

DISTRIBUTIONS OF STABLE RANDOM FIELDS OF CHENTSOV TYPE

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§ 1. Introduction

In this paper we discuss the determinism of distributions of some stable random fields which are constructed through integral-geometric method. The determinism depends on the dimension of the parameter space \mathbf{R}^d .

We say that a family of random variables $\{X(t); t \in \mathbf{R}^d\}$ is a *symmetric α -stable* (abbreviated to *S α S*) random field on \mathbf{R}^d if every finite linear combination $\sum_{i=1}^n a_i X(t_i)$ has a symmetric stable distribution of index α . Let (E, \mathcal{B}, μ) be a measure space. We say that a family of random variables $\{Y(B); B \in \mathcal{B}, \mu(B) < \infty\}$ is the *S α S random measure* corresponding to (E, \mathcal{B}, μ) if (i) $E(\exp[izY(B)]) = \exp(-\mu(B)|z|^\alpha)$, for $z \in \mathbf{R}$ and $\mu(B) < \infty$, (ii) $Y(B_1), Y(B_2), \dots$ are independent whenever B_1, B_2, \dots are disjoint and $\mu(B_j) < \infty, j = 1, 2, \dots$, (iii) $Y(\bigcup_{j=1}^\infty B_j) = \sum_{j=1}^\infty Y(B_j)$ a.s. whenever B_1, B_2, \dots are disjoint and $\mu(\bigcup_{j=1}^\infty B_j) < \infty$.

We define a class of *S α S* random fields with a particular choice of E . Let E_0 be the set of all $(d-1)$ -dimensional spheres in \mathbf{R}^d . Any element of E_0 is expressed by a coordinate system (r, x) , where (r, x) corresponds to the sphere with radius $r \in \mathbf{R}_+ = (0, \infty)$ and center $x \in \mathbf{R}^d$. Thus we make the identification

$$(1.1) \quad E_0 = \{(r, x); r \in \mathbf{R}_+, x \in \mathbf{R}^d\}.$$

For $t \in \mathbf{R}^d$, let S_t be the set of all spheres which separate the point t and the origin O , namely

$$(1.2) \quad S_t = \{(r, x); d(x, O) \leq r\} \Delta \{(r, x); d(r, x) \leq r\},$$

where $A \Delta B$ denotes the symmetric difference of A and B and $d(a, b)$ denotes the Euclidean distance between a and b . Let \mathcal{B}_0 be the σ -algebra

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of Borel sets in E_0 . Given a measure μ on (E_0, \mathcal{B}_0) such that

$$(1.3) \quad \mu(S_t) < \infty \quad \text{for all } t \in \mathbf{R}^d,$$

we define an $S\alpha S$ random field by

$$(1.4) \quad X(t) = Y(S_t), \quad t \in \mathbf{R}^d,$$

using the $S\alpha S$ random measure $\{Y(B)\}$ corresponding to $(E_0, \mathcal{B}_0, \mu)$. We call this $\{X(t)\}$ $S\alpha S$ random field of Chentsov type on \mathbf{R}^d associated with μ .

Such a random field is viewed as an extension of N.N. Chentsov's representation $Y(S'_t)$ of Lévy's Brownian motion of \mathbf{R}^d -parameter. The $Y(S'_t)$ is defined by Chentsov through Gaussian random measure Y associated with a measure on the space E' of all hyperplanes of co-dimension 1 in \mathbf{R}^d and the defining set S'_t is the set of all hyperplanes which separate t and the origin O , [1], [3]. S. Takenaka, [7], applied this idea to stable case. Using E_0 in place of E' , he proves that if $d\mu_\beta(r, x) = r^{\beta-d-1}drdx$, $0 < \beta < 1$, then the Chentsov type $S\alpha S$ random field $X_{\alpha, \beta}(t)$ associated with (E_0, μ_β) is self-similar with exponent $H = \beta/\alpha$. For $d = 1$, $\{X_{\alpha, \beta}(t)\}$ presents a new example of $S\alpha S$, H -self-similar process with stationary increments in the area of α and H where no examples were known before.

The distributions of a Chentsov type $S\alpha S$ random field on \mathbf{R}^d have a characteristic property which depends on the dimension d of the parameters. We do not assume any condition other than (1.3) for the associated measure. The aim of this paper is to prove the following theorem.

THEOREM 1. *Let $0 < \alpha < 2$. Let μ be a measure on (E_0, \mathcal{B}_0) satisfying (1.3) and let $\{X(t); t \in \mathbf{R}^d\}$ be the $S\alpha S$ random field of Chentsov type on \mathbf{R}^d associated with μ . Then, for any $n > d + 1$ and for any distinct $t_1, \dots, t_n \in \mathbf{R}^d$, the distribution $(X(t_1), \dots, X(t_n))$ is determined by its $(d + 1)$ -dimensional marginal distributions.*

COROLLARY. *Let $0 < \alpha < 2$. Let μ and $\tilde{\mu}$ be measures on (E_0, \mathcal{B}_0) satisfying (1.3). Let $\{X(t); t \in \mathbf{R}^d\}$ and $\{\tilde{X}(t); t \in \mathbf{R}^d\}$ be the $S\alpha S$ random fields of Chentsov type associated with μ and $\tilde{\mu}$, respectively. If the $(d + 1)$ -dimensional distributions of $\{X(t)\}$ and $\{\tilde{X}(t)\}$ coincide, then $\{X(t)\}$ and $\{\tilde{X}(t)\}$ are equivalent, that is, the finite-dimensional distributions of $\{X(t)\}$ and $\{\tilde{X}(t)\}$ coincide.*

Remark 1. The number $d + 1$ in Theorem 1 is best possible in the following sense. There are two Chentsov type random fields $\{X(t)\}$ and

$\{\tilde{X}(t)\}$ associated with μ and $\tilde{\mu}$, respectively, such that, for some $T = (t_1, \dots, t_{d+1})$, the d -dimensional marginal distributions of $(X(t_1), \dots, X(t_{d+1}))$ and $(\tilde{X}(t_1), \dots, \tilde{X}(t_{d+1}))$ coincide but their $(d + 1)$ -dimensional distributions are different. (see Example 4.2)

Remark 2. If we take E' and S'_t instead of E_0 and S_t and define

$$X'(t) = Y(S'_t) \quad \text{for } t \in \mathbf{R}^d,$$

where Y is an $S\alpha S$ random measure with $0 < \alpha < 2$ associated with a measure μ' on E' satisfying $\mu'(S'_t) < \infty$, then we have determinism by d -dimensional marginal distributions instead of determinism by $(d + 1)$ -dimensional marginal distributions in Theorem 1. Namely, for any $n > d$ and any distinct $t_1, \dots, t_n \in \mathbf{R}^d$, the distribution of $(X'(t_1), \dots, X'(t_n))$ is determined by its d -dimensional marginal distributions. This fact can be proved by a similar method as Theorem 1.

