Can. J. Math., Vol. XXXII, No. 3, 1980, pp. 511–530

SEMIGROUPS WITH QUASI-ZEROES

S. A. RANKIN AND C. M. REIS

1. Introduction. Let S be a semigroup. An element $a \in S$ is said to be a *left quasi-zero* if $\langle a \rangle x \cap \langle a \rangle \neq \emptyset$ for all $x \in S$, where $\langle a \rangle$ denotes the cyclic sub-semigroup of S generated by a. In a recent study [**6**] of semigroups with a maximum right congruence, such elements proved to be useful in providing characterizations of these semigroups. Left quasi-zeroes have appeared in the literature under different names in a variety of situations. In the context of semigroup radicals, left quasi-zeroes are called right quasi-regular elements, where an element is defined to be right quasi-regular if it is not a left identity for any right congruence other than the universal congruence (see [**4**], [**5**], [**2**], [**7**], and [**8**]). In quite a different context, monoids having a left quasi-zero which is a right unit but not a left unit have been studied by M. Demlová [**1**].

The semigroups with left quasi-zeroes studied in [**6**] were of the following kinds: (i) S not right simple but with right invertible elements; (ii) $S^2 = S$, with no right invertible elements; (iii) $S^2 \neq S$; and (iv) S right simple. Of semigroups with a maximum right congruence, a type (i) semigroup is an ideal extension of a periodic semigroup consisting entirely of left quasi-zeroes by a right simple semigroup with zero; a type (ii) is the two element left zero semigroup; a type (iii) semigroup is nil cyclic; and a type (iv) semigroup is a cyclic group of prime order.

In this paper semigroups with one or two-sided quasi-zeroes are studied (an element $a \in S$ is a right quasi-zero if $x\langle a \rangle \cap \langle a \rangle \neq \emptyset$ for all $x \in S$, and an element which is both a left and a right quasi-zero is called a two-sided quasi-zero). In particular, the structure of semigroups which consist entirely of left quasi-zeroes is described, thus improving the result obtained in [**6**] for type (i) semigroups with a maximum right congruence.

Throughout this paper, $\mathbf{Z}$ will denote the group of integers, $\mathbf{N}$ the cyclic semigroup of natural numbers, $\mathbf{N}^0$ the cyclic monoid of non-negative integers.

2. Aperiodic left quasi-zeroes. The theory of semigroups with aperiodic left quasi-zeroes rests in large measure on the following

(2.1) THEOREM. *Let S be a semigroup and $a \in S$ an aperiodic left quasi-*

Received October 13, 1977 and in revised form September 28, 1979. This work has been supported by grants A8218 and A4596 by the NRC to the respective authors.

zero. Then there exists a homomorphism $\psi_a: S \to Z$ *such that* $\psi_a(a) = 1$, *whence* $\psi_a(S) = \mathbf{N}, \mathbf{N}^0$ *or* $\mathbf{Z}$.

Proof. For each $x \in S$, $a^i x = a^j$ for some $i, j \in \mathbf{N}$. Suppose that $a^k x = a^l$ for $k, l \in \mathbf{N}$. We may assume that $k \geqq i$. Then

$$a^{k-i}a^i x = a^{k-i}a^j = a^l$$

and since a is aperiodic, $k - i + j = l$ or $l - k = j - i$. Define $\psi_a(x) = j - i$. It is easily seen that ψ_a is a homomorphism and that $\psi_a(a) = 1$, whence $\mathbf{N} \subset \psi_a(S)$. Thus $\psi_a(S) = \mathbf{N}, \mathbf{N}^0$ or $\mathbf{Z}$.

The dual result holds for right quasi-zeroes and we shall denote the corresponding homomorphism by ψ^a.

The above theorem establishes the existence of a homomorphism ψ_a from S to $\mathbf{Z}$. The next result shows that ψ_a is "essentially" the only homomorphism from S to $\mathbf{Z}$.

(2.2) THEOREM. *Let S be a semigroup with an aperiodic left quasi-zero a and let $\psi: S \to \mathbf{Z}$ be any homomorphism. Then $\psi = \psi(a)\psi_a$.*

Proof. For $x \in S$, $a^i x = a^j$ where $i, j \in \mathbf{N}$. Thus

$$\psi_a(x) = j - i \text{ and}$$

$$i\psi(a) + \psi(x) = j\psi(a)$$

and so

$$\psi(x) = (j - i)\psi(a) = \psi(a)\psi_a(x).$$

(2.3) COROLLARY. *Let a be an aperiodic quasi-zero of S. Then $\psi_a = \psi^a$.*

Proof. $\psi^a = \psi^a(a)\psi_a = \psi_a$ since $\psi^a(a) = 1$.

(2.4) COROLLARY. *Let $a_1, a_2,, \ldots, a_n$ be aperiodic left quasi-zeroes of S. Then*

$$\psi_{a_1}(a_2)\psi_{a_2}(a_3) \ldots \psi_{a_n}(a_1) = 1.$$

Proof. For aperiodic left quasi-zeroes a and b we have $\psi_a = \psi_a(b)\psi_b$. Thus if $c \in S$,

$$\psi_a(c) = \psi_a(b)\psi_b(c)$$

whence

$$\psi_{a_1}(a_2)\psi_{a_2}(a_3) \ldots \psi_{a_n}(a_1) = \psi_{a_1}(a_1) = 1.$$

We observe that if $\langle a \rangle$ is a right ideal of S then a is a left quasi-zero. As an immediate consequence of this we have

(2.5) LEMMA. *The only (left) quasi-zero of* $\mathbf{N}$ *and* $\mathbf{N}^0$ *is* 1.

https://doi.org/10.4153/CJM-1980-040-x Published online by Cambridge University Press

Proof. By the remark above, 1 is a left quasi-zero. Suppose $n \in \mathbf{N}^0$ is a left quasi-zero. Then $nk + 1 = nl$ for some $k, l \in \mathbf{N}$ whence $n = 1$.

(2.6) LEMMA. *If G is a group, then $a \in G$ is a left quasi-zero (aperiodic or periodic) if and only if a generates G.*

Proof. Suppose $a \in G$ is a left quasi-zero. Then for $x \in G$, $a^i x = a^j$ for some $i, j \in N$, whence $x = a^{j-i}$. Thus a generates G. Conversely, if a is a generator of G, then for $x \in G$, $x = a^i$ where $i \in \mathbf{Z}$. Thus for large enough $k \in \mathbf{N}$, $k + i \in \mathbf{N}$ and $a^k x = a^{k+i}$ whence a is a (left) quasi-zero.

(2.7) COROLLARY. *The (left) quasi-zeroes of $\mathbf{Z}$ are ± 1.*

It is appropriate at this point to ask whether periodic and aperiodic left quasi-zeroes can coexist. It is clear that left quasi-zeroes are preserved under epimorphisms and that if a is an aperiodic left quasi-zero and x a periodic element, $\psi_a(x) = 0$. These remarks in conjunction with Lemma 2.5 and Corollary 2.7 yield

(2.8) COROLLARY. *A semi-group cannot contain both periodic and aperiodic left quasi-zeroes.*

(2.9) COROLLARY. *Let a and b be aperiodic left quasi-zeroes. Then*

$$\psi_a(ab) = \psi_b(ab) \in \{0, 2\}$$

and thus ab is not a left quasi-zero. In particular, a^n is a left quasi-zero only for $n = 1$.

Proof. By Corollary 2.4, $\psi_a(b)\psi_b(a) = 1$ whence the result.

