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Spectral stability of constrained solitary waves for a generalized Ostrovsky equation

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We consider the existence and stability of constrained solitary wave solutions to the generalized Ostrovsky equation

$$\partial_x \left(\partial_t u + \alpha\partial_x u + \partial_x(f(u)) + \beta\partial_x^3 u\right) = \gamma u, \quad \|u\|_{L^2}^2 = \lambda > 0,$$

where the homogeneous nonlinearities $f(s) = \alpha_0|s|^p + \alpha_1|s|^{p-1}s$, with $p > 1$. If $\alpha_0, \alpha_1 > 0$, $\alpha \in \mathbb{R}$, and $\gamma < 0$ satisfying $\beta\gamma = -1$, we show that for $1 < p < 5$, there exists a constrained ground state traveling wave solution with travelling velocity $\omega > \alpha - 2$. Furthermore, we obtain the exponential decay estimates and the weak non-degeneracy of the solution. Finally, we show that the solution is spectrally stable. This is a continuation of recent work [1] on existence and stability for a water wave model with non-homogeneous nonlinearities.

Keywords: solitary wave; spectral stability; the generalized Ostrovsky equation

2010 *Mathematics Subject Classification:* Primary: 35Q35
Secondary: 76B25; 35B35

1. Introduction

In this paper, we consider the existence and stability of solitary wave solutions of the generalized Ostrovsky equation

$$\partial_x \left(\partial_t u + \alpha\partial_x u + \partial_x(f(u)) + \beta\partial_x^3 u\right) = \gamma u, \tag{1.1}$$

where $u = u(t, x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is the wave shape distribution; the homogeneous nonlinearities $f(u) = \alpha_0|u|^p + \alpha_1|u|^{p-1}u$, with degree $p > 1$; and $\alpha, \alpha_0, \alpha_1, \beta$, and γ are some parameters that arise during the derivation of the evolution equation. This study was inspired by the work of Levandosky [8] on the existence and stability of solitary waves of (1.1) with L^p-norm constraints, and in this paper, we consider the case of solutions with L^2-norm constraints. This is a continuation of recent work of Chen, Gao, and Han [1] for existence and stability for a water wave model with non-homogeneous nonlinearities.

If $f(u) = u^2$, $\alpha = 0$, and $\beta = \gamma = 1$, (1.1) is the classical Ostrovsky equation (see [18]):

$$\partial_x \left(\partial_t u - \partial_x^3 u - \partial_x (u^2) \right) = u,$$

which describes the unidirectional propagation of weakly nonlinear long surface and internal waves with small amplitude in rotating fluids. The spectral, orbital, and weak orbital stabilities of the solitary wave solutions have been proved in [14, 15, 17]. If $f(u) = |u|^2 u$, $\alpha = 0$, and $\beta = \gamma = 1$, (1.1) is the Ostrovsky–Vakhnenko model or the short pulse model:

$$\partial_x \left(\partial_t u - \partial_x^3 u - \partial_x (u^3) \right) = u,$$

which appears in the studies of water waves with Coriolis forces and the amplitude of short pulses in optical fibres, see, e.g., [2, 18, 19, 24]. If $f(u) = u^p$ or $|u|^{p-1}u$, then letting $u = v_x$, where v satisfies $v, v_x \to 0$, $|x| \to +\infty$, we get

$$\partial_x \left(\partial_t v - \partial_x^3 v - (|v_x|^p) \right) = v \quad \text{or} \quad \partial_x \left(\partial_t v - \partial_x^3 v - (|v_x|^{p-1} v_x) \right) = v.$$

Their local and global well-posedness (see, e.g., [3, 12, 21, 24–26]) and blowup solutions [16] have been established. Considering the solitary wave of form $v(t, x) = \phi(x - \omega t)$ yields the profile equation

$$\phi'''' + \omega \phi'' + \phi + (|\phi'|^p)' = 0 \quad \text{or} \quad \phi'''' + \omega \phi'' + \phi + (|\phi'|^{p-1}\phi')' = 0.$$

The existence of variational solutions can be found in [8–10], etc. When $p = 2$, the solution is unique (see [27]). Recently, Posukhovskyi and Stefanov [22, 23] considered the existence of solitary waves, with the L^2-norm constraint. In detail, they proved the existence and spectral stability for (1.1) with $f(u) = |u|^p$ $(1 < p < 3)$ or $f(u) = |u|^{p-1}u$ $(1 < p < 5)$, which satisfy $\|u\|_{L^2}^2 = \lambda > 0$. These results are different from those of Levandosky and Liu [9, 10] who considered the existence of solitary waves with L^{p+1}-norm constraints; meanwhile, they proved that the solitary waves are unstable when p is sufficiently large.

In this paper, we consider that $f(u)$ is the homogeneous nonlinearity with degree $p > 1$:

$$f(u) = |u|^p + |u|^{p-1}u.$$

Levandosky [8] proved that for $2 \le p < 5$, there exists an L^p-norm constrained solitary wave and it is stable. The purpose of this paper is to prove the existence and stability of L^2-norm constrained solitary waves. This is based on the recent work of Chen, Gao, and Han [1] on the existence and stability of L^2-norm constrained solitary waves in the intracoastal zone, which has a non-homogeneous nonlinearity. We consider the existence and stability of solitary waves with the L^2-norm constraint for (1.1). Let $u = \partial_x v$, then (1.1) becomes

$$\partial_x \left(\partial_t \partial_x v + \partial_x \left(\alpha \partial_x v + \alpha_0 |\partial_x v|^p + \alpha_1 |\partial_x v|^{p-1} \partial_x v \right) + \beta \partial_x^4 v \right) = \gamma \partial_x v,$$

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where

$$\lim_{|x|\to+\infty} v = \lim_{|x|\to+\infty} \partial_x v = 0. \tag{1.2}$$

Integrating the above equation with respect to x, we get

$$\partial_x \left(\partial_t v + \alpha\partial_x v + \alpha_0|\partial_x v|^p + \alpha_1|\partial_x v|^{p-1}\partial_x v + \beta\partial_x^3 v\right) = \gamma v. \tag{1.3}$$

The purpose of this paper is to construct stable solitary wave solutions of (1.3) of the form

$$v(t,x) = \phi(x-\omega t). \tag{1.4}$$

1.1. Problem setting

Substituting (1.4) into (1.3), we get ϕ that satisfies the profile equation

$$(\alpha-\omega)\phi'' + \alpha_0\left(|\phi'|^p\right)' + \alpha_1\left(|\phi'|^{p-1}\phi'\right)' + \beta\phi'''' - \gamma\phi = 0. \tag{1.5}$$

To state our problem, we introduce some notations. Denote $\|\cdot\|_{L^p}$ by the usual norm of Lebesgue spaces $L^p = L^p(\mathbb{R})$, with $p \geq 1$. For $u(x) \in L^1$, define the Fourier transform and its Fourier inverse transform as

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u(x)e^{-\mathrm{i}x\xi}dx, \quad u(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{u}(\xi)e^{\mathrm{i}x\xi}dx.$$

Define the norms in the Sobolev spaces $H^k := W^{k,2}(\mathbb{R})$ with $k \in \mathbb{N}$ and $k \in \mathbb{R}$ by

$$\|u\|_{H^k} = \sum_{\alpha=0}^{k}\|\partial_x^\alpha u\|_{L^2}, \quad \|u\|_{H^k} = \left(\int_{\mathbb{R}}\left(1+|\xi|^2\right)^k|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}},$$

respectively. Define the semi-morn on the homogeneous Sobolev space $\dot{H}^k$ as

$$\|u\|_{\dot{H}^k} = \left(\int_{\mathbb{R}}|\xi|^{2k}|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

The dual space $\dot{H}^{-k}$ with $k \in \mathbb{N}$ is defined by

$$\dot{H}^{-k} = \left\{f \in \mathcal{S}'(\mathbb{R}) : f = \partial_x^k g, \|f\|_{\dot{H}^{-k}} = \|g\|_{L^2}\right\},$$

where $\mathcal{S}'(\mathbb{R})$ is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$.