Theorem 1 will be reduced to a geometric theorem concerning an intersection property of a family of cones in $\mathbf{R}_+ \times \mathbf{R}^d$. The proof of this geometric theorem is an essential part of our argument. For $t \in \mathbf{R}^d$, set

$$(1.5) \quad C_t = \{(r, x); d(x, t) \leq r\}.$$

Then, C_t is a right cone in $\mathbf{R}_+ \times \mathbf{R}^d$ with vertex $(0, t)$. Note that the point $(0, t)$ is not included in the space $\mathbf{R}_+ \times \mathbf{R}^d$. Hereafter we simply call C_t the cone with vertex t . The relation

$$S_t = C_0 \triangle C_t$$

shows that, instead of S_t 's, we may study C_t 's. Given m cones C_{t_1}, \dots, C_{t_m} , we consider the partition of the set $\bigcup_{i=1}^m C_{t_i}$ generated by $\{C_{t_i}, i = 1, \dots, m\}$. Now set

$$(1.6) \quad \mathcal{E}_m = \{e = (e_1, \dots, e_m); e_i = 0 \text{ or } 1 \text{ for } i = 1, \dots, m\} \setminus \{(0, \dots, 0)\}.$$

We call $e \in \mathcal{E}_m$ a label of size m and \mathcal{E}_m the label set. With the notation

$$C_t^1 = C_t \quad \text{and} \quad C_t^0 = C_t^c = (\mathbf{R}_+ \times \mathbf{R}^d) \setminus C_t,$$

we define

$$(1.7) \quad C(T, e) = \bigcap_{i=1}^m C_{t_i}^{e_i}$$

for $T = (t_1, \dots, t_m) \in (\mathbf{R}^d)^m$ and $e = (e_1, \dots, e_m) \in \mathcal{E}_m$. Then $C(T, e)$, $e \in \mathcal{E}_m$, are disjoint sets and

$$(1.8) \quad \bigcup_{i=1}^m C_{t_i} = \bigcup_{e \in \mathcal{E}_m} C(T, e).$$

For $e = (e_1, \dots, e_m) \in \mathcal{E}_m$, the complementary label e^* of e is defined by

$$e^* = (e_1^*, \dots, e_m^*), \quad e_i + e_i^* = 1 \quad \text{for any } i.$$

THEOREM 2. *If $m \geq d + 3$, then, for any $T = (t_1, \dots, t_m) \in (\mathbf{R}^d)^m$, there exists a label $e \in \mathcal{E}_m$ such that $C(T, e) = \emptyset$ and $C(T, e^*) = \emptyset$.*

In § 2 we reduce Theorem 1 to Theorem 2, which is proved in § 3 after a series of lemmas. Concluding remarks are given in § 4. The particular cases $d = 1$ and 2 of Theorem 1 have been treated in the author’s paper [4], [5]. Determinism under different defining sets in one-dimensional case will be discussed in a joint paper with S. Takenaka [6].

§ 2. Reduction of Theorem 1 to Theorem 2

Let $0 < \alpha < 2$. Let μ be a measure on (E_0, \mathcal{B}_0) satisfying (1.3) and let $\{X(t); t \in \mathbf{R}^d\}$ be the $S\alpha S$ random field of Chentsov type on \mathbf{R}^d associated with μ . For $t \in \mathbf{R}^d$ let S_t be the set defined by (1.2). For $T = (t_1, \dots, t_n) \in (\mathbf{R}^d)^n$ and $e = (e_1, \dots, e_n) \in \mathcal{E}_n$, we write

$$(2.1) \quad S_{t_k}^{e_k} = \begin{cases} S_{t_k} & \text{if } e_k = 1 \\ S_{t_k}^c & \text{if } e_k = 0 \end{cases}$$

$$(2.2) \quad \tilde{S}_{t_k}^{e_k} = \begin{cases} S_{t_k} & \text{if } e_k = 1 \\ \mathbf{R}_+ \times \mathbf{R}^d & \text{if } e_k = 0 \end{cases}$$

$$(2.3) \quad S(T, e) = \bigcap_{k=1}^n S_{t_k}^{e_k}$$

$$(2.4) \quad \tilde{S}(T, e) = \bigcap_{k=1}^n \tilde{S}_{t_k}^{e_k}.$$

$S(T, e)$ is an element (labelled with e) of the partition of the set $\bigcup_{k=1}^n S_{t_k}$ generated by $\{S_{t_k}; k = 1, \dots, n\}$.

DEFINITION 2.1. We say that $T = (t_1, \dots, t_n) \in (\mathbf{R}^d)^n$ satisfies Condition (I) if there exists a label $e \in \mathcal{E}_n$ such that $S(T, e) = \emptyset$.

Remark. Suppose that $T = (t_1, \dots, t_n)$ satisfies Condition (I). Then $T' = (t_1, \dots, t_n, t_{n+1}, \dots, t_m)$ satisfies Condition (I) for any $t_{n+1}, \dots, t_m \in \mathbf{R}^d$. In fact, suppose that $S(T, e) = \emptyset$ for $e = (e_1, \dots, e_n) \in \mathcal{E}_n$. If we take a label $\tilde{e} = (\tilde{e}_1, \dots, \tilde{e}_m)$ such that $\tilde{e}_1 = e_1, \dots, \tilde{e}_n = e_n$, then $S(T', \tilde{e}) = \emptyset$.

LEMMA 2.2. *Let t_1, \dots, t_n be different points in \mathbf{R}^d . If $T = (t_1, \dots, t_n)$*

satisfies Condition (I), then the distribution of $X_T = (X(t_1), \dots, X(t_n))$ is determined by the system of $(n - 1)$ -dimensional marginal distributions of X_T .

Proof. The characteristic function of X_T is written, for $z = (z_1, \dots, z_n) \in \mathbf{R}^n$, as

$$\begin{aligned}
 \varphi_T(z) &= E \exp \left\{ i \sum_{k=1}^n z_k Y(S_{t_k}) \right\} \\
 &= E \exp \left\{ i \sum_{k=1}^n z_k \sum_{\substack{e \in \mathcal{E}_n \\ e_k=1}} Y(S(T, e)) \right\} \\
 (2.5) \quad &= E \exp \left\{ i \sum_{e \in \mathcal{E}_n} \left(\sum_{k=1}^n e_k z_k \right) Y(S(T, e)) \right\} \\
 &= \exp \left\{ - \sum_{e \in \mathcal{E}_n} \left| \sum_{k=1}^n e_k z_k \right|^\alpha \mu(S(T, e)) \right\} . \\
 &= \exp \left\{ - \sum_{e \in \mathcal{E}_n} |\xi(e) \cdot z|^\alpha (\#e)^{\alpha/2} \mu(S(T, e)) \right\} ,
 \end{aligned}$$

where $e = (e_1, \dots, e_n)$, $\#e = \sum_{i=1}^n e_i$, and

$$(2.6) \quad \xi(e) = (1/(\#e)^{1/2})e .$$

On the other hand it is known that the characteristic function of an n -dimensional $S\alpha S$ distribution, $0 < \alpha < 2$, has the following unique canonical representation:

$$(2.7) \quad \varphi(z) = \exp \left\{ - c \int_{S^{n-1}} |\xi \cdot z|^\alpha \lambda(d\xi) \right\} ,$$

where $c > 0$ and λ is a symmetric probability measure on the unit sphere S^{n-1} [2]. Comparing the last expression of (2.5) to (2.7) and noticing that $\xi(e)$ of (2.6) belongs to S^{n-1} for any e , we see that the last expression of (2.5) gives the canonical form of $\varphi_T(z)$. So, we see that $\varphi_T(z)$ is determined by the values of $\mu(S(T, e))$, $e \in \mathcal{E}_n$, and that, conversely, $\mu(S(T, e))$, $e \in \mathcal{E}_n$, are determined by $\varphi_T(z)$.