Thus the set of all left quasi-zeroes in a semigroup with aperiodic left quasi-zeroes is a mutant [**3**]. However, the product of more than two left quasi-zeroes can be a left quasi-zero as $\mathbf{Z}$ illustrates.

We have seen that if a is an aperiodic left quasi-zero of S then S has $\mathbf{N}$, $\mathbf{N}^0$ or $\mathbf{Z}$ as a homomorphic image. For any left zero semigroup E, $E \times \mathbf{N}$, $E \times \mathbf{N}^0$ and $E \times \mathbf{Z}$ are examples of semigroups with aperiodic left quasi-zeroes having the respective homomorphic images $\mathbf{N}$, $\mathbf{N}^0$ and $\mathbf{Z}$.

We now give an example of a semigroup generated by an aperiodic left quasi-zero and an aperiodic right quasi-zero having $\mathbf{Z}$ as a homomorphic image but which does not contain a copy of $\mathbf{Z}$.

(2.10) *Example.* Let $\mathscr{B}_0 = \langle x_0, \bar{x}_0 | \, x_0\bar{x}_0 = 1 \rangle$ be the bicyclic semigroup. x_0 is an aperiodic left quasi-zero and $\bar{x}_0$ an aperiodic right quasi-zero. Since $\psi_{x_0}(\bar{x}_0) = -1$,

$$\psi_{x_0}(\mathscr{B}_0) = \mathbf{Z}.$$

However, $\mathscr{B}_0$ does not contain a copy of $\mathbf{Z}$. The left quasi-zeroes of $\mathscr{B}_0$

are exactly the elements $\bar{x}_0^{i-1}x_0^{\ i}$ for $i \in \mathbf{N}$. For

$$\psi_{x_0}(\bar{x}_0^{\ i}x_0^{\ j}) = \pm 1 \Leftrightarrow i - j = \pm 1$$

whence

$$i = j \pm 1.$$

Consider first elements of the form $\bar{x}_0^{\ i}x_0^{\ i+1}$. For $k \in \mathbf{N}$,

$$(\bar{x}_0^{\ i}x_0^{\ i+1})^k = \bar{x}_0^{\ i}x_0^{\ i+k}$$

and so

$$(\bar{x}_0^{\ i}x_0^{\ i+1})^k(\bar{x}_0^{\ r}x_0^{\ s}) = (\bar{x}_0^{\ i}x_0^{\ i+1})^l$$

where $r, s \in N$, $k > r$ and $l = k - r + j$. Thus $\bar{x}_0^{\ i}x_0^{\ i+1}$ is a left quasi-zero of $\mathscr{B}_0$. On the other hand

$$(\bar{x}_0^{\ i}x_0^{\ i-1})^k = \bar{x}_0^{\ i+k}x_0^{\ i-1}$$

and so $\bar{x}_0^{\ i}x_0^{\ i-1}$ is not a left quasi-zero. It is however a right quasi-zero.

(2.11) *Definition*. An aperiodic left quasi-zero $a \in S$ is said to be a *left quasi-zero of type* $\mathbf{N}$, $\mathbf{N}^0$ *or* $\mathbf{Z}$ and referred to as an $\mathbf{N}$-*left quasi-zero*, an $\mathbf{N}^0$-*left quasi-zero* or a $\mathbf{Z}$-*left quasi-zero*, if $\psi_a(S) = \mathbf{N}, \mathbf{N}^0$ or $\mathbf{Z}$ respectively.

In Corollary 2.8 it was observed that a semigroup cannot contain both periodic and aperiodic left quasi-zeroes. We may be more precise and state the following

(2.12) THEOREM. *Let S be a semigroup with an aperiodic left quasi-zero. Then all left quasi-zeroes are of the same type.*

Proof. Let a and b be left quasi-zeroes of S. By Theorem 2.2,

$$\psi_a = \psi_a(b)\psi_b = \pm\psi_b.$$

(2.13) *Definition*. Let S be a semigroup with a homomorphism $\psi:S \to \mathbf{Z}$ such that each homomorphism $\chi:S \to \mathbf{Z}$ factors through ψ. Then S is said to be a *semigroup of type* $\mathbf{N}$, $\mathbf{N}^0$ or $\mathbf{Z}$ if $\psi(S) = \mathbf{N}, \mathbf{N}^0$ or $\mathbf{Z}$ respectively.

Thus if S has an aperiodic left quasi-zero of type $\mathbf{N}$, $\mathbf{N}^0$ or $\mathbf{Z}$, then S is a semigroup of type $\mathbf{N}$, $\mathbf{N}^0$ or $\mathbf{Z}$ respectively. In fact, we can characterize semigroups with aperiodic left quasi-zeroes in terms of homomorphisms.

(2.14) THEOREM. *A semigroup S has an aperiodic left quasi-zero if and only if S has the following property: there exists a homomorphism $\psi:S \to \mathbf{Z}$ such that for each homomorphism $\chi:S \to T$, T any semigroup, there exists $a \in T$ with the property that for $x \in S$,*

$$a^n\chi(x) = a^{\psi(x)+n}$$

for sufficiently large $n \in \mathbf{N}$.

https://doi.org/10.4153/CJM-1980-040-x Published online by Cambridge University Press

Proof. Suppose S has an aperiodic left quasi-zero b and let $\psi = \psi_b$. Then for $x \in S$, $b^i x = b^j$ for some $i, j \in \mathbf{N}$ and $\psi_b(x) = j - i$. If $\chi:S \to T$ is a homomorphism choose $a = \chi(b) \in T$. Then

$$a^i\chi(x) = a^j = a^{j-i+i} = a^{\psi(x)+i}.$$

Conversely, suppose S has the property described above. Consider $1_S:S \to S$. Then there exists $a \in S$ such that for $x \in S$,

$$a^n x = a^{\psi(x)+n}$$

for sufficiently large n. Thus a is a left quasi-zero and $\psi(a) = 1$ whence a is aperiodic.

3. Periodic left quasi-zeroes.

(3.1) *Definition.* A periodic left quasi-zero is said to be *of type n* or an *n-left quasi-zero* if it is of period n, i.e., the group ideal of the cyclic subsemigroup generated by the left quasi-zero is a cyclic group of order n.

We have the following result which is entirely analogous to the aperiodic case:

(3.2) THEOREM. *Let a be an n-left quasi-zero of a semigroup S. Then there exists an epimorphism $\psi_a:S \to Z_n$ for which $\psi_a(a) = [1]_n$.*

We remark that a semigroup S has an n-left quasi-zero a if and only if S has a minimal right ideal which is a cyclic group of order n. The following theorem is therefore a consequence of well-known results.

(3.3) THEOREM. *All left quasi-zeroes of a semigroup S are of the same type.*

(3.4) THEOREM. *Let a be an n-left quasi-zero of S and let ψ be a homomorphism of S to $\mathbf{Z}_n$. Then $\psi = \psi(a)\psi_a$.*

As an immediate consequence of the above we have

(3.5) COROLLARY. *Let a and b be periodic left quasi-zeroes of a semigroup S. Then*

$$\psi_a = \psi_a(b)\psi_b.$$

(3.6) COROLLARY. *Let $a_1, a_2, \ldots, a_n$ be m-left quasi-zeroes of a semigroup S. Then*

$$\psi_{a_1}(a_2)\psi_{a_2}(a_3) \ldots \psi_{a_n}(a_1) = [1]_m.$$

In contrast to the aperiodic case, we have the following result:

(3.7) THEOREM. *Let $a \in S$ be an n-left quasi-zero. Then a^r is a left quasi-zero if and only if $(r, n) = 1$.*

Proof. Since a is an n-left quasi-zero, there is an epimorphism $\psi_a:S \to \mathbf{Z}_n$ with $\psi_a(a) = [1]_n$. Thus $\psi_a(a^r) = [r]_n$. If a^r is a left quasi-zero then $[r]_n$ is a left quasi-zero of $\mathbf{Z}_n$ and so by Lemma 2.6 $[r]_n$ is a generator of $\mathbf{Z}_n$, i.e., $(r, n) = 1$.