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We consider the solutions of the minimization problem with respect to (1.3):

$$\begin{cases} E[u] = -\frac{\alpha_0}{p+1}\int_{\mathbb{R}} |u'|^p u' dx \\ \qquad -\frac{\alpha_1}{p+1}\int_{\mathbb{R}} |u'|^{p+1} dx + \frac{\beta}{2}\int_{\mathbb{R}} |u''|^2 dx - \frac{\gamma}{2}\int_{\mathbb{R}} |u|^2 \mathrm{d}x \to \min, \\ \int_{\mathbb{R}} |u'(x)|^2 dx = \lambda > 0, \end{cases} \tag{1.6}$$

and

$$\begin{cases} \mathcal{E}[u] = -\frac{\alpha_0}{p+1}\int_{\mathbb{R}} |u|^p u dx \\ \qquad -\frac{\alpha_1}{p+1}\int_{\mathbb{R}} |u|^{p+1} dx + \frac{\beta}{2}\int_{\mathbb{R}} |u'|^2 dx - \frac{\gamma}{2}\int_{\mathbb{R}} |\partial_x^{-1} u|^2 dx \to \min, \\ \int_{\mathbb{R}} |u(x)|^2 \mathrm{d}x = \lambda > 0, \quad u \in \dot{H}^{-1}. \end{cases} \tag{1.7}$$

Here, $' = \partial_x$. Notice that $E[u] = \mathcal{E}[u']$. The Euler–Lagrange equations corresponding to the constrained functionals $E[u]$ and $\mathcal{E}[u]$ are derived in appendix A.

To study the stability of solutions, we linearize the solution $v(t,x)$ of (1.3) near $\phi(x-\omega t)$, where ϕ is the minimizer of (1.6). Then, we get the linearized equation

$$\partial_t\partial_x v - (\omega-\alpha)\partial_x^2 v + (\alpha_0+\alpha_1)\partial_x\left(|\phi'|^{p-2}\phi'\partial_x v\right) + \beta\partial_x^4 v = \gamma v.$$

Let $v(t,x) = \mathrm{e}^{t\mu} z(x)$, we get the eigenvalue problem

$$L_+ z = \mu\partial_x z, \tag{1.8}$$

where

$$L_+ = (\omega-\alpha)\partial_x^2 - (\alpha_0+\alpha_1)\partial_x\left(|\phi'|^{p-2}\phi'\partial_x(\cdot)\right) - \beta\partial_x^4 + \gamma\,\mathrm{Id}\,.$$

Here, Id is the identity operator. Thus, L_+ is a self-adjoint unbounded operator in L^2 and $\mathcal{D}(L_+) = H^4$. Spectral instability is to study the existence of nontrivial pairs (μ, z) for problem (1.8) with $\Re\mu > 0$ and $z \neq 0$ for $z \in \mathcal{D}(L_+)$. On the contrary, the spectral stability means that no such pair (μ, z) exists. Let

$$L_+ = -\partial_x\mathcal{L}_+\partial_x,$$

where

$$\mathcal{L}_+ = -(\omega-\alpha)\,\mathrm{Id} + \alpha_0 p|\phi'|^{p-2}\phi' + \alpha_1 p|\phi'|^{p-1} + \beta\partial_x^2 - \gamma\partial_x^{-2}.$$

Here, $\mathcal{D}(\mathcal{L}_+) = H^2 \cap \dot{H}^{-2}$. Thus, (1.8) becomes

$$-\partial_x\mathcal{L}_+(\partial_x z) = \mu\partial_x z. \tag{1.9}$$

Using (1.2), we obtain that (1.9) is equivalent to $(-\mathcal{L}_+\partial_x)z = \mu z$, that is, the eigenvalue μ of $-\mathcal{L}_+\partial_x$. Let ν be the eigenvalue of self-adjoint operator $\partial_x\mathcal{L}_+ = (-\mathcal{L}_+\partial_x)^*$, i.e.,

$$\partial_x\mathcal{L}_+ z = \nu z. \tag{1.10}$$

Thus, the spectral stability of travelling wave solutions is to prove that the eigenvalue problem (1.10) has no nontrivial solutions (ν, z) with $\Re\nu > 0$ and $z \neq 0$.

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1.2. Main results

To state the main results, we define the weak non-degeneracy and spectral stability.

DEFINITION 1.1. *The wave ϕ is weak non-degenerate, if $\phi \perp \mathrm{Ker}[\mathcal{L}_+]$. We call the solution of (1.7) to be spectrally stable, if the eigenvalue problem (1.10) has no nontrivial solution (ν, z) with $\Re\nu > 0$, $z \neq 0$.*

The first result is existence and decay estimates of constrained solitary waves.

THEOREM 1.2 *Assume that $\lambda, \alpha_0, \alpha_1 > 0$, $\gamma < 0$ satisfy $\beta\gamma = -1$, $\alpha \in \mathbb{R}$ and $\omega > \alpha - 2$. Then, for $1 < p < 5$, the constrained variational problems (1.6) and (1.7) exist solutions*

$$\phi = \phi_\lambda \in H^4, \quad \psi = \psi_\lambda \in H^2 \cap \dot{H}^{-2},$$

respectively, which satisfy

$$\begin{aligned}
&\phi' = \psi,\\
&(\alpha - \omega)\phi'' + \alpha_0 \left(|\phi'|^p\right)' + \alpha_1 \left(|\phi'|^{p-1}\phi'\right)' + \beta\phi'''' - \gamma\phi = 0,\\
&(\alpha - \omega)\psi + \alpha_0|\psi|^p + \alpha_1|\psi|^{p-1}\psi + \beta\psi'' - \gamma\partial_x^{-2}\psi = 0,\\
&|\phi(x)| + |\phi'(x)| + |\psi(x)| \le Ce^{-k_\omega \cdot |x|},
\end{aligned}$$

where $C = C(\alpha, \omega, \beta, \gamma) > 0$ and

$$k_\omega = \begin{cases} \sqrt{\frac{\omega-\alpha-\sqrt{(\alpha-\omega)^2-4}}{2\beta}}, & \omega > \alpha + 2,\\ \sqrt{\frac{\omega-\alpha}{4\beta} + \frac{1}{2}\sqrt{\frac{-\gamma}{\beta}}}, & \alpha - 2 < \omega < \alpha + 2. \end{cases}$$

The second result is weak non-degeneracy and spectral stability of solutions in theorem 1.2.

THEOREM 1.3 *The minimizer $\phi = \phi_\lambda$ of the constrained variational problem (1.6) constructed in theorem 1.2 is weakly non-degenerate. Furthermore, if we additionally assume that*

$$\left\langle \mathcal{L}_+^{-1}\phi, \phi \right\rangle \neq 0,$$

then ϕ is spectrally stable.

Here are some comments on the theorems.