Since μ is a measure, the following consistency condition holds:

$$(2.8) \quad \mu(\tilde{S}(T, e)) = \sum_{e' \in \mathcal{E}'_n(e)} \mu(S(T, e')) \quad \text{for each } e \in \mathcal{E}_n ,$$

where, for $e = (e_1, \dots, e_n) \in \mathcal{E}_n$, $\mathcal{E}'_n(e)$ is the subset of \mathcal{E}_n defined by

$$(2.9) \quad \mathcal{E}'_n(e) = \{e' = (e'_1, \dots, e'_n); e'_k \geq e_k \text{ for } k = 1, \dots, n\} .$$

Since the number of the elements of \mathcal{E}_n is $2^n - 1$, (2.8) consists of $2^n - 1$ equations. But one of them, that is, the case $e = (1, \dots, 1)$, is a trivial

equation. So we consider

$$(2.10) \quad \sum_{e' \in \mathcal{E}_n'(e)} \mu(S(T, e')) = \mu(\tilde{S}(T, e)) \quad \text{for } e \in \mathcal{E}_n \setminus \{(1, \dots, 1)\}$$

as a system of $2^n - 2$ simultaneous linear equations in which the unknowns are the values of $\mu(S(T, e))$, $e \in \mathcal{E}_n$, (the number of them is $2^n - 1$) and the given right-hand sides are the values of $\mu(\tilde{S}(T, e))$, $e \in \mathcal{E}_n \setminus \{(1, \dots, 1)\}$. These right-hand sides are determined by the $(n - 1)$ -dimensional marginal distributions of X_T . We write the matrix representation of the system (2.10) as

$$(2.11) \quad M_n x = \mathbb{b},$$

where M_n is a $(2^n - 2) \times (2^n - 1)$ -matrix, whose components are 0 or 1. Let $M_n(k)$ be the $(2^n - 2) \times (2^n - 2)$ -matrix obtained from M_n by deleting the k -th column. Then we can prove that

$$(2.12) \quad M_n(k) \text{ is invertible for any } k = 1, \dots, 2^n - 1.$$

Proof of (2.12) will be given at the end of this section. By the assumption that T satisfies Condition (I), there exists a label e such that $S(T, e) = \emptyset$. This implies $\mu(S(T, e)) = 0$ for the label e . So, the number of the unknowns is reduced to $2^n - 2$. Suppose that the $\mu(S(T, e))$ corresponding to the label e is the k -th component of the column vector of the unknowns. Then our system of simultaneous linear equations is equivalent to the system having $M_n(k)$ as its coefficient matrix. By virtue of (2.12) the system of equations has a unique solution. Thus, all the unknowns are determined. □

Now we need to study the problem when T satisfies Condition (I).

Let $T = (t_1, \dots, t_n) \in (\mathbf{R}^d)^n$ and $e = (e_1, \dots, e_n) \in \mathcal{E}_n$. The set $S(T, e)$ is partitioned into two disjoint subsets:

$$(2.13) \quad S(T, e) = \{S(T, e) \cap C_0\} \cup \{S(T, e) \cap C_0^c\},$$

where C_0 is the cone with vertex 0.

LEMMA 2.3. *We have*

$$(2.14) \quad S(T, e) \cap C_0 = C(T, e^*) \cap C_0,$$

$$(2.15) \quad S(T, e) \cap C_0^c = C(T, e) \cap C_0^c,$$

where e^* is the complementary element of e .

Proof. We have

$$S(T, e) \cap C_0 = \left(\bigcap_{i=1}^n S_{t_i}^{e_i} \right) \cap C_0 = \bigcap_{i=1}^n (S_{t_i}^{e_i} \cap C_0).$$

If $e_i = 1$, then

$$S_{t_i}^{e_i} \cap C_0 = S_{t_i} \cap C_0 = (C_{t_i} \triangle C_0) \cap C_0 = C_{t_i}^c \cap C_0 = C_{t_i}^{e_i^*} \cap C_0.$$

If $e_i = 0$, then

$$S_{t_i}^{e_i} \cap C_0 = S_{t_i}^c \cap C_0 = (C_{t_i} \triangle C_0)^c \cap C_0 = C_{t_i} \cap C_0 = C_{t_i}^{e_i^*} \cap C_0.$$

Hence

$$\bigcap_{i=1}^n (S_{t_i}^{e_i} \cap C_0) = \bigcap_{i=1}^n (C_{t_i}^{e_i^*} \cap C_0) = \left(\bigcap_{i=1}^n C_{t_i}^{e_i^*} \right) \cap C_0 = C(T, e^*) \cap C_0.$$

This proves (2.14). The relation (2.15) is obtained more easily. □

Using Lemma 2.3 we can reduce Condition (I) to the following Condition (II).

DEFINITION 2.4. We say that $T = (t_1, t_2, \dots, t_m) \in (\mathbf{R}^d)^m$ satisfies Condition (II) if there exists a label $e \in \mathcal{E}_m$ such that both $C(T, e) = \emptyset$ and $C(T, e^*) = \emptyset$ hold.

LEMMA 2.5. $T = (t_1, \dots, t_n) \in (\mathbf{R}^d)^n$ satisfies Condition (I) if and only if $\tilde{T} = (0, t_1, \dots, t_n) \in (\mathbf{R}^d)^{n+1}$ satisfies Condition (II).

Proof. (i) Suppose that $T = (t_1, \dots, t_n)$ satisfies Condition (I). Let $e = (e_1, \dots, e_n) \in \mathcal{E}_n$ be the label such that $S(T, e) = \emptyset$. It follows from (2.13) and Lemma 2.3 that $C(T, e^*) \cap C_0 = \emptyset$ and $C(T, e) \cap C_0^c = \emptyset$. Put $\tilde{e} = (0, e_1, \dots, e_n) \in \mathcal{E}_{n+1}$. Then

$$\begin{aligned} C(T, e^*) \cap C_0 &= C(\tilde{T}, \tilde{e}^*), \\ C(T, e) \cap C_0^c &= C(\tilde{T}, \tilde{e}). \end{aligned}$$

Hence \tilde{T} satisfies Condition (II).

(ii) Suppose that $\tilde{T} = (0, t_1, \dots, t_n)$ satisfies Condition (II). Let $\tilde{e} = (e_0, e_1, \dots, e_n) \in \mathcal{E}_{n+1}$ be the label such that $C(\tilde{T}, \tilde{e}) = \emptyset$ and $C(\tilde{T}, \tilde{e}^*) = \emptyset$. Let $e = (e_1, \dots, e_n)$. If $e_0 = 0$, then $e_0^* = 1$ and

$$\begin{aligned} \emptyset &= C(\tilde{T}, \tilde{e}) = C(T, e) \cap C_0^c, \\ \emptyset &= C(\tilde{T}, \tilde{e}^*) = C(T, e^*) \cap C_0, \end{aligned}$$

which implies $S(T, e) = \emptyset$ by (2.13) and Lemma 2.3. If $e_0 = 1$, then $e_0^* = 0$ and, taking account of $(e^*)^* = e$, we have $S(T, e^*) = \emptyset$. In either case, $T = (t_1, \dots, t_n)$ satisfies Condition (I). □

for $n \geq 3$. Let a_j be the j -th column vector and a_{ij} be the (i, j) -component of M_n . If we delete the last column in M_n , then we get an upper triangular matrix $M_n(N_n)$ with diagonal elements 1. Hence

$$(2.17) \quad a_1, \dots, a_{N_n-1}$$

are linearly independent. Now we claim the following.

$$(2.18) \quad \text{If } c_1 a_1 + c_2 a_2 + \dots + c_{N_n} a_{N_n} = 0 \text{ with } c_k = 0 \text{ for some } k \neq N_n, \text{ then } c_{N_n} = 0.$$

Suppose that (2.18) is true. Then we see that $M_n(k)$ is invertible for every k . Indeed, if $k = N_n$, then $M_n(k)$ is invertible by (2.17). If $k \neq N_n$, then (2.17) and (2.18) show that the column vectors of $M_n(k)$ are linearly independent.