Conversely, suppose $(r, n) = 1$ and let $x \in S$. Then $a^i x = a^j$ for some $i, j \in \mathbf{N}$. Choose i and j sufficiently large so that a^i and a^j belong to the group ideal of $\langle a \rangle$ and such that $i = rk$ for some $k \in \mathbf{N}$. Since $(r, n) = 1$, there exists $s \in \mathbf{N}$ such that $rs \equiv 1$ [mod n] and so

$$j \equiv rsj \text{ [mod } n].$$

Hence $a^{rk}x = a^{rjs}$.

(3.8) *Definition.* A semigroup S is said to be *of type n* if there exists an epimorphism $\psi:S \to \mathbf{Z}_n$ such that every homomorphism $\chi:S \to \mathbf{Z}_n$ factors through ψ.

It is immediate that if S has an n-left quasi-zero, then S is a semigroup of type n.

The following characterization of semigroups with periodic left quasi-zeroes in terms of homomorphisms is analogous to Theorem 2.14.

(3.9) THEOREM. *S has an n-left quasi-zero if and only if there exists an epimorphism $\psi:S \to \mathbf{Z}_n$ such that for every semigroup T and every homomorphism $\chi:S \to T$ there exists a homomorphism $\bar{\chi}:\mathbf{Z}_n \to T$ with $\bar{\chi}(0)\chi = \bar{\chi}\psi$. Here 0 is the identity of $\mathbf{Z}_n$.*

Proof. If S has an n-left quasi-zero b, then choose $\psi = \psi_b$. For G_b, the group ideal of $\langle b \rangle$, $\psi|G_b$ is an isomorphism between $\mathbf{Z}_n$ and G_b. If $\chi:S \to T$ is a homomorphism, define $\bar{\chi}:\mathbf{Z}_n \to T$ by

$$\bar{\chi} = \chi(\psi|G_b)^{-1}.$$

For some $m \in \mathbf{N}$, b^m is the identity of G_b. For each $x \in S$, $b^m x \in G_b$ and

$$\chi(b^m x) = \bar{\chi}\psi(b^m x) = \bar{\chi}\psi(x).$$

But

$$\chi(b^m x) = \chi(b^m)\chi(x) = \bar{\chi}\psi(b^m)\chi(x) = \bar{\chi}(0)\chi(x).$$

Conversely, consider the homomorphism $1_S:S \to S$. Then there exists $\bar{\chi}:Z_n \to S$ such that $\bar{\chi}(0)1_S = \bar{\chi}\psi$. Let $b = \bar{\chi}([1]_n)$. Thus $\bar{\chi}(0) = b^n$ and so for each $x \in S$,

$$b_n x = \bar{\chi}\psi(x).$$

Let $\psi(x) = [k]_n$. Then

$$b^n x = \bar{\chi}([k]_n) = \bar{\chi}(k[1]_n) = b^k$$

and so b is an n-left quasi-zero.

(3.10) *Example.* If E is a left zero semigroup then $E \times \mathbf{Z}_n$ has n-left quasi-zeroes.

(3.11) THEOREM. *Let S be a semigroup with an n-left quasi-zero. Then S has a right group kernel K isomorphic to $E(K) \times \mathbf{Z}_n$.*

Proof. Since S has an n-left quasi-zero, S has a minimal right ideal which is a cyclic group G of order n. Thus S has a kernel K which is the disjoint union of the minimal right ideals of S where each such right ideal is a group isomorphic to G.

We now characterize semigroups each of whose elements is a left quasi-zero. To this end we need the following

(3.12) *Definition.* A semigroup S is said to be *combinatorial* if for each $x \in S$ there exists $n \in \mathbf{N}$ such that $x^n = x^{n+1}$.

(3.13) LEMMA. *Let S be a semigroup and $a \in S$ be such that each power of a is a left quasi-zero of S. Then $a^n = a^{n+1}$ for some $n \in \mathbf{N}$.*

Proof. By Corollary 2.9 a is periodic and by Theorem 3.7 the group ideal of $\langle a \rangle$ is trivial.

The above yields immediately

(3.14) COROLLARY. *Let S be a semigroup. Then S consists entirely of left quasi-zeroes if and only if S is combinatorial and each idempotent of S is a left zero.*

Every element except the identity of a cyclic group of prime order is a left quasi-zero. The following is a generalization of this.

(3.15) THEOREM. *A semigroup S is an ideal extension of a right group $K \approx E \times \mathbf{Z}_p$ by a nil semigroup of index not greater than a prime $p > 1$ if and only if S is archimedean and the set of left quasi-zeroes of S is $S \backslash E(S) \neq \emptyset$.*

Proof. Let $p > 1$ be a prime and suppose that S is an ideal extension of a right group $K \approx E \times \mathbf{Z}_p$ by a nil semigroup of index not more than p. For any $a \in S\backslash E(S)$, a is a pth power only if $a \in K$ and so for some $n \in \mathbf{N}$, a^n is a generator of one of the groups in K. Thus a is a p-left quasi-zero of S. Since $p > 1$, no idempotent can be a left quasi-zero. Thus the set of left quasi-zeroes of S is $S\backslash E(S) \neq \emptyset$. Finally, any periodic semigroup which is an ideal extension of a right group by a nil semigroup is archimedean.

Conversely, suppose that S is an archimedean semigroup for which the set of left quasi-zeroes is $S\backslash E(S) \neq \emptyset$. Then S is an ideal extension of a right group $K \approx E \times Z_n$ for some $n \in N$. Since every non-idempotent element of each group in K is a left quasi-zero, n is a prime. Now for

https://doi.org/10.4153/CJM-1980-040-x Published online by Cambridge University Press

each $x \in S \backslash K$, x^n is not a left quasi-zero by Theorem 3.7 and so x^n is an idempotent. Since S is archimedean with a right group kernel, all idempotents of S belong to K. Thus S/K is nil of index not greater than n.

(3.16) THEOREM. *Let S be an archimedean semigroup. Then S is an ideal extension of a right group $K \approx E \times G$ if and only if the following hold:*
(i) *S is the disjoint union of archimedean homogroups H_e, $e \in E(S)$,*
(ii) *the group ideal G_e of each H_e is a right ideal of S.*

Proof. Suppose that S is archimedean and is an ideal extension of a right group $K \approx E \times G$. Since S is archimedean, $\langle x \rangle \cap K \neq \emptyset$ for all $x \in S$. In particular,

$$E(S) = E(K) \approx E.$$

For each $e \in E(S)$, define

$$H_e = \{x \in S | \langle x \rangle \cap G_e \neq \emptyset\}.$$

We show that $\langle x \rangle \cap G_e \neq \emptyset$ if and only if $xG_e = G_e$, in which case the result follows. If $\langle x \rangle \cap G_e \neq \emptyset$ and $xG_e = G_f$, then $x^{n+1}G_f = G_f$ for n such that $x^n \in G_e$. Thus

$$G_f = x^{n+1}G_f = ex^{n+1}G_f = eG_f = G_e$$

and so $xG_e = G_e$. Conversely, if $xG_e = G_e$ then $x^nG_e = G_e$ for all n. If $x^n \in G_f$, then

$$G_f = fG_e = fx^nG_e = x^nG_e = G_e$$

whence $x^n \in G_e$.