REMARK 1.4. If we consider the variational problems (1.6) without the L^2-norm constraints, the restriction $\omega > \alpha - 2$ in theorem 1.2 is optimal. Indeed, by (2.12),

$$E[\phi] < \lambda = \int_{\mathbb{R}} |\phi'|^2 dx,$$

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i.e.,

$$\frac{\beta}{2}\int_{\mathbb{R}}|\phi''|^2dx-\frac{\gamma}{2}\int_{\mathbb{R}}|\phi|^2dx$$
$$<\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|\phi'|^p\phi' dx+\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|\phi'|^{p+1}dx+\int_{\mathbb{R}}|\phi'|^2dx. \tag{1.11}$$

Using the Pohozaev identity deduced in appendix B, we have

$$\begin{aligned}
&\frac{\beta}{2}\int_{\mathbb{R}}|\phi''|^2dx-\frac{\gamma}{2}\int_{\mathbb{R}}|\phi|^2dx\\
=&\frac{\beta}{2}\left(\frac{(2p-1)\alpha_0}{2(p+1)\beta}\int_{\mathbb{R}}|\phi'|^p\phi' dx+\frac{(2p-1)\alpha_1}{(2(p+1))\beta}\int_{\mathbb{R}}|\phi'|^{p+1}dx\right)\\
&-\frac{3}{4}\left(-\frac{2(\alpha-\omega)}{3}\int_{\mathbb{R}}|\phi'|^2dx-\frac{(3-2p)\alpha_0}{3(p+1)}\int_{\mathbb{R}}|\phi'|^p\phi' dx-\frac{(3-2p)\alpha_1}{3(p+1)}\int_{\mathbb{R}}|\phi'|^{p+1}dx\right)\\
=&\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|\phi'|^p\phi' dx+\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|\phi'|^4dx\\
&+\frac{1}{2}(\alpha-\omega)\int_{\mathbb{R}}|\phi'|^2dx+\frac{(2p-1)\alpha_1}{4(p+1)}\int_{\mathbb{R}}|\phi'|^{p+1}dx.
\end{aligned}$$

This combined with lemma 2.6 shows that (1.11) becomes

$$\left[\frac{1}{2}(\alpha-\omega)-1\right]\int_{\mathbb{R}}|\phi'|^2dx\le-\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|\phi'|^{p+1}dx<0.$$

This implies that $\omega>\alpha-2$. However, it is not clear whether this condition is optimal when considering the L^2-norm constraints. Moreover, it is not clear whether the solution obtained in theorem 1.2 is unique. Finally, theorem 1.2 implies that there exists ω,α satisfying $\omega>\alpha-2$ such that the solution exists, and it is not clear whether there exists a solution $\phi=\phi_\lambda$ for any $\omega>\alpha-2$.

Remark 1.5. Levandosky has proved the existence and stability of weak solutions with L^p-norm constraints with $2\le p<5$ for (1.1) with $\alpha=0$ (see Main Result (i) in [8]). Compare with his results, we consider the L^2-norm constraints in this paper; the weak solution obtained in theorem 1.2 is actually a strong solution (see proposition 2.2); moreover, we obtain a fine decay estimate of the solution.

2. Existence of constrained solitary waves

In this section, we consider the existence and decay estimates of constrained solitary waves of (1.3).

2.1. Decay estimates

We first define the weak solutions of (1.5).

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DEFINITION 2.1. *We call $\phi \in H^2$ a weak solution of (1.5), if*

$$\left\langle (\alpha-\omega)\phi'' + \alpha_0\left(|\phi'|^p\right)' + \alpha_1\left(|\phi'|^{p-1}\phi'\right)' - \gamma\phi, \psi \right\rangle + \langle \beta\phi'', \psi''\rangle = 0, \tag{2.1}$$

for any $\psi(x) \in C_c^\infty(\mathbb{R})$, where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2,L^2}$.

The weak solution defined above is actually a strong solution.

PROPOSITION 2.2. *Assume that $\beta > 0$ and $\gamma < 0$, then the weak solution $\phi \in H^2$ of the profile equation (1.5) defined by (2.1) actually satisfies $\phi \in H^4$.*

Proof. The proof is based on the bootstrap argument. Since $\beta > 0$ and $\gamma < 0$, the formal solution of (1.5) is

$$\tilde{\phi} = -\left(\beta\partial_x^4 - \gamma\,\mathrm{Id}\right)^{-1}\left((\alpha-\omega)\phi'' + \alpha_0\left(|\phi'|^p\right)' + \alpha_1\left(|\phi'|^{p-1}\phi'\right)'\right) \in L^2. \tag{2.2}$$

Since $(\beta\partial_x^4 - \gamma\,\mathrm{Id})^{-1} : L^2 \to H^4$, we get $\tilde{\phi} \in H^3$. Using (1.5), we have

$$\begin{aligned}\left\langle \left(\beta\partial_x^4 - \gamma\,\mathrm{Id}\right)\phi, \psi\right\rangle &= -\left\langle \left((\alpha-\omega)\phi'' + \alpha_0\left(|\phi'|^p\right)' + \alpha_1\left(|\phi'|^{p-1}\phi'\right)'\right), \psi\right\rangle \\ &= \left\langle \left(\beta\partial_x^4 - \gamma\,\mathrm{Id}\right)\tilde{\phi}, \psi\right\rangle, \quad \forall\psi \in C_c^\infty(\mathbb{R}),\end{aligned}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-2},H^2}$. So,

$$\left\langle \phi, \left(\beta\partial_x^4 - \gamma\,\mathrm{Id}\right)\psi\right\rangle = \left\langle \tilde{\phi}, \left(\beta\partial_x^4 - \gamma\,\mathrm{Id}\right)\psi\right\rangle, \quad \forall\psi \in C_c^\infty(\mathbb{R}).$$

Thus, we have $\phi = \tilde{\phi}$ in the distribution sense, which means $\phi \in H^3$. Since ϕ is a weak solution, we obtain

$$\alpha_0\left(|\phi'|^p\right)' + \alpha_1\left(|\phi'|^{p-1}\phi'\right)' \in L^2.$$

Thus, by (2.2), we obtain $\phi \in H^4$. □

Next, we consider the decay estimates of solutions for the profile equation (1.5).

PROPOSITION 2.3. *Suppose $\beta > 0$, $\gamma < 0$, and $\omega > \alpha - 2\sqrt{-\beta\gamma}$, assume that $\phi \in H^4$ is a solution of (1.5). Then,*

$$|\phi(x)| + |\phi'(x)| \le Ce^{-k_\omega\cdot|x|}, \tag{2.3}$$

where $C = C(\alpha, w, \beta, \gamma) > 0$ and

$$k_\omega = \begin{cases} \sqrt{\dfrac{\omega-\alpha-\sqrt{(\alpha-\omega)^2+4\beta\gamma}}{2\beta}}, & \omega > \alpha + 2\sqrt{-\beta\gamma}, \\ \sqrt{\dfrac{\omega-\alpha}{4\beta} + \dfrac{1}{2}\sqrt{\dfrac{-\gamma}{\beta}}}, & \alpha - 2\sqrt{-\beta\gamma} < \omega < \alpha + 2\sqrt{-\beta\gamma}. \end{cases}$$

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Proof. According to $\beta>0$, $\gamma<0$, and $\omega>\alpha-2\sqrt{-\beta\gamma}$, we obtain that $-(\alpha-\omega)\xi^2+\beta\xi^4-\gamma>0$ for any $\xi\in\mathbb{R}$. Thus, $\left((\alpha-\omega)\partial_x^2+\beta\partial_x^4-\gamma\,\mathrm{Id}\right)^{-1}$ is a bounded operator in L^2. Therefore, the solution of (1.5) is

$$\phi=-\left((\alpha-\omega)\partial_x^2+\beta\partial_x^4-\gamma\,\mathrm{Id}\right)^{-1}\partial_x\left(\alpha_0|\phi'|^p+\alpha_1|\phi'|^{p-1}\phi'\right). \tag{2.4}$$

The asymptotic behaviour (1.2) yields that

$$\lim_{|x|\to+\infty}\left(\alpha_0|\phi'|^p+\alpha_1|\phi'|^{p-1}\phi'\right)(x)=0, \tag{2.5}$$

for any $\phi\in H^4\subset C_0(\mathbb{R})$; meanwhile,

$$\left((\alpha-\omega)\partial_x^2+\beta\partial_x^4-\gamma\,\mathrm{Id}\right)^{-1}g(x)=\int_{\mathbb{R}}G_{\alpha,\omega,\beta,\gamma}(x-y)g(y)dy,$$

where $G_{\alpha,\omega,\beta,\gamma}(x)$ is the fundamental solution of $\left((\alpha-\omega)\partial_x^2+\beta\partial_x^4-\gamma\,\mathrm{Id}\right)\phi=0$, satisfying

$$\widehat{G}_{\alpha,\omega,\beta,\gamma}(\xi)=\frac{1}{-(\alpha-\omega)\xi^2+\beta\xi^4-\gamma}.$$