It remains to prove the assertion (2.18). It suffices to show that the relation

$$(2.19) \quad c_1 a_1 + c_2 a_2 + \dots + c_{N_n} a_{N_n} = 0 \text{ with } c_{N_n} = 1$$

implies that

$$(2.20) \quad c_k \neq 0 \text{ for } k = 1, \dots, N_n - 1.$$

Note that, by (2.17), all of c_1, \dots, c_{N_n-1} are determined uniquely by (2.19). Denote the row vector

$$c_n = (c_1, c_2, \dots, c_{N_n}).$$

Using the column vectors of M_{n-1} in place of those of M_n , we get the row vector c_{n-1} in place of c_n . Let

$$c_{n-1} = (\gamma_1, \gamma_2, \dots, \gamma_{N_{n-1}}), \quad \gamma_{N_{n-1}} = 1.$$

We write (2.19) componentwise:

$$(2.21) \quad c_1 a_{i1} + c_2 a_{i2} + \dots + c_{N_n} a_{iN_n} = 0, \quad i = 1, \dots, N_n - 1, \text{ and } c_{N_n} = 1.$$

For $i = N_{n-1} + 2, \dots, N_n - 1$, the relation between M_n and M_{n-1} in (2.16) shows that (2.21) implies

$$(2.22) \quad (c_{N_{n-1}+2}, \dots, c_{N_n}) = (\gamma_1, \dots, \gamma_{N_{n-1}}).$$

For $i = N_{n-1}$, (2.21) reduces to $c_{N_{n-1}} + c_{N_n} = 0$ by virtue of (2.16). Hence

$$(2.23) \quad c_{N_{n-1}} = -1.$$

For $i = 1, \dots, N_{n-1} - 1$, taking account of (2.16) and using (2.22) and (2.23), we have

$$(2.24) \quad (c_1, c_2, \dots, c_{N_{n-1}}) = (-\gamma_1, -\gamma_2, \dots, -\gamma_{N_{n-1}}).$$

It follows from (2.22) and (2.24) that

$$(2.25) \quad c_n = (-c_{n-1}, c_{N_{n-1}+1}, c_{n-1}).$$

We have

$$(2.26) \quad c_2 = (-1, -1, 1)$$

explicitly from M_2 . For $i = N_{n-1} + 1$, (2.21) reduces to

$$c_{N_{n-1}+1} + c_{N_{n-1}+2} + \dots + c_{N_n} = 0$$

by (2.16). Hence, noticing (2.22) and using (2.26) or (2.25) for $n - 1$ in place of n , we get

$$(2.27) \quad c_{N_{n-1}+1} = \begin{cases} 1 & \text{for } n = 3 \\ -\gamma_{N_{n-2}+1} & \text{for } n \geq 4. \end{cases}$$

Now, from (2.25) and (2.27) we see that each component of c_n is 1 or -1 . This proves (2.20). □

§ 3. Proof of Theorem 2

We prepare lemmas.

LEMMA 3.1. Let $t_i \in \mathbf{R}^d$, $i = 1, \dots, n + m$. Let $A = \{t_1, \dots, t_n\}$ and $B = \{t_{n+1}, \dots, t_{n+m}\}$.

$$(3.1) \quad \bigcap_{t_i \in A} C_{t_i} \subset \bigcup_{t_j \in B} C_{t_j}$$

if and only if

$$(3.2) \quad \max_{t_i \in A} d(t_i, x) \geq \min_{t_j \in B} d(t_j, x) \quad \text{for any } x \in \mathbf{R}^d.$$

We denote the relation (3.1) by $A < B$. This means

$$\left(\bigcap_{t_i \in A} C_{t_i}\right) \cap \left(\bigcap_{t_j \in B} C_{t_j}^c\right) = \emptyset.$$

Proof. Suppose (3.1). Let $x \in \mathbf{R}^d$. Let $r = \max_{t_i \in A} d(x, t_i)$. Then $d(x, t_i) \leq r$ for any $t_i \in A$, that is, $(r, x) \in \bigcap_{t_i \in A} C_{t_i}$. Hence $(r, x) \in \bigcup_{t_j \in B} C_{t_j}$. This means $d(x, t_j) \leq r$ for some $t_j \in B$. Hence (3.2) holds. Conversely, assume (3.2). Let $(r, x) \in \bigcap_{t_i \in A} C_{t_i}$. That means $d(x, t_i) \leq r$ for every $t_i \in A$.

It follows from (3.2) that there exists t_{j_0} such that $d(t_{j_0}, x) \leq r$. So, $(r, x) \in C_{t_{j_0}}$. □

LEMMA 3.2. *Let $1 \leq k \leq d + 1$. Let t_1, \dots, t_k , and t_{k+1}, \dots, t_{d+2} in \mathbf{R}^d be such that there is no hyperplane of co-dimension 1 containing $t_1, \dots, t_k, t_{k+1}, \dots, t_{d+1}$. Suppose that there exist positive constants p_1, \dots, p_k and q_{k+1}, \dots, q_{d+2} such that*

$$(3.3) \quad \sum_{i=1}^k p_i = \sum_{j=k+1}^{d+2} q_j = 1,$$

$$(3.4) \quad \sum_{i=1}^k p_i t_i = \sum_{j=k+1}^{d+2} q_j t_j.$$

Then at least one of the following holds:

$$(3.5) \quad \bigcap_{i=1}^k C_{t_i} \subset \bigcup_{j=k+1}^{d+2} C_{t_j} \quad (\text{that is } A \prec B),$$

$$(3.6) \quad \bigcup_{i=1}^k C_{t_i} \supset \bigcap_{j=k+1}^{d+2} C_{t_j} \quad (\text{that is } A \succ B),$$

where $A = \{t_1, \dots, t_k\}$ and $B = \{t_{k+1}, \dots, t_{d+2}\}$.