Now if S satisfies conditions (i) and (ii), then S has a right group kernel $K \approx E(S) \times G$ where $G = G_e$ for $e \in E(S)$. For any $x, y \in S$, $x^n \in G_e$ for some $n \in \mathbf{N}$, $e \in E(S)$, and so $x^n = x^n(ey)^{-1}ey$ where the inverse refers to the group G_e. Thus $x^n \in S^1yS^1$ and so S is archimedean.

(3.17) COROLLARY. *Let S be an archimedean semigroup. Then S has an n-left quasi-zero if and only if S is the disjoint union of archimedean homogroups H_α, each with group ideal $G_\alpha \approx Z_n$ and each G_α is a right ideal of S.*

Since this partition induces a left zero congruence on the kernel of S, it is natural to ask whether the partition itself is a left zero congruence. However, this is rarely the case. Consider for example the left zero semigroup $\{a, b, c\}$ and adjoin d by defining $d^2 = a$, $dc = b$. Then $H_a = \{a, d\}$, $H_b = \{b\}$, $H_c = \{c\}$ and $H_aH_c = \{a, b\}$.

4. Semigroups generated by left quasi-zeroes. In this section we apply the theory developed so far to presentations of semigroups generated by left quasi-zeroes. In particular, we construct examples of such semigroups to show the extent to which their structure can differ.

If $X = \{a_i | \, i \in I\}$ and for each $j \in J$, R_j is a relation on X^+, then the quotient semigroup X^+ modulo the congruence generated by these relations is denoted by $\langle a_i | \, R_j, i \in I, j \in J\rangle$.

(4.1) THEOREM. *Let* $X = \{a_1, a_2, \ldots, a_n\}$ *and let* k_i, $l_i \in \mathbf{N}$ *for* $i = 1, 2, \ldots, n$. *For* $i = 1, 2, \ldots, n$ *define a relation*

$$R_i{:}\, a_i^{k_i} a_{i+1} = a_i^{l_i} \text{ (where } a_{n+1} = a_1).$$

Then $\langle a_1, a_2, \ldots, a_n | R_1, R_2, \ldots R_n\rangle$ *is a semigroup generated by the left quasi-zeroes* $a_1, a_2, \ldots, a_n$.

Proof. It is clear that $\langle a_i\rangle a_j \cap \langle a_i\rangle \neq \emptyset$ for all $i, j = 1, 2, \ldots, n$.

The next theorem enables us to tell at a glance what type of left quasi-zeroes the generators $a_1, \ldots, a_n$ are.

(4.2) THEOREM. *Let* $X = \{a_1, a_2, \ldots, a_n\}$ *and* $R_i{:}\, a_i^{j_i} a_{i+1} = a_i^{j_i + k_i}$ *for* j_i, $j_i + k_i \in \mathbf{N}$ *and* $k_i \in \mathbf{Z}$ (*again* $a_{n+1} = a_1$). *Let*

$$k = \left| \prod_{i=1}^{n} k_i - 1 \right| .$$

Then $S = \langle a_1, a_2, \ldots, a_n | R_1, R_2, \ldots, R_n\rangle$ *is a semigroup of type* k *if* $k \neq 0$ *and a semigroup of type* $\mathbf{N}$ *or* $\mathbf{Z}$ *if* $k = 0$.

Proof. By (4.1) we know that S is generated by left quasi-zeroes. By (2.4),

$$\phi_{a_1}(a_2)\phi_{a_2}(a_3) \ldots, \phi_{a_n}(a_1) = 1$$

if the left quasi-zeroes are aperiodic. Thus if $k \neq 0$, the left quasi-zeroes are periodic and hence S is a semigroup of type n for some $n \in \mathbf{N}$. But then by (3.6),

$$\phi_{a_1}(a_2) \ldots, \phi_{a_n}(a_1) \equiv 1 \pmod{n}$$

whence $n | k$. We show now that $k | n$. Define a correspondence $\{a_1, \ldots, a_n\}$ to Z_k as follows:

$$a_1 \mapsto [1]_k = b_1, \quad a_i \mapsto \left[\prod_{j=1}^{i-1} k_j\right]_k = b_i \quad \text{for } i = 2, 3, \ldots, n.$$

It is easy to show that the b_i satisfy the same relations as the a_i. Thus the above correspondence may be extended to a homomorphism of S to $\mathbf{Z}_k$. In particular, $\langle a_1\rangle$ maps onto $\mathbf{Z}_k$ and so the group ideal of $\langle a_1\rangle$ maps onto $\mathbf{Z}_k$. Thus $k | n$ and so $k = n$.

If $k = 0$ then, for the same reasons as above, a homomorphism from

S to $\mathbf{Z}$ may be defined by

$$a_1 \mapsto 1, \quad a_i \mapsto \prod_{j=1}^{i-1} k_j, \quad i = 2, 3, \ldots, n.$$

It is then clear that the semigroup is of type $\mathbf{Z}$ if and only if $k_i = -1$ for some i. If $k_i = 1$ for all i then S is a semigroup of type $\mathbf{N}$. In particular, S has no periodic elements in this case.

We remark that the left quasi-zeroes of a semigroup S of type $\mathbf{N}$ must belong to any generating set of S.

(4.3) LEMMA. *Let a, b be aperiodic left quasi-zeroes of a semigroup S. If $a^n = b^m$ for some $m, n \in \mathbf{N}$, then $m = n$ and there exists $k_0 \in \mathbf{N}$ such that for $k \geqq k_0$, $a^k = b^k$.*

Proof. If $a^n = b^m$ then $n = m\phi_a(b)$. Since $|\phi_a(b)| = 1$ we must have $\phi_a(b) = 1$ and so $m = n$. Thus $a^{nt} = b^{nt}$ for all $t \geqq 1$. Furthermore, $a^i b = a^{i+1}$ for some $i \in \mathbf{N}$. If $n \geqq i$ then $a^n b = a^{n+1}$ and so $b^{n+1} = a^{n+1}$ and, by induction, $a^k = b^k$ for all $k \geqq n$. If $n < i$, choose t so that $nt \geqq i$. As before $a^k = b^k$ for $k \geqq nt$.

Let k_0 denote the least element of $\mathbf{N}$ for which $a^k = b^k$ for $k \geqq k_0$. It can happen that $a^n = b^n$ with $n < k_0$. For example,

$$S = \langle a, b \mid a^2 = b^2, a^n = b^n \text{ for } n \geqq 4\rangle$$

is generated by aperiodic left quasi-zeroes a, b but $a^3 \neq b^3$, while $k_0 = 4$. The verification that $a^3 \neq b^3$ is obtained by representing S by the matrix semigroup generated by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We note that since S is a semigroup of type N, S has no other left quasi-zeroes.

(4.4) *Example.* Let $S = \langle a, b \mid a^2 b = a, b^2 a = b\rangle$. S is generated by the two aperiodic left quasi-zeroes a and b. However both ab and ba are idempotent. Probably the simplest example of such a semigroup is $\mathbf{Z}$.

The following example shows that a semigroup finitely generated by periodic left quasi-zeroes is not necessarily periodic.