Let h_1 and h_2 be the roots of the polynomial $-(\alpha-\omega)h^2+\beta h^4-\gamma$ with respect to h, then

$$h_1^2=\frac{\alpha-\omega+\sqrt{(\alpha-\omega)^2+4\beta\gamma}}{2\beta},\quad h_2^2=\frac{\alpha-\omega-\sqrt{(\alpha-\omega)^2+4\beta\gamma}}{2\beta},$$

$$\frac{\sqrt{2\pi}}{2h}\widehat{e^{-h|x|}}(\xi)=\frac{1}{2h}\left(\int_{-\infty}^{0}e^{(h-\mathrm{i}\xi)x}dx+\int_0^{+\infty}e^{-(h+\mathrm{i}\xi)x}dx\right)=\frac{1}{h^2+\xi^2},$$

$$\Re h=\begin{cases}\sqrt{\frac{\omega-\alpha-\sqrt{(\alpha-\omega)^2+4\beta\gamma}}{2\beta}}, & \omega>\alpha+2\sqrt{-\beta\gamma},\\ \sqrt{\frac{\omega-\alpha}{4\beta}+\frac{1}{2}\sqrt{\frac{-\gamma}{\beta}}}, & \alpha-2\sqrt{-\beta\gamma}<\omega<\alpha+2\sqrt{-\beta\gamma},\end{cases}$$

and

$$G_{\alpha,\omega,\beta,\gamma}(x)=\frac{\sqrt{2\pi}}{2h\sqrt{(\alpha-\omega)^2+4\beta\gamma}}e^{-\Re h\cdot|x|},\quad \omega>\alpha-2\sqrt{-\beta\gamma}.$$

Thus,

$$\left|G^{(k)}_{\alpha,\omega,\beta,\gamma}(x)\right|\le\begin{cases}Ce^{-\sqrt{\frac{\omega-\alpha-\sqrt{(\alpha-\omega)^2+4\beta\gamma}}{2\beta}}|x|}, & \omega>\alpha+2\sqrt{-\beta\gamma},\\ Ce^{\sqrt{\frac{\omega-\alpha}{4\beta}+\frac{1}{2}\sqrt{\frac{-\gamma}{\beta}}}}, & \alpha-2\sqrt{-\beta\gamma}<\omega<\alpha+2\sqrt{-\beta\gamma},\end{cases}$$

where $k\in\mathbb{N}$ and $C=C(\alpha,\omega,\beta,\gamma)>0$ is a constant.

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According to (2.5), for any $\epsilon = \epsilon(\alpha_0, \alpha_1, \alpha, \omega, \beta, \gamma) > 0$, there exists sufficiently large N, such that when $|x| > N$,

$$\left|\left(\alpha_0|\phi'|^p + \alpha_1|\phi'|^{p-1}\phi'\right)(x)\right| < \epsilon|\phi'(x)|.$$

Thus, using (2.4), we obtain

$$\begin{aligned}\phi' = &-\int_{|y|>N} G''_{\alpha,\omega,\beta,\gamma}(x-y)\left(\alpha_0|\phi'|^p + \alpha_1|\phi'|^{p-1}\phi'\right)(y)dy \\ &-\int_{|y|\le N} G''_{\alpha,\omega,\beta,\gamma}(x-y)\left(\alpha_0|\phi'|^p + \alpha_1|\phi'|^{p-1}\phi'\right)(y)dy.\end{aligned} \tag{2.6}$$

We consider the integral equation on $L^\infty(\{x : |x| > N\})$:

$$\mathcal{F}\phi'(x) = \chi_{\{x:|x|>N\}}\left[\phi'(x) + \int_{|y|\le N} G''_{\alpha,\omega,\beta,\gamma}(x-y)\left(\alpha_0|\phi'|^p + \alpha_1|\phi'|^{p-1}\phi'\right)(y)dy\right],$$

where

$$\mathcal{F}\phi'(x) = -\chi_{\{x:|x|>N\}}\int_{|y|>N} G''_{\alpha,\omega,\beta,\gamma}(x-y)\left(\alpha_0|\phi'|^p + \alpha_1|\phi'|^{p-1}\phi'\right)(y)dy.$$

Let

$$\mathcal{H}_m = \left\{u(x) : \|u\|_{\mathcal{H}_m} := \sup_{|x|>N}|u(x)|e^{m|x|} < +\infty,\ m \ge 0\right\},$$

then for any $m \in [0, \Re h]$ and $\phi'(x) \in \mathcal{H}_m$,

$$\begin{aligned}|\mathcal{F}\phi'(x)| = &\left|-\chi_{\{x:|x|>N\}}\int_{|y|>N} G''_{\alpha,\omega,\beta,\gamma}(x-y)\left(\alpha_0|\phi'|^p + \alpha_1|\phi'|^{p-1}\phi'\right)(y)dy\right| \\ \le &C\epsilon\int_{\mathbb{R}} |G''_{\alpha,\omega,\beta,\gamma}(x-y)||\phi'(y)|dy \\ \le &C\epsilon\|\phi'\|_{\mathcal{H}_m}\int_{\mathbb{R}} e^{-\Re h\cdot|x-y|}e^{-m|y|}dy \le C\epsilon\|\phi'\|_{\mathcal{H}_m}e^{-m|x|}.\end{aligned}$$

Thus, $\mathcal{F} : \mathcal{H}_m \to \mathcal{H}_m$ satisfies $\|\mathcal{F}\|_{\mathcal{L}(\mathcal{H}_m)} \le C\epsilon$. Selecting $\epsilon > 0$ sufficiently small such that $C\epsilon < 1$, we obtain that $\mathrm{Id} - \mathcal{F}$ is bounded and invertible; moreover,

$$(\mathrm{Id} - \mathcal{F})^{-1} = \sum_{k=0}^{\infty}\mathcal{F}^k, \quad \left\|(\mathrm{Id} - \mathcal{F})^{-1}\right\|_{\mathcal{H}_m} \le \frac{1}{1 - \|\mathcal{F}\|_{\mathcal{H}_m}},$$

where $\mathcal{F}^0 = \mathrm{Id}$. Thus, using (2.6) and taking $m = 0$, we obtain the von Neumann series

$$\phi'(x) = \sum_{k=0}^{\infty}\mathcal{F}^k\left[-\int_{|y|\le N} G''_{\alpha,\omega,\beta,\gamma}(x-y)\left(\alpha_0|\phi'|^p + \alpha_1|\phi'|^{p-1}\phi'\right)(y)dy\right].$$

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This combined with

$$\left| -\int_{|y|\le N} G''_{\alpha,\omega,\beta,\gamma}(x-y)\left(\alpha_0|\phi'|^p+\alpha_1|\phi'|^{p-1}\phi'\right)(y)dy\right| \le Ce^{-\Re h\cdot|x|}$$

gives $\phi' \in \mathcal{H}_{\Re h}$. By the definition of $\mathcal{H}_m$, we get $\sup_{\{x:|x|>N\}}|\phi'(x)| \le Ce^{-\Re h\cdot|x|}$. This combined with the boundedness of $\phi'(x)$ gives

$$\sup_{\mathbb{R}} |\phi'(x)| \le Ce^{-\Re h\cdot|x|}.$$

In addition, $\phi(x)$ has the same decay estimate. In fact, note that $\lim\limits_{|x|\to+\infty}\phi = 0$, then

$$\phi(x) = \int_{-\infty}^{x}\phi'(y)dy = -\int_{x}^{\infty}\phi'(y)dy,$$

and $\phi(x)$ has a decay estimate with the same order as $\phi'(x)$ at $x = \pm\infty$. □

REMARK 2.4. Consider the zero eigenvalue problems of L_+ and $\mathcal{L}_+$ defined in (1.8); we find that the solutions w of $L_+w = 0$ and $\mathcal{L}_+w = 0$ have similar estimates as (2.3) by using proposition 2.3.