Proof. Let D be the $(d - 1)$ -dimensional sphere on which the points $t_1, \dots, t_k, t_{k+1}, \dots, t_{d+1}$ lie. Without loss of generality, we assume that the center of D is $O = (0, \dots, 0) \in \mathbf{R}^d$. Let r be the radius of D . Suppose that $|t_{d+2}| \leq r$. We will show that (3.5) holds. By Lemma 3.1 it is enough to show that, for any $x \in \mathbf{R}^d$,

$$(3.7) \quad \max_{i=1, \dots, k} d(t_i, x) - \min_{j=k+1, \dots, d+2} d(t_j, x) \geq 0.$$

Taking account of (3.3) and (3.4) we have

$$(3.8) \quad \min_{i=1, \dots, k} (t_i, x) \leq \sum_{i=1}^k p_i (t_i, x) = \sum_{j=k+1}^{d+2} q_j (t_j, x) \leq \max_{j=k+1, \dots, d+2} (t_j, x)$$

where (x, y) denotes inner product of \mathbf{R}^d . Let i_0 and j_0 be the elements which attain the minimum and the maximum in (3.8), respectively. Then

$$(3.9) \quad (t_{j_0} - t_{i_0}, x) \geq 0.$$

On the other hand,

$$(3.10) \quad \begin{aligned} & \max_{i=1, \dots, k} \{d(t_i, x)\}^2 - \min_{j=k+1, \dots, d+2} \{d(t_j, x)\}^2 \\ & \geq \{d(t_{i_0}, x)\}^2 - \{d(t_{j_0}, x)\}^2 \\ & = \{|t_{i_0}|^2 + |x|^2 - 2(t_{i_0}, x)\} - \{|t_{j_0}|^2 + |x|^2 - 2(t_{j_0}, x)\} \\ & = |t_{i_0}|^2 - |t_{j_0}|^2 + 2(t_{j_0} - t_{i_0}, x). \end{aligned}$$

(3.9) and the assumption $|t_{i_0}| = r \geq |t_{j_0}|$ imply that the last term of (3.10) is non-negative. So, (3.7) is proved.

Suppose that $|t_{a+2}| \geq r$. Then we prove that

$$(3.11) \quad \max_{j=k+1, \dots, d+2} d(t_j, x) - \min_{i=1, \dots, k} d(t_i, x) \geq 0 \quad \text{for any } x \in R^d,$$

which implies (3.6) by Lemma 3.1. In fact, let

$$(t_{i_0}, x) = \max_{i=1, \dots, k} (t_i, x), \quad (t_{j_0}, x) = \min_{j=k+1, \dots, d+2} (t_j, x).$$

Then

$$\begin{aligned} & \max_{j=k+1, \dots, d+2} \{d(t_j, x)\}^2 - \min_{i=1, \dots, k} \{d(t_i, x)\}^2 \\ & \geq \{d(t_{j_0}, x)\}^2 - \{d(t_{i_0}, x)\}^2 \\ & = |t_{j_0}|^2 - |t_{i_0}|^2 + 2(t_{i_0} - t_{j_0}, x) \geq 0 \end{aligned}$$

which is (3.11). □

LEMMA 3.3. *Let $t_1, \dots, t_{d+1} \in \mathbf{R}^d$. Suppose that no hyperplane of co-dimension 1 contains them and that any d vectors out of t_1, \dots, t_{d+1} are linearly independent. Let $t_{d+2} = 0$. Then the set $\{t_1, \dots, t_{d+1}, t_{d+2}\}$ is uniquely partitioned into two disjoint sets A, B such that $A \neq \emptyset, B \ni t_{d+2}$ and there exist positive constants p_i 's and q_j 's satisfying*

$$(3.12) \quad \sum_{t_i \in A} p_i t_i = \sum_{t_j \in B} q_j t_j,$$

$$(3.13) \quad \sum_{t_i \in A} p_i = \sum_{t_j \in B} q_j = 1.$$

Proof. Since t_1, \dots, t_{d+1} are linearly dependent, there exist constants c_1, \dots, c_{d+1} such that $(c_1, \dots, c_{d+1}) \neq (0, \dots, 0)$ and $\sum_{i=1}^{d+1} c_i t_i = 0$. Notice that $c_i \neq 0$ for any i by the assumption that any d out of t_1, \dots, t_{d+1} are linearly independent. Moreover, c_1, \dots, c_{d+1} are unique up to constant multiple. We have $\sum_{i=1}^{d+1} c_i \neq 0$, because, if it is zero, then $\sum_{i=1}^d c_i (t_i - t_{d+1}) = 0$ and t_1, \dots, t_{d+1} are on a hyperplane of co-dimension 1. So, we may assume that $\sum_{i=1}^{d+1} c_i > 0$. Let $A = \{t_i; c_i > 0\}$ and $B = \{t_i; c_i < 0\} \cup \{t_{d+2}\}$. Let $p_i = c_i$ for $c_i > 0$, $q_j = -c_j$ for $c_j < 0$, and $q_{d+2} = \sum_{i=1}^{d+1} c_i$. Then $\sum_{t_i \in A} p_i - \sum_{t_j \in B} q_j = 0$ and (3.12) holds. Multiplication of some constant yields (3.13). Uniqueness of A and B is obvious from this argument. □

COROLLARY 3.4. *Let $t_i \in \mathbf{R}^d, i = 1, \dots, d + 2$. Assume that no $d + 1$ points out of them are contained in a hyperplane of co-dimension 1 in \mathbf{R}^d .*

Then the set $\{t_1, \dots, t_{d+2}\}$ is partitioned into two disjoint non-empty sets A and B such that, for some $p_i > 0$ and $q_j > 0$,

$$(3.14) \quad \sum_{t_i \in A} p_i t_i = \sum_{t_j \in B} q_j t_j, \quad \sum_{t_i \in A} p_i = \sum_{t_j \in B} q_j = 1.$$

The partition is unique up to the naming of A and B .

Proof. Let $u_i = t_i - t_{d+2}$ and apply Lemma 3.3 to u_1, \dots, u_{d+2} . □

We call A, B in Corollary 3.4 the *natural partition* of $\{t_1, \dots, t_{d+2}\}$.

The corollary above is rephrased geometrically as follows. For a finite set $C = \{t_1, \dots, t_n\} \subset \mathbf{R}^d$, denote by \bar{C} the solid simplex having C as the set of vertices, that is,

$$\bar{C} = \left\{ \sum_{i=1}^n p_i t_i; \sum_{i=1}^n p_i = 1, p_i \geq 0, i = 1, \dots, n \right\}.$$

COROLLARY 3.5. *Let $t_i, i = 1, \dots, d + 2$, be as in Corollary 3.4. Then there are two disjoint non-empty sets A, B such that $A \cup B = \{t_1, \dots, t_{d+2}\}$, $A \cap B = \emptyset$, and $\bar{A} \cap \bar{B} \neq \emptyset$. The sets A, B are unique up to naming of A and B . The set $\bar{A} \cap \bar{B}$ consists of only one point.*

Combining Lemma 3.1 and Corollary 3.4, we get the following proposition.

PROPOSITION 3.6. *For any $T = (t_1, \dots, t_{d+2}) \in (\mathbf{R}^d)^{d+2}$ such that no $d + 1$ points out of t_1, \dots, t_{d+2} are contained in a hyperplane of co-dimension 1 in \mathbf{R}^d , there exists a label $e \in \mathcal{E}_{d+2}$ which satisfies $C(T, e) = \emptyset$.*

Now we deal with a set of $d + 3$ points in \mathbf{R}^d in order to discuss Condition (II). Consider a set $\Gamma = \{t_1, \dots, t_{d+3}\}$ in \mathbf{R}^d . Assume that Γ is non-degenerate in the sense that

$$(3.15) \quad \begin{array}{l} \text{no } d + 1 \text{ points out of } t_1, \dots, t_{d+3} \text{ are contained in} \\ \text{a hyperplane of co-dimension 1 in } \mathbf{R}^d. \end{array}$$

For each i , apply Corollary 3.4 to $\Gamma \setminus \{t_i\}$ and let

$$(3.16) \quad \Gamma \setminus \{t_i\} = A_i \cup B_i$$

be the natural partition of $\Gamma \setminus \{t_i\}$. By Lemma 3.2, at least one of $A_i \prec B_i$ and $A_i \succ B_i$ holds.