(4.5) *Example.* Let $S = \langle a, b \mid a^3 b = a^2, b^2 a = b^2\rangle$. Then $a^3 = a^2$ and $b^3 = b^2$ whence $a^2 b = a^2$ and $b^2 a = b^2$. Thus we have symmetry in the defining relations of S. It is easy to show that the elements of S are of the form displayed in the table below where each column contains all powers of the top element and ab and ba are aperiodic elements.

TABLE 1.

						ab	ba							
						$(ab)^2$	$(ba)^2$							
						$(ab)^3$	$(ba)^3$							
						$\vdots$	$\vdots$							
	$(ab)^2a$		$(ab)a$		a			b		$(ba)b$		$(ba)^2b$		
$\dots$	$(ab)^2a^2$	$(ab)^2b$	$(ab)a^2$	$(ab)b$	a^2			b^2	$(ba)a$	$(ba)b^2$	$(ba)^2a$	$(ba)^2b^2$	$\dots$	$\dots$

It is a matter of routine computation to verify that any 2 distinct words from Table 1 can be distinguished by at least one of the following four homomorphisms.

(1) $$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Define

$$\phi_1 : S \rightarrow \langle A, B \rangle \text{ by } \phi_1(a) = A,\ \phi_1(b) = B \text{ and}$$
$$\phi_2 : S \rightarrow \langle A, B \rangle \text{ by } \phi_2(b) = A,\ \phi_2(a) = B.$$

(2) Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Define

$$\phi_3 : S \rightarrow \langle A, B \rangle \text{ by } \phi_3(a) = A,\ \phi_3(b) = B \text{ and}$$
$$\phi_4 : S \rightarrow \langle A, B \rangle \text{ by } \phi_4(a) = B,\ \phi_4(b) = A.$$

The kernel of the above semigroup is the (infinite) bottom row of the table. We observe that if any further relations are added to ensure periodicity, the semigroup becomes finite. This does not happen in general.

(4.6) *Example*. There is an infinite periodic semigroup finitely generated by left quasi-zeroes. For A. Thue [**9**] has shown that there exist infinitely many square-free words on an alphabet of three letters. If

$$X = \{a, b, c\} \text{ and } I = \{uv^2w \mid u, w \in X^*, v \in X^+\},$$

then $S = X^+/I$ is a nil semigroup generated by the left quasi-zeroes a, b and c and S is infinite.

To conclude this section, we show that a finite semigroup generated by left quasi-zeroes is not archimedean in general, i.e., it is not necessarily the case that every idempotent belongs to the kernel.

(4.7) *Example.* The following semigroup was obtained as a subsemigroup of a transformation semigroup on 5 letters.

Let

$$S = \langle a, b \mid a^3b = a^2, b^2a = b^3, b^3a = b^2, aba = a, bab = b, ab^2a = b^2ab\rangle.$$

Then

$$S = \{a, a^2, a^3, b, b^2, b^3, ab, ba, ab^2, ab^3\}.$$

This semigroup is displayed in the form of a table below, where each column contains all powers of the top element and the idempotents are enclosed within the dotted rectangle.

	a		b	
ab	a^2	ab^3	b^2	ba
	a^3	ab^2	b^3	

The kernel of this semigroup is $\{a^2, a^3, ab^3, ab^2, b^2, b^3\}$ which is a right group with group isomorphic to $\mathbf{Z}_2$. The elements ab and ba are idempotents.

5. A sequence of generalizations of the bicyclic semigroup. In (2.10) we examined the bicyclic semigroup $\mathscr{B}_0 = \langle x_0, \bar{x}_0 \mid x_0\bar{x}_0 = 1\rangle$ for left and right quasi-zeroes. In this section we introduce a sequence of generalizations of the bicyclic semigroup and investigate their properties. The set of all left quasi-zeroes of each such semigroup is determined and by symmetry the right quasi-zeroes of each are also obtained.

(5.1) *Definition.* For each $n \geqq 0$, let

$$\mathscr{B}_n = \langle x_n, \bar{x}_n \mid x_n^{n+1}\bar{x}_n = x_n^{\,n}, x_n\bar{x}_n^{\,n+1} = \bar{x}_n^{\,n}\rangle.$$

For each $n \geqq 0$, it is immediate that of the generators of $\mathscr{B}_n$, $[x_n]$ is a left quasi-zero while $[\bar{x}_n]$ is a right quasi-zero. We are going to see that only $\mathscr{B}_0$ and $\mathscr{B}_1$ have infinitely many left (right) quasi-zeroes, while for $n \geqq 2$, $\mathscr{B}_n$ has exactly one left (right) quasi-zero. The exceptional behaviour of $\mathscr{B}_0$ and $\mathscr{B}_1$ does not end here. It is well known that $\mathscr{B}_0$ can be described as a semigroup on the set $\mathbf{N}_0 \times \mathbf{N}_0$. In much the same way, $\mathscr{B}_1$ can be described as a semigroup on $\mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{N}_0$. There does not appear

https://doi.org/10.4153/CJM-1980-040-x Published online by Cambridge University Press

to be any such representation of $\mathscr{B}_n$ for $n \geqq 2$. The Green's relations for $\mathscr{B}_1$ can be described readily by utilizing this representation.

The first result to be established is that $\mathscr{B}_2$ has exactly one left quasi-zero, whence by a similar argument, it can be shown that $\mathscr{B}_2$ has exactly one right quasi-zero.

In order to prove this result we shall require the notion of a reduced word (relative to the congruence relation defining $\mathscr{B}_2$, i.e., $x_2\bar{x}_2{}^3 = \bar{x}_2{}^2$ and $x_2{}^3\bar{x}_2 = x_2{}^2$). We shall delete any reference to the subscript 2 on the generators for this discussion.

(5.2) *Definition.* A word in X^*, $X = \{x, \bar{x}\}$, is said to be *reduced* if it is of the form

$$\bar{x}^{i_1}x^{i_2}\bar{x}^{i_3} \ldots \bar{x}^{i_{k-1}}x^{i_k} \text{ for } k \text{ even,}$$

with

$$0 < i_2, i_3, \ldots, i_{k-1} \leqq 2 \text{ if } k \geqq 4, \text{ and } i_1, i_k \geqq 0.$$

(5.3) LEMMA. *Each word of X^* is congruent to a reduced word by the congruence generated by the relations $x\bar{x}^3 = \bar{x}^2$, $x^3\bar{x} = x^2$.*

Proof. Let $u \in X^*$ and suppose that u is not reduced. Then u is of the form

$$\bar{x}^{i_1}x^{i_2} \ldots \bar{x}^{i_{k-1}}x^{i_k} \text{ for } k \geqq 3, i_1 \geqq 0, i_k \geqq 0 \text{ and}$$
$$i_2, i_3, \ldots, i_{k-1} > 0.$$

It follows that for some $1 < j < k$, $i_j > 2$. If j is even, form the word u_1 by deleting $x\bar{x}$ from $x^{i_j}\bar{x}^{i_{j+1}}$ in u, while if j is odd, form u_1 by deleting $x\bar{x}$ from $x^{i_{j-1}}\bar{x}^{i_j}$ in u. Then $u \equiv u_1$ and $|u_1| = |u| - 2$. If u_1 is reduced, we are done. Otherwise, repeat the process. By induction, we obtain that u is congruent to a reduced word.