2.2. Variational properties

Recalling the previous constrained variational problems (1.6) and (1.7), we introduce the following cost functions:

$$M_E(\lambda) = \inf_{\substack{u\in H^2\\ \|u'\|_{L^2}^2=\lambda}} \Big\{ -\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|u'|^p u' dx - \frac{\alpha_1}{p+1}\int_{\mathbb{R}}|u'|^{p+1}dx \tag{2.7}$$

$$+\frac{\beta}{2}\int_{\mathbb{R}}|u''|^2dx - \frac{\gamma}{2}\int_{\mathbb{R}}|u|^2dx\Big\},$$

$$M_{\mathcal{E}}(\lambda) = \inf_{\substack{u\in H^1\cap\dot{H}^{-1}\\ \|u\|_{L^2}^2=\lambda}} \Big\{ -\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|u|^p u dx - \frac{\alpha_1}{p+1}\int_{\mathbb{R}}|u|^{p+1}dx \tag{2.8}$$

$$+\frac{\beta}{2}\int_{\mathbb{R}}|u'|^2dx - \frac{\gamma}{2}\int_{\mathbb{R}}|\partial_x^{-1}u|^2dx\Big\}.$$

If they exist, then they correspond to the infimums of the constrained variational functionals (1.6) and (1.7).

We first study some properties of the functional $E[u]$ and $\mathcal{E}[u]$.

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LEMMA 2.5. *If $\gamma < 0$ and $1 < p < 5$, then the functional (1.6) is bounded from below, i.e., $M_E(\lambda) > -\infty$. In addition, $M_E(\lambda) = M_{\mathcal{E}}(\lambda)$; moreover, if ϕ_λ is a minimizer of $M_E(\lambda)$, then ϕ'_λ is a minimizer of $M_{\mathcal{E}}(\lambda)$.*

Proof. Using the Gagliardo–Nirenberg–Sobolev inequality

$$\|v\|_{L^p}^p \leq C_p \|v\|_{L^2}^{(1-\beta_p)p} \|\nabla v\|_{L^2}^{\beta_p p}, \quad \beta_p = \frac{1}{2} - \frac{1}{p}, \tag{2.9}$$

where $C_p > 0$ is a constant, we get

$$\begin{aligned} E[u] =& -\frac{\alpha_0}{p+1}\int_{\mathbb{R}} |\partial_x u|^p \partial_x u dx - \frac{\alpha_1}{p+1}\int_{\mathbb{R}} |\partial_x u|^{p+1} dx + \frac{\beta}{2}\int_{\mathbb{R}} |\partial_x^2 u|^2 dx - \frac{\gamma}{2}\int_{\mathbb{R}} |u|^2 dx \\ \geq& -\frac{|\alpha_0| + |\alpha_1|}{p+1} \|\partial_x u\|_{L^{p+1}}^{p+1} + \frac{\beta}{2}\|\partial_x^2 u\|_{L^2}^2 - \frac{\gamma}{2}\|u\|_{L^2}^2 \\ \geq& -C\frac{|\alpha_0| + |\alpha_1|}{p+1} \|\partial_x u\|_{L^2}^{1-\beta_{p+1}(p+1)} \|\partial_x^2 u\|_{L^2}^{\beta_{p+1}(p+1)} + \frac{\beta}{2}\|\partial_x^2 u\|_{L^2}^2 - \frac{\gamma}{2}\|u\|_{L^2}^2 \\ \geq& -C(\alpha_0, \beta)\|\partial_x u\|_{L^2}^{\frac{(1-\beta_{p+1})(\beta_{p+1}(p+1)-2)}{\beta_{p+1}}} - \frac{\gamma}{2}\|u\|_{L^2}^2 \\ =& -C(\alpha_0, \beta)\lambda^{\frac{(1-\beta_{p+1})(\beta_{p+1}(p+1)-2)}{2\beta_{p+1}}} - \frac{\gamma}{2}\|u\|_{L^2}^2. \end{aligned}$$

Since $\gamma < 0$, $M_E(\lambda) > -\infty$.

Denote $\mathbf{S}$ by the set of $\phi \in \mathcal{S}(\mathbb{R})$ such that $\|\phi\|_{L^2}^2 = \lambda$, $\hat{\phi}$ has a compact support, and there exists $\delta > 0$ such that $\hat{\phi}(\xi) = 0$ for $|\xi| < \delta$. Clearly, $\mathbf{S}$ is dense in $\{\phi \in H^1 : \|\phi\|_{L^2}^2 = \lambda\}$, and $\partial_x^{-1}\phi$ is well-defined. Thus,

$$\begin{aligned} M_{\mathcal{E}}(\lambda) &= \inf_{\phi \in \mathbf{S}} \mathcal{E}[\phi] = E[\partial_x^{-1}\phi] \geq M_E(\lambda), \\ M_{\mathcal{E}}(\lambda) &= \inf_{\phi \in \mathbf{S}} \mathcal{E}[\phi] \leq \inf_{\phi \in \mathbf{S}\cap H^2} \mathcal{E}[\phi] = M_E(\lambda). \end{aligned}$$

which implies $M_E(\lambda) = M_{\mathcal{E}}(\lambda)$. Moreover, if ϕ_λ is a minimizer of (2.7), then $E[\phi_\lambda] = \mathcal{E}[\phi'_\lambda]$. □

Theorem 2.5 implies the equivalence of $M_E(\lambda)$ and $M_{\mathcal{E}}(\lambda)$. Next, let $\{u_k\}_{k=0}^{\infty}$ be a minimizing sequence of $\mathcal{E}[u]$ constrained on $\{u : \|u\|_{L^2}^2 = \lambda\}$, i.e.,

$$\lim_{k\to\infty} \mathcal{E}[u_k] = M_{\mathcal{E}}, \quad \|u_k\|_{L^2}^2 = \lambda. \tag{2.10}$$

Then, there exists a subsequence of $\{u_k\}_{k=0}^{\infty}$ (still denoted as $\{u_k\}_{k=0}^{\infty}$), such that

$$\begin{aligned} \int_{\mathbb{R}} |u_k|^p u_k dx \to \mathcal{E}_1, \quad \int_{\mathbb{R}} |u_k|^{p+1} dx \to \mathcal{E}_2, \\ \int_{\mathbb{R}} |\partial_x u_k|^2 dx \to \mathcal{E}_3, \quad \int_{\mathbb{R}} |\partial_x^{-1} u_k|^2 dx \to \mathcal{E}_4, \end{aligned} \qquad k \to +\infty. \tag{2.11}$$

We will prove that $\mathcal{E}_1$ and $\mathcal{E}_2$ are positive, which is crucial for proving strict subadditivity of $M_E(\lambda)$ in § 2.3.

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Lemma 2.6. *If α_0, $\alpha_1 > 0$, $\beta\gamma = -1$, and $1 < p < 5$, then for any minimizing sequence satisfying (2.11), we have $\mathcal{E}_1, \mathcal{E}_2 > 0$.*

Proof. $\mathcal{E}_2 \geq 0$ is obvious, and we claim that $\mathcal{E}_1 \geq 0$. If not, note that $\alpha_0 > 0$ and the other terms of $\mathcal{E}[u]$ are symmetric with respect to u. Let $u \to -u$ and then $\mathcal{E}[-u] < \mathcal{E}[u]$, i.e., $-u$ is closer to $M_{\mathcal{E}}(\lambda)$.