Let $i \neq j$. We say that t_i and t_j *link together* if the restrictions to $\Gamma \setminus \{t_i, t_j\}$ of the natural partitions of $\Gamma \setminus \{t_i\}$ and $\Gamma \setminus \{t_j\}$ coincide.

LEMMA 3.7. *Let $i \neq j$ and suppose that t_i and t_j link together. Let A_i, B_i and A_j, B_j be the natural partitions of $\Gamma \setminus \{t_i\}$ and $\Gamma \setminus \{t_j\}$, respectively. If*

$$(3.17) \quad A_i < B_i, \quad A_j > B_j, \quad A_i \cap A_j \neq \emptyset,$$

then $T = (t_1, \dots, t_{d+3})$ satisfies Condition (II).

Proof. Without loss of generality we assume $i = 1, j = 2$. Keeping $A_1 \cap A_2 \neq \emptyset$ in mind, we can find A and B satisfying $A \cup B = \Gamma \setminus \{t_1, t_2\}$ and $A \cap B = \emptyset$ such that one of the following four conditions holds:

- (a) $A_1 = A \cup \{t_2\}, B_1 = B, A_2 = A \cup \{t_1\}, B_2 = B;$
- (b) $A_1 = A \cup \{t_2\}, B_1 = B, A_2 = A, B_2 = B \cup \{t_1\};$
- (c) $A_1 = A, B_1 = B \cup \{t_2\}, A_2 = A, B_2 = B \cup \{t_1\};$
- (d) $A_1 = A, B_1 = B \cup \{t_2\}, A_2 = A \cup \{t_1\}, B_2 = B.$

We may assume that $A = \{3, \dots, k\}$ and $B = \{k + 1, \dots, d + 3\}$ where $3 \leq k \leq d + 3$ ($B = \emptyset$ if $k = d + 3$).

Case (a). We have

$$C(T, e) = \emptyset \quad \text{with} \quad e = (e_1, \underbrace{1, 1, \dots, 1}_{k-1}, \underbrace{0, \dots, 0}_{d+3-k}),$$

$$C(T, e') = \emptyset \quad \text{with} \quad e = (0, e'_2, \underbrace{0, \dots, 0}_{k-2}, \underbrace{1, \dots, 1}_{d+3-k}),$$

whatever e_1 and e'_2 are. Letting $e_1 = 1$ and $e'_2 = 0$, we get a complementary pair e, e' . Hence T satisfies Condition (II).

Case (b). We have

$$C(T, e) = \emptyset \quad \text{with} \quad e = (e_1, \underbrace{1, 1, \dots, 1}_{k-1}, \underbrace{0, \dots, 0}_{d+3-k})$$

$$C(T, e') = \emptyset \quad \text{with} \quad e' = (1, e'_2, \underbrace{0, \dots, 0}_{k-2}, \underbrace{1, \dots, 1}_{d+3-k}),$$

whatever e_1 and e'_2 are. Letting $e_1 = 0$ and $e'_2 = 0$, we obtain a complementary pair.

Cases (c) and (d) are treated similarly to (a) and (b), respectively. \square

Remark. Another sufficient condition for T to satisfy Condition (II) is that there exists i such that $A_i < B_i$ and $A_i > B_i$. But we will not use this condition.

We see easily that, to prove Theorem 2, it is enough to prove it for $m = d + 3$. In order to prove it for $m = d + 3$ under the condition that

$\{t_1, \dots, t_{d+3}\}$ are non-degenerate in the sense of (3.15), we will show the existence of i and j which satisfy the condition of Lemma 3.7. Applying Corollary 3.4 to $\Gamma \setminus \{t_1\}$ and $\Gamma \setminus \{t_2\}$, we have

$$(3.18) \quad \sum_{k=1}^{d+3} c_{1k} t_k = 0 \quad \text{with} \quad c_{11} = 0, \quad \sum_{k=1}^{d+3} c_{1k} = 0, \quad c_{1k} \neq 0 \quad (k \neq 1),$$

and

$$(3.19) \quad \sum_{k=1}^{d+3} c_{2k} t_k = 0 \quad \text{with} \quad c_{22} = 0, \quad \sum_{k=1}^{d+3} c_{2k} = 0, \quad c_{2k} \neq 0 \quad (k \neq 2).$$

The representations are unique up to constant multiplication. We assume $c_{12} > 0$ and $c_{21} > 0$. We set, for $i \geq 3$,

$$(3.20) \quad \begin{cases} \lambda_i = c_{2i}/c_{1i} \\ c_{ik} = c_{2k} - \lambda_i c_{1k}. \end{cases}$$

Then we get the relations for $\Gamma \setminus \{i\}$, $i = 3, \dots, d + 3$, that

$$(3.21) \quad \sum_{k=1}^{d+3} c_{ik} t_k = 0 \quad \text{with} \quad c_{ii} = 0, \quad \sum_{k=1}^{d+3} c_{ik} = 0.$$

Obviously we have, for $i \geq 3$,

$$(3.22) \quad \begin{cases} c_{i1} = c_{21} > 0 \\ c_{i2} = -\lambda_i c_{12} \\ c_{ik} = c_{1k}(\lambda_k - \lambda_i) = c_{2k}(1 - \lambda_i/\lambda_k), \quad \text{for } k \geq 3. \end{cases}$$

Moreover we see that λ_i , $i = 3, \dots, d + 3$, are distinct and $c_{ik} \neq 0$ for $k \neq i$, because, if otherwise, some $d + 1$ points in Γ are contained in a hyperplane of co-dimension 1. Without loss of generality we assume $\lambda_i < \lambda_{i+1}$ for $i = 3, \dots, d + 3$. Let

$$(3.23) \quad L_- = \{i \geq 3; \lambda_i < 0\}, \quad L_+ = \{j \geq 3; \lambda_j > 0\}.$$

We see that $c_{i2} > 0$ for $i \in L_-$ and $c_{j2} < 0$ for $j \in L_+$. Using the relations in (3.22) and noticing that the natural partition of $\Gamma \setminus \{i\}$ is made according to the signs of c_{ik} , we get

LEMMA 3.8. *If both i and $i + 1$ belong to L_- , then t_i and t_{i+1} link together. If both j and $j + 1$ belong to L_+ , then t_j and t_{j+1} link together.*

Now we assume that

$$(3.24) \quad L_- \neq \emptyset \quad \text{and} \quad L_+ \neq \emptyset.$$

The case without this assumption will be treated later. Let

$$L_- = \{3, 4, \dots, r\}, \quad L_+ = \{r + 1, \dots, d + 3\}.$$

Then we get the following lemma.

LEMMA 3.9. *The following pairs link together:*

- (1) t_1 and t_3 ;
- (2) t_1 and t_{d+3} ;
- (3) t_2 and t_r ;
- (4) t_2 and t_{r+1} .