(5.4) THEOREM. *If u and v are reduced words of X^* and $u \equiv v$, then $u = v$.*

Proof. Suppose that $u = \bar{x}^{i_1}x^{i_2} \ldots \bar{x}^{i_{k-1}}x^{i_k}$ is reduced. Then there exists a finite sequence $u = u_1 \equiv u_2 \equiv \ldots \equiv u_m = v$ where u_{t+1} is obtained from u_i by inserting or deleting $x\bar{x}$ (by using the relations $x^3\bar{x} = x^2$, $x\bar{x}^3 = \bar{x}^2$). To form u_2, $x\bar{x}$ must be inserted into u_1 and so $|u_2| = |u_1| + 2$. Define

$$F = \{x^t\bar{x}^t | \, t \geqq 0\}.$$

Let k_1 be the first number for which

$$|u_{k_1+1}| = |u_{k_1}| - 2$$

(that k_1 exists is clear). Then

$$u_{k_1} = w_1\bar{x}^{i_1}w_2x^{i_2} \ldots w_kx^{i_k}w_{k+1}$$

where each $w_i \in F^*$. Since u_{k_1} is not reduced, $u_{k_1} \neq v$. Let k_2 be the least subscript greater than k_1 for which $|u_{k_2+1}| = |u_{k_2}| + 2$. Then each of the words u_i, $k_1 < i \leqq k_2$ is obtained from u_{i-1} by deleting $x\bar{x}$. Each such deletion can occur only within the subwords of the w_i which remain at any particular step. This is easily seen to be the case since each such subword begins with x and ends with $\bar{x}$. If $m = k_1 + k_2$, then $v = u_{k_2}$. But u_{k_2} is reduced only if no subwords of the w_i remain, i.e., only if $u_{k_2} = u_1$. Thus $u = v$. If $m > k_1 + k_2$, we repeat the process, beginning with u_{k_2} where

$$u_{k_2} = w_1'\bar{x}^{i_1}w_2'x^{i_2} \ldots w_k'x^{i_k}w_{k+1}'$$

with each $w_i' \in F^*$. But in this manner we see that for each j,

$$u_j = y_1\bar{x}^{i_1}y_2x^{i_2} \ldots y_kx^{i_k}y_{k+1}$$

for some $y_1, y_2, \ldots, y_{k+1} \in F^*$. Thus in particular, u_m is such a word and since such a word is reduced only if $y_1 = y_2 = \ldots = y_{k+1} = 1$,

$$v = u_m = \bar{x}^{i_1}x^{i_2} \ldots x^{i_k} = u_1 = u.$$

Thus each class of the congruence on X^* generated by the relations $x^3\bar{x} = x^2$, $x\bar{x}^3 = \bar{x}^2$ has a unique reduced representative.

(5.5) THEOREM. *$\mathscr{B}_2$ has exactly one left quasi-zero, namely the congruence class $[x]$.*

Proof. Let $u \in X^*$ be a reduced representative of a left quasi-zero of $\mathscr{B}_2$. Then

$$u = \bar{x}^{i_1}x^{i_2} \ldots x^{i_k}$$

with k even and

$$\sum i_{2j} - \sum i_{2j+1} = 1.$$

If $k = 2$, we have $u = \bar{x}^ix^{i+1}$ for some $i \geqq 0$. If $i = 0$ we have $u = x$. Otherwise, $i > 0$ and then by an inductive argument, one can show that the reduced representative of u^t is

$$\bar{x}^i(x^2\bar{x})^{t-1}x^{i+1}.$$

But then u^tx has reduced representative

$$\bar{x}^i(x^2\bar{x})^{t-1}x^{i+2},$$

whence $u^tx \not\equiv u^r$ for any $t, r \in \mathbf{N}$. Thus we obtain a contradiction and so for $k = 2$ we obtain $u = x$ and $[x]$ is a left quasi-zero of $\mathscr{B}_2$. Now consider $k \geqq 4$. There are four possible cases. We shall show that none of the cases can occur.

https://doi.org/10.4153/CJM-1980-040-x Published online by Cambridge University Press

Case 1. $i_1, i_k > 0$. Consider $u^2 = \bar{x}^{i_1} \ldots x^{i_n}\bar{x}^{i_1} \ldots x^{i_n}$. If $x^{i_n}\bar{x}^{i_1}$ is reduced, then the reduced representative for u^r ends in $\bar{x}x^{i_n}$ for all r and so $u^r x \not\equiv u^s$ for any $r, s \in \mathbf{N}$. But this implies that $[u]$ is not a left quasi-zero of $\mathscr{B}_2$, a contradiction. Thus $x^{i_n}\bar{x}^{i_1}$ is not reduced. If $i_1 \geqq i_n$, then the reduction of u^2 will be accomplished by making deletions of $x\bar{x}$ factors to the left of the right factor $\bar{x}^2$ in $\bar{x}^{i_1}$. Once again, the reduced representative of u^r ends in $\bar{x}x^{i_n}$ for all r, a contradiction. Thus $i_1 < i_n$ and the reduction of u^2 is accomplished by deleting $x\bar{x}$ factors to the right of the left factor x^2 in x^{i_n}. If these deletions do not reach the right factor $\bar{x}x^{i_n}$ in u^2, again the reduced representative for u^r ends in $\bar{x}x^{i_n}$ for any $r \in \mathbf{N}$ whence $[u]$ is not a left quasi-zero of $\mathscr{B}_2$, a contradiction. Thus the reduced representative of u^2 is

$$\bar{x}^{i_1}x^{i_2} \ldots \bar{x}^{i_{n-1}}x^t \text{ where}$$
$$t = i_n - i_{n-1} + i_{n-2} - \ldots - i_1 + i_n.$$

However, for the deletions to have been possible, we must have had $i_n \geqq i_1 + 2$, $i_n - i_1 + i_2 \geqq i_3 + 2$, and so on, finishing with $i_n - i_1 + i_2 - i_3 + \ldots + i_{n-2} \geqq i_{n-1} + 2$. But this implies that

$$\sum i_{2j} - \sum i_{2j+1} \geqq 2,$$

a contradiction. Thus at least one of i_1 or i_n is zero.

Case 2. $i_1 = 0$, $i_n > 0$ whence $u = x^{i_2}\bar{x}^{i_3} \ldots \bar{x}^{i_{n-1}}x^{i_n}$ and $0 < i_2, i_3, \ldots, i_{n-1} \leqq 2$. The arguments of Case (1) can be applied here by putting $i_1 = 0$ to conclude that $[u]$ is not a left quasi-zero of $\mathscr{B}_2$, a contradiction.

Case 3. $i_1 > 0$, $i_n = 0$ and so $n = \bar{x}^{i_1}x^{i_2} \ldots \bar{x}^{i_{n-1}}$. When we consider u^2 in this case, we see that the only reduction that could possibly occur is to the left of the right factor $\bar{x}^2$ in $\bar{x}^{i_{n-1}+i_1}$, leaving the reduced representative of u^r for any $r \in \mathbf{N}$ to end in $\bar{x}^{i_{n-1}}$. Since $u^r x \not\equiv u^s$ for any r, s in such a case, $[u]$ would not be a left quasi-zero of $\mathscr{B}_2$, a contradiction.

Case 4. $i_1 = i_n = 0$. Then no reduction in u^2 can occur whence u^2 is reduced and thus u^r is reduced for all $r \in \mathbf{N}$. Obviously $[u]$ would not be a left quasi-zero of $\mathscr{B}_2$, a contradiction.

By similar arguments, it can be shown that $\mathscr{B}_2$ has exactly one right quasi-zero, namely $[\bar{x}]$.