Next, we claim that $\mathcal{E}_1, \mathcal{E}_2 \neq 0$. If not, using the Hölder inequality and the embedding $H^{s-1} \subset L^\infty$, $s > 2$, we get $\mathcal{E}_1 = \mathcal{E}_2 = 0$. Since $\beta\gamma = -1$,

$$
\begin{aligned}
M_{\mathcal{E}}(\lambda) &= \inf_{\|u\|_{L^2}^2 = \lambda} \left\{ \frac{\beta}{2} \int_{\mathbb{R}} |\partial_x u|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}} |\partial_x^{-1} u|^2 dx \right\} \\
&= \inf_{\|u\|_{L^2}^2 = \lambda} \left\{ \frac{\beta}{2} \int_{\mathbb{R}} \xi^2 |\hat{u}(\xi)|^2 d\xi - \frac{\gamma}{2} \int_{\mathbb{R}} \frac{1}{\xi^2} |\hat{u}(\xi)|^2 d\xi \right\} \\
&= \inf_{\|u\|_{L^2}^2 = \lambda} \left\{ \frac{\beta}{2} \int_{\mathbb{R}} \frac{1}{\xi^2} \left(\xi^2 - \frac{1}{\beta} \right)^2 |\hat{u}(\xi)|^2 d\xi + \int_{\mathbb{R}} |\hat{u}(\xi)|^2 d\xi \right\} \geq \lambda.
\end{aligned}
$$

The above inequality is actually an equality. In fact, it is necessary to select $u(x)$ such that $\hat{u}(\xi)$ is concentrated at $\{\xi : \xi = \frac{1}{\sqrt{\beta}}\}$. Next, in order to derive a contradiction and complete the proof, it is only necessary to show

$$
M_{\mathcal{E}}(\lambda) < \lambda. \tag{2.12}
$$

Following the spirits of [22], let $\omega_\epsilon(x) \in L^1$ such that

$$
\begin{aligned}
\widehat{\omega_\epsilon}(\xi) = \frac{1}{\sqrt{\epsilon}} \Bigg\{ &\hat{\chi}\left(\frac{\xi - \frac{1}{\sqrt{\beta}}}{\epsilon} \right) + \hat{\chi}\left(\frac{\xi + \frac{1}{\sqrt{\beta}}}{\epsilon} \right) \\
&+ \epsilon^{1-\sigma} \left[\hat{\chi}\left(\frac{\xi - \frac{2}{\sqrt{\beta}}}{\epsilon} \right) + \hat{\chi}\left(\frac{\xi + \frac{2}{\sqrt{\beta}}}{\epsilon} \right) \right] \Bigg\}
\end{aligned}
$$

and

$$
\|\omega_\epsilon(\cdot)\|_{L^2}^2 = \lambda,
$$

where $\epsilon > 0$ and $0 < \sigma \ll 1$ are sufficiently small, satisfying $\epsilon \varrho \sqrt{\beta} < 1$ and $\chi \in \mathcal{S}(\mathbb{R})$ is a non-negative function, such that $\hat{\chi}$ is an even C^∞ bump function and $\operatorname{supp} \hat{\chi} \subset (-\varrho, \varrho)$, $\varrho > 0$. Thus,

$$
\begin{aligned}
\omega_\epsilon(x) =& \sqrt{\epsilon} \chi(\epsilon x) \left[e^{\mathrm{i} \frac{x}{\sqrt{\beta}}} + e^{-\mathrm{i} \frac{x}{\sqrt{\beta}}} + \epsilon^{1-\sigma} \left(e^{\mathrm{i} \frac{2x}{\sqrt{\beta}}} + e^{-\mathrm{i} \frac{2x}{\sqrt{\beta}}} \right) \right] \\
=& 2\sqrt{\epsilon} \chi(\epsilon x) \left(\cos\left(\frac{x}{\sqrt{\beta}} \right) + \epsilon^{1-\sigma} \cos\left(\frac{2x}{\sqrt{\beta}} \right) \right) \in \mathbb{R}.
\end{aligned}
$$

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Since $\epsilon\varrho\sqrt{\beta} < 1$, we have

$$\operatorname{supp}\hat{\chi}\left(\frac{\xi-\frac{1}{\sqrt{\beta}}}{\epsilon}\right)\bigcap\operatorname{supp}\hat{\chi}\left(\frac{\xi+\frac{1}{\sqrt{\beta}}}{\epsilon}\right)\bigcap\operatorname{supp}\hat{\chi}\left(\frac{\xi-\frac{2}{\sqrt{\beta}}}{\epsilon}\right)\bigcap\operatorname{supp}\hat{\chi}\left(\frac{\xi+\frac{2}{\sqrt{\beta}}}{\epsilon}\right)=\emptyset.$$

Thus,pt

$$\begin{aligned}
&\frac{\beta}{2}\int_{\mathbb{R}}\frac{1}{\xi^2}\left(\xi^2-\frac{1}{\beta}\right)^2|\widehat{\omega_\epsilon}(\xi)|^2d\xi\\
=&\frac{\beta}{2}\int_{\mathbb{R}}\frac{1}{\xi^2}\left(\xi^2-\frac{1}{\beta}\right)^2\left[\left|\frac{1}{\sqrt{\epsilon}}\hat{\chi}\left(\frac{\xi-\frac{1}{\sqrt{\beta}}}{\epsilon}\right)\right|^2+\left|\frac{1}{\sqrt{\epsilon}}\hat{\chi}\left(\frac{\xi+\frac{1}{\sqrt{\beta}}}{\epsilon}\right)\right|^2\right]d\xi\\
&+\frac{\beta}{2}\int_{\mathbb{R}}\frac{1}{\xi^2}\left(\xi^2-\frac{1}{\beta}\right)^2\left[\left|\frac{1}{\sqrt{\epsilon}}\hat{\chi}\left(\frac{\xi-\frac{2}{\sqrt{\beta}}}{\epsilon}\right)\right|^2+\left|\frac{1}{\sqrt{\epsilon}}\hat{\chi}\left(\frac{\xi+\frac{2}{\sqrt{\beta}}}{\epsilon}\right)\right|^2\right]d\xi\\
=&\frac{\beta}{2}\int_{\mathbb{R}}\frac{\left(\epsilon\xi+\frac{2}{\sqrt{\beta}}\right)^2(\epsilon\xi)^2}{\left(\epsilon\xi+\frac{1}{\sqrt{\beta}}\right)^2}|\hat{\chi}(\xi)|^2d\xi+\frac{\beta}{2}\int_{\mathbb{R}}\frac{\left(\epsilon\xi-\frac{2}{\sqrt{\beta}}\right)^2(\epsilon\xi)^2}{\left(\epsilon\xi-\frac{1}{\sqrt{\beta}}\right)^2}|\hat{\chi}(\xi)|^2d\xi\\
&+\frac{\beta}{2}\epsilon^{2(1-\sigma)}\int_{\mathbb{R}}\frac{\left(\epsilon\xi+\frac{3}{\sqrt{\beta}}\right)^2\left(\epsilon\xi+\frac{1}{\sqrt{\beta}}\right)^2}{\left(\epsilon\xi+\frac{2}{\sqrt{\beta}}\right)^2}|\hat{\chi}(\xi)|^2d\xi\\
&+\frac{\beta}{2}\epsilon^{2(1-\sigma)}\int_{\mathbb{R}}\frac{\left(\epsilon\xi-\frac{3}{\sqrt{\beta}}\right)^2\left(\epsilon\xi-\frac{1}{\sqrt{\beta}}\right)^2}{\left(\epsilon\xi-\frac{2}{\sqrt{\beta}}\right)^2}|\hat{\chi}(\xi)|^2d\xi\\
=&O\left(\epsilon^{2(1-\sigma)}\right).
\end{aligned}$$