Proof. Again use (3.22) and the fact that the natural partition of $\Gamma \setminus \{t_i\}$ is decided by the signs of c_{ik} , $k \neq i$. [

It follows from Lemma 3.7 that, if L_- or L_+ contains adjacent elements i, j satisfying (3.17), then $T = (t_1, \dots, t_{d+3})$ satisfies Condition (II). So, let us consider the situation that neither L_- nor L_+ contains adjacent elements satisfying (3.17). In the naming of A_i, B_i in the natural partition (3.16) of $\Gamma \setminus \{t_i\}$, we make $A_i \ni t_1$ for $i = 2, 3, \dots, d + 3$, and $A_1 \ni t_2$. Recalling that the natural partitions are made by the signs of c_{ik} , we see that $t_2 \in A_i$ for $i \in L_-$ and that $t_2 \in B_i$ for $i \in L_+$. We note that $A_{i-1} \cap A_i \neq \emptyset$ for $i \in L_- \cup L_+$. Hence Lemma 3.7 yields that we have one of the following situations:

- (1) $A_i < B_i$ for $i \in L_- \cup L_+$;
- (2) $A_i < B_i$ for $i \in L_-$ and $A_j > B_j$ for $j \in L_+$;
- (3) $A_i > B_i$ for $i \in L_- \cup L_+$;
- (4) $A_i > B_i$ for $i \in L_-$ and $A_j < B_j$ for $j \in L_+$.

We will prove that in each case at least one of pairs (1), (2), (3), (4) of Lemma 3.9 satisfies the condition of Lemma 3.7.

Case (1). If $A_1 < B_1$, then t_1 and t_{d+3} satisfy the condition of Lemma 3.7, because $A_{d+3} < B_{d+3}$ and $A_1 \cap B_{d+3} \ni t_2$. If $A_1 > B_1$, then t_1 and t_3 satisfy (3.17), since $A_3 < B_3$ and $A_1 \cap A_3 \ni t_2$.

Case (2). If $A_2 < B_2$, then t_2 and t_{r+1} satisfy (3.17), since $A_{r+1} > B_{r+1}$ and $A_2 \cap A_{r+1} \ni t_1$. If $A_2 > B_2$, then t_2 and t_r satisfy (3.17), because $A_r < B_r$ and $A_2 \cap A_r \ni t_1$.

Case (3). Similarly to case (1), the pair t_1, t_3 or the pair t_1, t_{d+3} satisfies the condition of Lemma 3.7.

Case (4). Similar to case (2). The pair t_2, t_7 or the pair t_2, t_{7+1} satisfies (3.17).

Thus, under the assumption (3.24), $T = (t_1, \dots, t_{d+3})$ satisfies Condition (II).

Let us consider the case where L_- or L_+ is empty.

LEMMA 3.10. *If $L_- = \emptyset$, then each of the following pairs links together:*

$$t_1, t_2 \ ; \ t_1, t_{d+3} \ ; \ t_2, t_3 .$$

If $L_+ = \emptyset$, then each of the following pairs links together.

$$t_1, t_2 \ ; \ t_1, t_3 \ ; \ t_2, t_{d+3} .$$

Proof. Suppose that $L_- = \emptyset$. Let

$$A = \{i \geq 3; c_{1i} > 0, c_{2i} > 0\}, \quad B = \{i \geq 3; c_{1i} < 0, c_{2i} < 0\} .$$

Then $A \cup B = \{3, \dots, d + 3\}$, and hence t_1 and t_2 link together. If $L_+ = \emptyset$, then letting

$$A = \{i \geq 3; c_{1i} > 0, c_{2i} < 0\}, \quad B = \{i \geq 3; c_{1i} < 0, c_{2i} > 0\} ,$$

we see that $A \cup B = \{3, \dots, d + 3\}$ and that t_1 and t_2 link together. The other assertions are proved in the same way by use of (3.22). \square

As before we make the naming of A_i, B_i in the natural partition (3.16) in such a way that $A_i \ni t_1$ for $i = 2, 3, \dots, d + 3$, and $A_1 \ni t_2$. We have $t_2 \in A_i$ for $i \in L_-$ and $t_2 \in B_i$ for $i \in L_+$.

Suppose that $L_- = \emptyset$. If L_+ contains adjacent elements i, j satisfying (3.17), then, by Lemmas 3.7 and 3.8, $T = (t_1, \dots, t_{d+3})$ satisfies Condition (II). So, suppose that L_+ does not contain adjacent elements satisfying (3.17). Then we have one of the following:

- (1) $A_i < B_i$ for $i \geq 3$, (2) $A_i > B_i$ for $i \geq 3$.

Case (1). If $A_1 < B_1$, then t_1, t_{d+3} satisfy condition of Lemma 3.7 since $A_{d+3} < B_{d+3}$ and $A_1 \cap B_{d+3} \ni t_2$. If $A_2 > B_2$, then t_2, t_3 satisfy (3.15), since $A_3 < B_3$ and $A_2 \cap A_3 \ni t_1$. In the remaining case, suppose that $A_1 > B_1$ and $A_2 < B_2$. If $c_{1k} > 0$ for some $k \geq 3$, then $c_{2k} > 0$ and $A_1 \cap A_2 \ni t_k$. If $c_{1k} < 0$ for some $k \geq 3$, then $c_{2k} < 0$ and $B_1 \cap B_2 \ni t_k$. So, t_1, t_2 satisfy the condition of Lemma 3.7. We made use of Lemma 1.10.

Case (2). If $A_1 > B_1$, then t_1, t_{d+3} satisfy the condition of Lemma 3.7,

since $A_{d+3} \succ B_{d+3}$ and $A_1 \cap B_{d+3} \ni t_2$. If $A_2 \prec B_2$, then t_2, t_3 satisfy the condition, because $A_3 \succ B_3$ and $A_2 \cap A_3 \ni t_1$. If $A_1 \prec B_1$ and $A_2 \succ B_2$, then t_1, t_2 satisfy the condition by same reason as case (1).

Suppose that $L_+ = \emptyset$. Then we can make similar discussion. Namely, suppose that L_- does not contain adjacent elements satisfying (3.15). Then (1) or (2) holds. In either case we can find the following pair satisfying the condition of Lemma 3.7.

Case (1). If $A_1 \succ B_1$, then t_1, t_2 . If $A_2 \succ B_2$, then t_2, t_{d+3} . If $A_1 \prec B_1$ and $A_2 \prec B_2$, then t_1, t_2 .

Case (2). If $A_1 \prec B_1$, then t_1, t_3 . If $A_2 \prec B_2$, then t_2, t_{d+3} . If $A_1 \succ B_1$ and $A_2 \succ B_2$, then t_1, t_2 .

Therefore, in the case that L_- or L_+ is empty, $T = (t_1, \dots, t_{d+3})$ satisfies Condition (II). This finishes proof of Theorem 2 for $m = d + 3$ under the assumption that t_1, \dots, t_{d+3} are non-degenerate in the sense of (3.15).

If $d + 1$ points are on a hyperplane of co-dimension 1 and no $d + 2$ points are on a hyperplane of co-dimension 1, then we can apply Lemma 3.2 again and similar argument can be made. If $d + 2$ points are on a hyperplane of co-dimension 1, then, taking account of the remark to Definition 2.1, we see that the situation is reduced to $(d - 1)$ -dimensional case.

§ 4. Concluding remarks

In order to construct an example mentioned in Remark 1 of §1, we prepare a lemma.

LEMMA 4.1. *Let $T = (t_1, \dots, t_{d+2}) \in (\mathbb{R}^d)^{d+2}$, where t_1, \dots, t_{d+2} are distinct and no $d + 1$ points of them are on a hyperplane of codimension 1. Let D be the $(d - 1)$ -dimensional sphere on which the points t_1, \dots, t_{d+1} lie. Assume that t_{d+2} is situated inside of D and, moreover, that $\bar{A} \cap \bar{B} \neq \emptyset$ for $A = \{t_{d+1}, t_{d+2}\}$ and $B = \{t_1, \dots, t_d\}$, using the notation introduced before Corollary 3.5. Then there is no label e of size $d + 2$ such that $C(T, e) = C(T, e^*) = \emptyset$.*

Proof. For $e = (e_1, \dots, e_{d+2}) \in \mathcal{E}_{d+2}$, let $A_e = \{t_i; e_i = 1\}$ and $B_e = \{t_i; e_i = 0\}$. In order to prove our assertion, it is enough to consider only e such that $A_e \ni t_{d+2}$. We separate our discussion into three cases.