(5.6) COROLLARY. *For each $n \geqq 2$, $\mathscr{B}_n$ has exactly one left quasi-zero and one right quasi-zero, namely $[x_n]$ and $[\bar{x}_n]$ respectively.*

Proof. The assignment $[\bar{x}_n] \to [\bar{x}_{n-1}]$ and $[x_n] \to [x_{n-1}]$ determines a surjective homomorphism of $\mathscr{B}_n$ onto $\mathscr{B}_{n-1}$ whence quasi-zeroes are mapped to quasi-zeroes (of the same type, left or right). The result is thus established upon observing that $[x_n] = \{x_n\}$ and $[\bar{x}_n] = \{\bar{x}_n\}$.

(5.7) LEMMA. *Every word in* X^*, $X = \{x, \bar{x}\}$, *is congruent to a word of the form* $\bar{x}^k(x\bar{x})^l x^n$ *for some* $(k, l, n) \in \mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{N}_0$ *by the congruence defining* $\mathscr{B}_1$, *i.e., the congruence determined by the relations* $x^2\bar{x} = x$, $x\bar{x}^2 = \bar{x}$.

Proof. We shall proceed by induction on the length of words in X^*. The statement is obviously true for words of length 0, 1 or 2. Suppose then that any word of length less than n is congruent to a word of the required form, with $n \geqq 2$. Let $u \in X^*$ be a word of length n. Then either $u = (x\bar{x})^t$ with $n = 2t$, or else x or $\bar{x}$ appears in u to a power greater than 1. In the first instance, $u = \bar{x}^0(x\bar{x})^t x^0$, while in the second, either $u = w_1x^iw_2$ or else $u = v_1\bar{x}^iv_2$ for some $i > 1$ and with $w_1 \in X^*\backslash X^*x$, $w_2 \in X^*\backslash xX^*$, $v_1 \in X^*\backslash X^*\bar{x}$, $v_2 \in X^*\backslash \bar{x}X^*$ and $i, j \geqq 2$. Consider the case $u = w_1x^iw_2$. If $w_2 = 1$ it follows from the induction hypothesis, while if $w_2 \neq 1$, then $w_2 = \bar{x}w_3$ and so

$$u = w_1x^i\bar{x}w_3 \equiv w_1x^{i-1}w_3 = v.$$

But $|v| < n$ and so the result follows from the induction hypothesis. A similar argument can be presented if $u = v_1\bar{x}^iv_2$.

By observing how the multiplication of congruence classes works, as determined by representatives of the form described in (5.7), we are led to define a binary operation $*$ on the set $\mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{N}_0$ as follows: for $(k, l, m), (r, s, t) \in \mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{N}_0$,

$$(k, l, m)*(r, s, t) = \begin{cases} (k + r - m, s, t) & m < r \\ (k, l + s + 1, t) & m = r \neq 0 \\ (k, l + s, t) & m = r = 0 \\ (k, l, t + m - r) & m > r \end{cases}$$

Sixteen computations are required to verify that this binary operation is associative, and these computations have been carried out. Thus $(\mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{N}_0, *)$ is a semigroup.

(5.8) *Definition.* Let $\bar{B}_1$ denote the semigroup $(\mathbf{N}_0 \times \mathbf{N}_0 \times \mathbf{N}_0, *)$.

(5.9) THEOREM. $\mathscr{B}_1 \simeq \bar{B}_1$.

Proof. Let $X = \{x, \bar{x}\}$. Define a homomorphism $\mu : X^* \to \bar{B}_1$ by putting $\mu(x) = (0, 0, 1)$ and $\mu(\bar{x}) = (1, 0, 0)$. Since

$$(0, 0, 1)^2*(1, 0, 0) = (0, 0, 1) \text{ and}$$
$$(0, 0, 1)*(1, 0, 0)^2 = (1, 0, 0),$$

μ induces a homomorphism $\bar{\mu} : \mathscr{B}_1 \to \bar{B}_1$ for which $\bar{\mu}([x]) = (0, 0, 1)$ and $\bar{\mu}([\bar{x}]) = (1, 0, 0)$. Now define a function $\eta : \bar{B}_1 \to \mathscr{B}_1$ by

$$\eta(k, l, n) = [\bar{x}^k(x\bar{x})^l x^n].$$

It is readily verified that η is a homomorphism and that

$$\eta \circ \bar{\mu}([x]) = [x], \ \eta \circ \bar{\mu}([\bar{x}]) = [\bar{x}]$$

whence $\eta \circ \bar{\mu}$ is the identity on $\mathscr{B}_1$. As well one can readily show by computation that

$$\bar{\mu} \circ \eta(k, l, n) = (k, l, n)$$

whence $\bar{\mu} \circ \eta$ is the identity on $\bar{B}_1$. Thus $\bar{\mu}$ and η are isomorphisms.

By means of this representation of $\mathscr{B}_1$, we are readily able to describe the set of right quasi-zeroes and the set of left quasi-zeroes for $\mathscr{B}_1$ and, as well, we can determine the Green's relations on $\mathscr{B}_1$.

(5.10) THEOREM. *The set of left quasi-zeroes of $\bar{B}_1$ is*

$$\{(k, l, k + 1| \ k, l \in \mathbf{N}_0\}.$$

Proof. It is easily established by induction that

$$(k, l, k + 1)^n = (k, l, k + n)$$

for any $n \in \mathbf{N}$ and so for $(r, s, t) \in \bar{B}_1$ we choose $n > r - t$ to obtain

$$(k, l, k + 1)^n * (r, s, t) = (k, l, k + n + t - r) = (k, l, k + 1)^{n+t-r}.$$

Thus $(k, l, k + 1)$ is a left quasi-zero for any $k, l \in \mathbf{N}_0$. In particular, $u = (0, 0, 1)$ is a left quasi-zero. Let $\phi_u : \bar{B}_1 \rightarrow \mathbf{Z}$ be the corresponding homomorphism. Then $\phi_u(1, 0, 0) = -1$ and since

$$(k, l, m) = (1, 0, 0)^k * ((0, 0, 1) * (1, 0, 0))^l * (0, 0, 1)^m,$$

we have

$$\phi_u(k, l, m) = m - k.$$

Thus $m - k = \pm 1$, and so we need only show each element of the form $(k + 1, l, k)$ is not a left quasi-zero. By computation, one can readily show that

$$(k + 1, l, k)^n * (0, 0, 1) \neq (k + 1, l, k)^r \text{ for any } n, r \in \mathbf{N}.$$

Thus $(k + 1, l, k)$ is not a left quasi-zero for any $k, l \in \mathbf{N}_0$.

Similarly, one can show that the set of right quasi-zeroes of $\bar{B}_1$ is $\{(k + 1, l, k)| \ k, l \in \mathbf{N}_0\}$.

We now demonstrate that for $n \neq m$, $\mathscr{B}_n \not\simeq \mathscr{B}_m$. Since the set of left quasi-zeroes of $\mathscr{B}_1$ and of $\mathscr{B}_0$ are both infinite, while the set of left quasi-zeroes of each $\mathscr{B}_n$, $n \geqq 2$, is a singleton, $\mathscr{B}_0$ and $\mathscr{B}_1$ are distinct from $\mathscr{B}_n$, $n \geqq 2$.

(5.11) THEOREM. $E(\mathscr{B}_1) = \{1\}$.

Proof. We prove that $E(\bar{B}_1) = \{(0, 0, 0)\}$. If $(k, l, m) \in \bar{B}_1$ is an idempotent, then

$$(k, l, m)*(k, l, m) = (k, l, m)$$

from which one concludes that $m = k$. But by definition of $*$, not only must $m = k$, but we must have $m = k = 0$.