Since $\chi \geq 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}}|\omega_\epsilon(x)|^4dx=&\int_{\mathbb{R}}\left|2\sqrt{\epsilon}\chi(\epsilon x)\left(\cos\left(\frac{x}{\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right)\right|^{p+1}dx\\
=&2^{p+1}\epsilon^{\frac{p-1}{2}}\int_{\mathbb{R}}\chi^{p+1}(x)\left|\cos\left(\frac{x}{\epsilon\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\epsilon\sqrt{\beta}}\right)\right|^{p+1}dx\\
\geq&2^{p+1}\epsilon^{\frac{p-1}{2}}\sum_{n=-\infty}^{\infty}\int_{2\pi n\epsilon\sqrt{\beta}}^{\left(2\pi n+\frac{\pi}{4}\right)\epsilon\sqrt{\beta}}\chi^{p+1}(x)\left|\min_x\cos\left(\frac{x}{\epsilon\sqrt{\beta}}\right)\right|^{p+1}dx
\end{aligned} \tag{2.13}$$

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$$=2^{p+1}\epsilon^{\frac{p-1}{2}}\left(\frac{\sqrt{2}}{2}\right)^{p+1}\sum_{n=-\infty}^{\infty}\int_{2\pi n\epsilon\sqrt{\beta}}^{\left(2\pi n+\frac{\pi}{4}\right)\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx$$
$$\geq C\epsilon^{\frac{p-1}{2}}\int_{\mathbb{R}}\chi^{p+1}(x)dx+O(\epsilon^{\frac{p+1}{2}}).$$

The last inequality here has used

$$\sum_{n=-\infty}^{\infty}\int_{2\pi n\epsilon\sqrt{\beta}}^{\left(2\pi n+\frac{\pi}{4}\right)\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx\geq C\int_{\mathbb{R}}\chi^{p+1}(x)dx+O(\epsilon), \tag{2.14}$$

which is proved in [22]; here, we give a modified version in appendix C.

Next, we show that

$$\int_{\mathbb{R}}|\omega_\epsilon(x)|^p\omega_\epsilon(x)dx\to 0,\quad \text{as } \epsilon\to 0^+. \tag{2.15}$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}}|\omega_\epsilon(x)|^p\omega_\epsilon(x)dx\\
=&2^{p+1}\epsilon^{\frac{p+1}{2}}\int_{\mathbb{R}}\chi^{p+1}(\epsilon x)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right|^p\\
&\times\left[\cos\left(\frac{x}{\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right]dx\\
=&2^{p+1}\epsilon^{\frac{p+1}{2}}\int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|\leq\epsilon^{\frac{1}{3}}}\chi^{p+1}(\epsilon x)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right|^p\\
&\qquad\cdot\left[\cos\left(\frac{x}{\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right]dx\\
&+2^{p+1}\epsilon^{\frac{p+1}{2}}\int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|>\epsilon^{\frac{1}{3}}}\chi^{p+1}(\epsilon x)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right|^p\\
&\qquad\cdot\left[\cos\left(\frac{x}{\sqrt{\beta}}\right)+\epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right]dx\\
=:&I_1+I_2,
\end{aligned}$$

where

$$\begin{aligned}
|I_1|\leq&2^{p+1}\epsilon^{\frac{p+1}{2}}\int_{\mathbb{R}}\chi^{p+1}(\epsilon x)\epsilon^{\frac{p}{3}}\epsilon^{\frac{1}{3}}dx\\
=&2^{p+1}\epsilon^{\frac{p+1}{2}+\frac{p+1}{3}}\int_{\mathbb{R}}\chi^{p+1}(x)dx\leq C\epsilon^{\frac{p}{2}+\frac{p+1}{3}},\quad \epsilon\to 0^+,
\end{aligned}$$

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and

$$
\begin{aligned}
I_2 =& 2^{p+1}\epsilon^{\frac{p+1}{2}} \int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|>\epsilon^{\frac{1}{3}}} \chi^{p+1}(\epsilon x) \left[\cos\left(\frac{x}{\sqrt{\beta}}\right) + \epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right] \\
&\qquad \cdot \left|\cos\left(\frac{x}{\sqrt{\beta}}\right) + \epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right|^p dx \\
=& 2^{p+1}\epsilon^{\frac{p+1}{2}} \int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|>\epsilon^{\frac{1}{3}}} \chi^{p+1}(\epsilon x) \left[\cos\left(\frac{x}{\sqrt{\beta}}\right) + \epsilon^{1-\sigma}\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right] \\
&\cdot \left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^p \left[1 + p\epsilon^{1-\sigma}\frac{\cos\left(\frac{2x}{\sqrt{\beta}}\right)}{\cos\left(\frac{x}{\sqrt{\beta}}\right)}\right] dx + O\left(\epsilon^{2\left(1-\sigma-\frac{1}{3}\right)+\frac{p-1}{2}}\right) \\
=& 2^{p+1}\epsilon^{\frac{p+1}{2}} \int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|>\epsilon^{\frac{1}{3}}} \chi^{p+1}(\epsilon x)\cos\left(\frac{x}{\sqrt{\beta}}\right)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^p dx \\
&+ 2^{p+1}(p+1)\epsilon^{\frac{p+1}{2}+1-\sigma} \int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|>\epsilon^{\frac{1}{3}}} \chi^{p+1}(\epsilon x)\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^p dx \\
&+ O\left(\epsilon^{\frac{p-1}{2}+2(1-\sigma)-\frac{2}{3}}\right) \\
=:& I_{21} + I_{22} + O\left(\epsilon^{\frac{p-1}{2}+2(1-\sigma)}\right).
\end{aligned}
$$

We can estimate that

$$
\begin{aligned}
I_{21} =& 2^{p+1}\epsilon^{\frac{p+1}{2}} \int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|>\epsilon^{\frac{1}{3}}} \chi^{p+1}(\epsilon x)\cos\left(\frac{x}{\sqrt{\beta}}\right)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^p dx \\
=& 2^{p+1}(p+1)\epsilon^{\frac{p+1}{2}} \int_{\mathbb{R}} \chi^{p+1}(\epsilon x)\cos\left(\frac{x}{\sqrt{\beta}}\right)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^p dx + O\left(\epsilon^{\frac{p+1}{2}}\right) \\
=& -2^{p+1}\epsilon^{\frac{p+1}{2}} \int_{\mathbb{R}} \chi^{p+1}\left(\epsilon\left(\frac{\pi}{2}-y\right)\right)\sin\left(\frac{y}{\sqrt{\beta}}\right)\left|\sin\left(\frac{y}{\sqrt{\beta}}\right)\right|^p dx + O\left(\epsilon^{\frac{p+1}{2}}\right) \\
=& -2^{p+1}\epsilon^{\frac{p+1}{2}} \int_{\mathbb{R}} \chi^{p+1}\left(\epsilon\left(\frac{\pi}{2}-y\right)\right)\left(8\partial_y\int_0^{\sin^2\left(\frac{y}{2\sqrt{\beta}}\right)}(s-s^2)^{\frac{p}{2}}ds\right)dx \\
&+ O\left(\epsilon^{\frac{p+1}{2}}\right) \\
=& -2^{p+1}\epsilon^{\frac{p+1}{2}} \int_{\mathbb{R}} \left(\chi^2\chi'\right)\left(\epsilon\left(\frac{\pi}{2}-y\right)\right)\cdot\left(\int_0^{\sin^2\left(\frac{y}{2\sqrt{\beta}}\right)}(s-s^2)ds\right)dx \\
&+ O\left(\epsilon^{\frac{p+1}{2}}\right) \\
=& O\left(\epsilon^{\frac{p+1}{2}}\right)
\end{aligned}
$$