- (a) A_e and B_e give the natural partition of $\{t_1, \dots, t_{d+2}\}$.

- (b) Either A_e or B_e is a one point set.
- (c) The remaining case.

Case (a). We have $A_e = A$ and $B_e = B$ by the assumption. From the proof of Lemma 3.2 we see that $A \succ B$. We do not have $A \prec B$. In fact, we can find a $(d - 1)$ -dimensional sphere D' such that $D' \supset B$ and that the points t_{d+1}, t_{d+2} and are inside of D' . Let x_0 be the center of D' . Then

$$\max_{t_i \in A} d(t_i, x_0) < \min_{t_j \in B} d(t_j, x_0).$$

It follows from Lemma 3.1 that $A \prec B$ does not hold. Hence $C(T, e) \neq \emptyset$.

Case (b). If A_e consists of only one point t_i , then $C(T, e)$ contains a point (ε, t_i) for sufficiently small $\varepsilon > 0$. If B_e consists of only one point, then $C(T, e^*) \neq \emptyset$.

Case (c). The sets A_e, B_e do not give the natural partition of $\{t_1, \dots, t_{d+2}\}$. So we have $\bar{A}_e \cap \bar{B}_e = \emptyset$ by the uniqueness of the natural partition. We can find a $(d - 1)$ -dimensional sphere D' such that $D' \supset B_e$ and all the points of A_e are inside of D' . Then $C(T, e) \neq \emptyset$, since

$$\max_{t_i \in A_e} d(t_i, x_0) < \min_{t_j \in B_e} d(t_j, x_0)$$

for the center x_0 of D' . □

EXAMPLE 4.2. Let $T_0 = (t_1, \dots, t_{d+1}) \in (\mathbf{R}^d)^{d+1}$ and $t_{d+2} = 0$. We choose and fix T_0 in such a way that $T = (t_1, \dots, t_{d+1}, t_{d+2})$ satisfies the assumption in Lemma 4.1. It follows from Lemmas 2.5 and 4.1 that $S(T, e) \neq \emptyset$ for every $e \in \mathcal{E}_{d+1}$. Let μ be a measure on $E = \mathbf{R}_+ \times \mathbf{R}^d$ satisfying (1.3) such that $\mu(S(T_0, e)) > 0$ for every $e \in \mathcal{E}_{d+1}$. Let us define $\tilde{\mu}$ in the following way. We make $\tilde{\mu} = \mu$ on $E \setminus \bigcup_{i=1}^{d+1} S_{t_i}$. First notice that μ satisfies the consistency condition (2.11) for $n = d + 1$. Using the notations in the proof of Lemma 2.2, let A be the matrix $M_{d+1}(2^{d+1} - 1)$ and b be the vector in (2.11). Let \mathfrak{c} be the $(2^{d+1} - 2)$ -vector every component of which is $\mu(\bigcap_{i=1}^{d+1} S_{t_i})$. Choose $\varepsilon \neq 0$ such that every component of the solution x of

$$Ax = \mathfrak{b} - (1 + \varepsilon)\mathfrak{c}$$

is positive. It suffices to make $|\varepsilon|$ small enough. Now let

$$\tilde{\mu}\left(\bigcap_{i=1}^{d+1} S_{t_i}\right) = (1 + \varepsilon)\mu\left(\bigcap_{i=1}^{d+1} S_{t_i}\right)$$

and let $\tilde{\mu}(S(T_0, e))$ for $e \in \mathcal{E}_{d+1} \setminus \{(1, \dots, 1)\}$ be given by the solution x . There exists a measure $\tilde{\mu}$ with these $\tilde{\mu}(S(T_0, e))$, $e \in \mathcal{E}_{d+1}$. We have $\tilde{\mu}(S_t) < \infty$ for all $t \in \mathbf{R}^d$. Let $\{X(t)\}$ and $\{\tilde{X}(t)\}$ be the Chentsov type $S\alpha S$ random fields associated with μ and $\tilde{\mu}$, respectively. From the construction

$$\tilde{\mu}(\tilde{S}(T_0, e)) = \mu(\tilde{S}(T_0, e)) \quad \text{for all } e \in \mathcal{E}_{d+1} \setminus \{(1, \dots, 1)\}.$$

It follows that $(X(t_1), \dots, X(t_{d+1}))$ and $(\tilde{X}(t_1), \dots, \tilde{X}(t_{d+1}))$ have different distributions but they have common d -dimensional marginal distributions.

EXAMPLE 4.3. An interesting problem is whether there are two measures μ and $\tilde{\mu}$ satisfying (1.3) such that the Chentsov type $S\alpha S$ random fields $\{X(t)\}$ and $\{\tilde{X}(t)\}$ on \mathbf{R}^d associated with μ and $\tilde{\mu}$, respectively, have identical d -dimensional distributions but different $(d+1)$ -dimensional distributions. We do not know the answer to this problem for general d yet. But, in case $d=1$, we can construct such measures.

Let $E = \mathbf{R}_+ \times \mathbf{R}^1$. Let μ be such that $\mu(S_t) = \mu(S_{-t}) < \infty$ and $\mu(S_t)$ is a continuous increasing function of $t > 0$. Suppose, further, that μ is mutually absolutely continuous with the Lebesgue measure. Let $\tilde{\mu}$ be a measure concentrated on $\mathbf{R}_+ \times \{0\}$ such that

$$\tilde{\mu}(S_t) = \tilde{\mu}(S_t \cap (\mathbf{R}_+ \times \{0\})) = \mu(S_t).$$

Then $\{X(t)\}$ and $\{\tilde{X}(t)\}$ have common 1-dimensional distributions. Let $0 < t_1 < t_2$. Then $\mu(S_{t_1} \cap S_{t_2}^c) > 0$ but $\tilde{\mu}(S_{t_1} \cap S_{t_2}^c) = 0$, which implies that $(X(t_1), X(t_2))$ and $(\tilde{X}(t_1), \tilde{X}(t_2))$ have different distributions.

Our technique in this paper works in finding determinism of random fields on \mathbf{R}^d of a similar sort.

THEOREM 4.4. *Let μ be a measure on $\mathbf{R}_+ \times \mathbf{R}^d$ satisfying $\mu(C_t) < \infty$ for every $t \in \mathbf{R}^d$ and let $Y(\cdot)$ be the $S\alpha S$ random measure associated with μ . Let*

$$X(t) = Y(C_t) \quad \text{for } t \in \mathbf{R}^d.$$

Then, for any $n > d$, any n -dimensional distribution of $\{X(t)\}$ is determined by its d -dimensional marginal distributions.

Proof. The non-degenerate case is dealt with Proposition 3.6 and Lemma 2.2. The degenerate case is obvious. \square

Finally we remark that, if μ is invariant under translation in \mathbf{R}^d ,

then $\{X(t); t \in \mathbf{R}^d\}$ in Theorem 4.4 is a homogeneous random field constructed geometrically.

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