But $(0, l, 0)^2 = (0, 2l, 0)$ and thus $2l = l$ whence $l = 0$.

(5.12) COROLLARY. $\mathscr{B}_0 \not\simeq \mathscr{B}_1$.

Let $\mathrm{NC}(k)$ denote the nil cyclic semigroup of order k.

(5.13) THEOREM. *There exists an epimorphism of $\mathscr{B}_n$ to $\mathrm{NC}(n)$ but no epimorphism of $\mathscr{B}_n$ to $\mathrm{NC}(n + 1)$.*

Proof. Define an epimorphism from $X^* \to \mathrm{NC}(n)$ by $x \to a$, $\bar{x} \to a$ where a is the generator of $\mathrm{NC}(n)$. Since the relations $x^{n+1}\bar{x} = x^n$, $x\bar{x}^{n+1} = \bar{x}^n$ define $\mathscr{B}_n$, this epimorphism determines an epimorphism $\phi_n:\mathscr{B}_n \to \mathrm{NC}(n)$ by

$$\phi_n([x]) = \phi_n([\bar{x}]) = a.$$

Suppose now that there does exist an epimorphism $\psi:\mathscr{B}_n \to \mathrm{NC}(n + 1)$. Then either $[x]$ or $[\bar{x}]$ maps to b where b is the generator of $\mathrm{NC}(n + 1)$. Suppose that $\psi([x]) = b$ and $\psi([\bar{x}]) = b^r$ for some $r \in \mathbf{N}$. Then

$$\psi([x]^{n+1}[\bar{x}]) = b^{n+1}b^r = 0$$

but $[x]^{n+1}[\bar{x}] = [x]^n$ whence

$$\psi([x]^{n+1}[\bar{x}]) = \psi([x]^n) = b^n.$$

Thus $b^n = 0$, a contradiction, and so no epimorphism from $\mathscr{B}_n$ to $\mathrm{NC}(n + 1)$ can exist.

(5.14) COROLLARY. *For all $n, m \in \mathbf{N}_0$, $n \neq m$, $\mathscr{B}_n \not\simeq \mathscr{B}_m$.*

Proof. It is only necessary to show that for $n > m \geqq 2$, $\mathscr{B}_n \not\simeq \mathscr{B}_m$. Suppose that for some such n, m, $\mathscr{B}_n \simeq \mathscr{B}_m$. Then since $n > m$, there is an epimorphism $\mathscr{B}_n \twoheadrightarrow \mathscr{B}_{m+1} \twoheadrightarrow \mathrm{NC}(m + 1)$ and since $\mathscr{B}_n \simeq \mathscr{B}_m$ we obtain an epimorphism of $\mathscr{B}_m \twoheadrightarrow \mathrm{NC}(m + 1)$, which is a contradiction by (5.13).

Finally, we determine the Green's relations for $\bar{B}_1$ and hence for $\mathscr{B}_1$. It is easily seen from the definition of $*$ that if two elements of $\bar{B}_1$ are R-related, their first and second components must be equal.

(5.15) THEOREM. *The R-classes of $\bar{B}_1$ are of two types, namely*

$$R_{ij} = \{(i, j, k) | k \in \mathbf{N}\} \text{ and } R^0_{ij} = \{(i, j, 0)\} \text{ for } i, j \in \mathbf{N}_0.$$

Proof. Let $k, k' \in \mathbf{N}$. Then

$$(i, j, k)*(k-1, 0, k'-1) = (i, j, k') \text{ and}$$
$$(i, j, k')*(k'-1, 0, k-1) = (i, j, k).$$

Thus $(i, j, k)R(i, j, k')$ for all $k, k' \in \mathbf{N}$. Moreover, if $(i, j, 0)R(i, j, k)$ for some $k \in \mathbf{N}$, then

$$(i, j, 0) = (i, j, k)*(x, y, z)$$

for some $x, y, z \in \mathbf{N}_0$. By definition of $*$, this is not possible.

(5.16) THEOREM. *The $\mathscr{L}$-classes of $\bar{B}_1$ are of two types, namely*

$$L_{jk} = \{(i, j, k)|i \in N\} \text{ and } L^0_{jk} = \{(0, j, k)\} \text{ for } j, k \in \mathbf{N}_0.$$

(5.17) COROLLARY. *The H-relation on $\bar{B}_1$ is trivial.*

(5.18) COROLLARY. *The D-relation for $\bar{B}_1$ has classes of four types, namely*

(1) $D_j = \{(i, j, k)|i, k \in \mathbf{N}\}, j \in \mathbf{N}_0;$

(2) $D_{0j} = \{(0, j, k)|k \in \mathbf{N}\}, j \in \mathbf{N}_0;$

(3) $D_{j0} = \{(i, j, 0)|i \in \mathbf{N}\}, j \in N_0$ *and*

(4) $D_{0j0} = \{(0, j, 0)\}, j \in \mathbf{N}_0.$

Thus in contrast to $\mathscr{B}_0$, $\mathscr{B}_1$ is not bisimple. Moreover, $\mathscr{B}_1$ is not even simple, but rather close to it.

(5.19) THEOREM. *The J-relation on $\bar{B}_1$ has two classes, namely $\{(0, 0, 0)\}$ and $\bar{B}_1 \backslash \{(0, 0, 0)\}$.*

Proof. For any $u, b \in \mathbf{N}$, $s, t \in \mathbf{N}_0$, we have

$$(0, b, 0) = (0, 0, s+1)*(s, t, u)*(u+1, t-1, 0) \text{ and}$$
$$(s, t, u) = (s, t, u)*(0, b, 0)*(0, 0, 0).$$

Thus $(0, b, 0)J(s, t, u)$ for all $u, b \in \mathbf{N}$, $s, t \in \mathbf{N}_0$. If we put

$$J_j = D_j \cup D_{0j} \cup D_{j0} \cup D_{0j0},$$

the above argument implies that $\cup_{j \in \mathbf{N}} J_j$, which is $\bar{B}_1 \backslash \{(0, 0, 0)\}$, is contained in a single J-class. Since it is clear that the J-class of $(0, 0, 0)$ is a singleton, the result follows.

REFERENCES

1. M. Demlová, *On groups of units in a special class of monoids*, Semigroup Forum *16* (1978), 443–454.

2. W. P. Jones, *Semigroups satisfying ring-like conditions*, Ph.D. Thesis, Univ. of Iowa (1969).

3. J. B. Kim, *No semigroup is a finite union of mutants*, Semigroup Forum *6* (1973), 360–361.
4. D. R. LaTorre, *An internal characterization of the O-radical of a semigroup*, Math. Nachr. *45* (1970), 279–281.
5. R. H. Oehmke, *Quasi-regularity in semigroups*, Seminaire Dubreil-Pisot (1969/70), Demigroups no. 1, 1–3.
6. S. Rankin, C. Reis and G. Thierrin, *Right local semigroups*, J. of Alg. *46* (1977), 134–147.
7. M. Satyanarayana, *On the O-radical of a semigroup*, Math. Nachr. *66* (1975), 231–234.
8. H. Seidel, *Über das Radikal einer Halbgruppe*, Math. Nachr. *29* (1965), 255–263.
9. A. Thue, *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihe*, Videnskabsselskabets Skrifter, I Mat. Nat. Kl., Kristiania (1912), 1–67.

University of Western Ontario,
London, Ontario