) \Big| \le 1, \quad \|w\| = 1,\ l_w \in T(w). \tag{2.14}$$

Moreover,

$$\Big\| \frac{1}{2!} D^2 f(0)(w, w) \Big\| \le 4, \quad \|w\| = 1,$$

and

$$\Big\| \frac{1}{k!} D^k f(0)(w^k) \Big\| < ek, \tag{2.15}$$

for $k \in \mathbb{N}$, $k \ge 3$, and $\|w\| = 1$.

Proof It suffices to prove the bounds (2.15). To this end, fix $k \in \mathbb{N}$, $k \ge 3$, and $w \in \mathbb{C}^n$, $\|w\| = 1$. Using the Cauchy formula

$$\frac{1}{k!} D^k f(0)(w^k) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta w)}{\zeta^{k+1}}\, d\zeta, \quad 0 < r < 1,$$

and taking into account Corollary 2.13, we easily obtain

$$\Big\| \frac{1}{k!} D^k f(0)(w^k) \Big\| \le \frac{1}{2\pi r^k} \int_0^{2\pi} \|f(re^{i\theta} w)\|\, d\theta \le \frac{1}{r^{k-1}(1-r)}.$$

Setting $r = 1 - 1/k$ in this inequality gives

$$\Big\| \frac{1}{k!} D^k f(0)(w^k) \Big\| \le k\Big(1 + \frac{1}{k-1}\Big)^{k-1} < ek.$$

This completes the proof.

Remark 2.20 It is well known that if $f \in K(B)$ and if $f(z) = z + \sum_{k=2}^{\infty} A_k(z)$, then the homogeneous polynomial $A_k(z)$ satisfies the following bounds (*cf.* [Fi-Th], [Go1], [Ha-Ko3], [Ko1], [Pf-Su3]):

$$\|A_k(w)\| \le 1, \quad \|w\| = 1,\ k \ge 2.$$

However, if $f \in S_g^0(B) \setminus K(B)$, with $g(\zeta) = 1 + \zeta$, then the above bound need not be satisfied. To see this, again consider the case $n = 2$ and $\mathbb{C}^2$ with the Euclidean structure. Also let $f(z) = (z_1 + az_2^2, z_2)'$ for $z = (z_1, z_2)' \in B$. In [Ro-Su2, Example 9] it is shown that if $|a| \le 3\sqrt{3}/4$, then $f \in \mathcal{G}$, and hence $f \in S_g^0(B)$ with $g(\zeta) = 1 + \zeta$.

However, if $|a| > 1/2$, then $f \notin K(B)$ by [Ro-Su2, Example 7]. If $|a| = 3\sqrt{3}/4$ and $w = (0,1)'$, then

$$\Big\| \frac{1}{2} D^2 f(0)(w^2) \Big\| = \frac{3\sqrt{3}}{4} > 1.$$

Finally, we remark that another case of special interest in the study of the set $S_g^0(B)$ is the case $g(\zeta) = (1 + c\zeta)/(1 - c\zeta)$, $\zeta \in U$, where $0 < c < 1$. Obviously, g is a univalent function on U that satisfies the assumptions of Definition 1.1 and moreover, the image of the unit disc is the disc centered at $\frac{1+c^2}{1-c^2}$ and of radius $\frac{2c}{1-c^2}$. Let $S_c^0(B)$ denote the set $S_g^0(B)$ when $g(\zeta) = (1 + c\zeta)/(1 - c\zeta)$, $c \in (0,1)$. In the case of one complex variable, a g-Loewner chain with $g(\zeta) = (1 + c\zeta)/(1 - c\zeta)$, is called a *c-chain* [Bec1]. The interest of such a chain $f(z,t)$ arises from the fact that its first element $f(z) = f(z,0)$ can be extended to a quasiconformal homeomorphism of $\mathbb{C}$ (see [Bec1]).

In the Euclidean case, Chuaqui [Chu] studied the following subset of $S^*(B)$, called the *set of strongly starlike mappings*. He proved that these mappings can be extended quasiconformally to $\mathbb{C}^n$. Also see [Ha-Ko4].

Definition 2.21 Let $z \in \mathbb{C}^n$, $\|z\| = 1$ and $f \in S^*(B)$. We say that f is *strongly starlike* if the values of

$$q(\zeta) = \frac{1}{\zeta} l_z\big([Df(\zeta z)]^{-1} f(\zeta z)\big), \quad |\zeta| < 1,$$

lie in a compact subset of the right half-plane, independent of z and $l_z \in T(z)$.

Next, let $c \in (0,1)$ and f be a normalized locally biholomorphic mapping on B such that

$$\left| \frac{1}{\|z\|} l_z\big([Df(z)]^{-1} f(z)\big) - \frac{1+c^2}{1-c^2} \right| < \frac{2c}{1-c^2}, \quad z \in B \setminus \{0\},\ l_z \in T(z). \tag{2.16}$$

Clearly if f satisfies the above assumption, then f is strongly starlike. Moreover, since $f(z,t) = e^t f(z)$ is a g-Loewner chain, with $g(\zeta) = (1+c\zeta)/(1-c\zeta)$, one deduces that $f \in S_c^0(B)$. Also, it is obvious that if f is strongly starlike, then f satisfies (2.16) for some $c \in (0,1)$ and thus, $f \in S_c^0(B)$.

From Theorems 2.2, 2.14 and Corollary 2.15 we obtain the following growth result and coefficient estimates for mappings in $S_c^0(B)$. In particular, if f satisfies the assumptions of Example 1.7 and $c \in (0,1)$, then $f \in S_c^0(B)$, and therefore satisfies the hypotheses of the result below.

Theorem 2.22 *Let $f \in S_c^0(B)$ with $c \in (0,1)$. Then*

$$\frac{\|z\|}{(1 + c\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1 - c\|z\|)^2}, \quad z \in B.$$

Moreover,

$$\Big| \frac{1}{2} l_w\big(D^2 f(0)(w,w)\big) \Big| \le 2c, \quad \|w\| = 1,\ l_w \in T(w),$$

and

$$\Big\| \frac{1}{2} D^2 f(0)(w,w) \Big\| \le 8c, \quad \|w\| = 1.$$

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