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and

$$\begin{aligned}I_{22} =&2^{p+1}\epsilon^{\frac{p+1}{2}+1-\sigma}\int_{\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|>\epsilon^{\frac{1}{3}}}\chi^{p+1}(\epsilon x)\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}dx\\ =&(p+1)2^{p+1}\epsilon^{\frac{p+1}{2}+1-\sigma}\int_{\mathbb{R}}\chi^{p+1}(\epsilon x)\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}dx\\ &+O\left(\epsilon^{\frac{p+1}{2}+2(1-\sigma)-\frac{2}{3}}\right)\\ =:&(p+1)2^{p+1}\epsilon^{\frac{p+1}{2}+1-\sigma}I_{221}+O\left(\epsilon^{\frac{p+1}{2}+2(1-\sigma)-\frac{2}{3}}\right).\end{aligned}$$

Let the intervals

$$\triangle_i=\left(\left(2\pi n+\frac{i\pi}{4}\right)\sqrt{\beta},\left(2\pi n+\frac{(i+1)\pi}{4}\right)\sqrt{\beta}\right),\quad n\in\mathbb{Z},\ i=0,1,\cdots,7,$$

then

$$|\triangle_i|=|\triangle_j|,\quad \cup_{i=0}^{7}\triangle_i=\left(2\pi n\sqrt{\beta},2\pi(n+1)\sqrt{\beta}\right).$$

Thus, we can calculate

$$\begin{aligned}I_{221}=&\sum_{n=-\infty}^{+\infty}\sum_{i=0}^{7}\int_{\triangle_i}\chi^{p+1}(\epsilon x)\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}dx\\ =&\sum_{n=-\infty}^{+\infty}\left\{\int_{\triangle_0\cup\triangle_3\cup\triangle_4\cup\triangle_7}\chi^{p+1}(\epsilon x)\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left[\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}\right.\right.\\ &\left.-\left|\sin\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}\right]dx\\ &+\int_{\triangle_0\cup\triangle_4}\left[\chi^{p+1}\left(\epsilon\left(x+\frac{\pi}{2}\sqrt{\beta}\right)\right)-\chi^{p+1}(\epsilon x)\right]\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\sin\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}dx\\ &\left.+\int_{\triangle_3\cup\triangle_7}\left[\chi^{p+1}\left(\epsilon\left(x-\frac{\pi}{2}\sqrt{\beta}\right)\right)-\chi^{p+1}(\epsilon x)\right]\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\sin\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}dx\right\}.\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{n=-\infty}^{+\infty}\int_{\triangle_i}\chi^{p+1}(\epsilon x)\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left[\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}-\left|\sin\left(\frac{x}{\sqrt{\beta}}\right)\right|^{p}\right]dx\\ =&\sum_{n=-\infty}^{+\infty}\int_{\triangle_i}\chi^{p+1}(\epsilon x)\left|\cos\left(\frac{2x}{\sqrt{\beta}}\right)\right|^{p}dx>0,\quad i=0,3,4,7;\end{aligned}$$

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thus,

$$\sum_{n=-\infty}^{+\infty}\int_{\triangle_i}\chi^{p+1}(\epsilon x)\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left[\left|\cos\left(\frac{x}{\sqrt{\beta}}\right)\right|^p-\left|\sin\left(\frac{x}{\sqrt{\beta}}\right)\right|^p\right]dx$$
$$\geq C\epsilon^{-1}\int_{\mathbb{R}}\chi^{p+1}(x)dx+O(1),\quad i=0,3,4,7.$$

In addition,

$$\sum_{n=-\infty}^{+\infty}\left|\int_{\triangle_i}\left[\chi^{p+1}\left(\epsilon\left(x+\frac{\pi}{2}\sqrt{\beta}\right)\right)-\chi^{p+1}(\epsilon x)\right]\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\sin\left(\frac{x}{\sqrt{\beta}}\right)\right|^p dx\right|$$
$$\leq\sum_{n=-\infty}^{+\infty}\left|\int_{\triangle_i}\int_{\epsilon x}^{\epsilon\left(x+\frac{\pi}{2}\sqrt{\beta}\right)}\left(\chi^{p+1}\right)'(y)dydx\right|$$
$$\leq C\int_{\mathbb{R}}\left|\left(\chi^{p+1}\right)'(y)\right|dy,\quad i=0,4,$$

$$\sum_{n=-\infty}^{+\infty}\left|\int_{\triangle_i}\left[\chi^{p+1}\left(\epsilon\left(x-\frac{\pi}{2}\sqrt{\beta}\right)\right)-\chi^{p+1}(\epsilon x)\right]\cos\left(\frac{2x}{\sqrt{\beta}}\right)\left|\sin\left(\frac{x}{\sqrt{\beta}}\right)\right|^p dx\right|$$
$$\leq\sum_{n=-\infty}^{+\infty}\left|\int_{\triangle_i}\int_{\epsilon x}^{\epsilon\left(x-\frac{\pi}{2}\sqrt{\beta}\right)}\left(\chi^{p+1}\right)'(y)dydx\right|$$
$$\leq C\int_{\mathbb{R}}\left|\left(\chi^{p+1}\right)'(y)\right|dy,\quad i=3,7.$$

In conclusion, we get (2.15). Combining (2.13) and (2.15), we get

$$\mathcal{E}[\omega_\epsilon]\leq O\left(\epsilon^{2(1-\sigma)}\right)+\lambda-C\epsilon^l,$$

where $l<2(1-\sigma)$. Thus, we have proved (2.12). This completes the proof. □

2.3. Existence of constrained solitary waves

In this section, we use the concentrated compactness principle to study the existence of solutions to the minimization problems (1.6) and (1.7). By lemma 2.5, we only need to establish the existence of solutions to the minimization problem (1.6).

First, we establish strict subadditivity of $M_E(\lambda)$.

Lemma 2.7. *Given $\lambda>0$, $M_E(\lambda)$ has strict subadditivity, i.e.,*

$$M_E(\lambda)<M_E(\alpha)+M_E(\lambda-\alpha),\quad\forall\alpha\in(0,\lambda).$$

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Proof. Let

$$\mathbf{T} : \mathbb{S}_\lambda \to \mathbb{S}_\alpha, \quad \forall \alpha \in (0, \lambda),$$
$$\|u_x(\cdot)\|_{L^2} \mapsto \left\| \sqrt{\frac{\alpha}{\lambda}} u_x(\cdot) \right\|_{L^2},$$

where $\mathbb{S}_\lambda := \{u(x) : \|u(\cdot)\|_{L^2}^2 = \lambda\}$. It follows from lemma 2.6 that

$$\begin{aligned} M_E(\lambda) = &\inf_{\|\partial_x u\|_{L^2}^2 = \lambda} \Big\{ - \frac{\alpha_0}{p+1} \int_{\mathbb{R}} |\partial_x u|^p \partial_x u dx \\ &\qquad - \frac{\alpha_1}{p+1} \int_{\mathbb{R}} |\partial_x u|^{p+1} dx + \frac{\beta}{2} \int_{\mathbb{R}} |\partial_x^2 u|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}} |u|^2 dx \Big\} \\ =&\frac{\lambda}{\alpha} \inf_{\|\partial_x u\|_{L^2}^2 = \alpha} \Big\{ - \frac{\alpha_0}{p+1} \left(\frac{\lambda}{\alpha}\right)^{\frac{p-1}{2}} \int_{\mathbb{R}} |\partial_x u|^p \partial_x u dx \\ &\qquad - \frac{\alpha_1}{p+1} \left(\frac{\lambda}{\alpha}\right)^{\frac{p-1}{2}} \int_{\mathbb{R}} |\partial_x u|^{p+1} dx + \frac{\beta}{2} \int_{\mathbb{R}} |\partial_x^2 u|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}} |u|^2 dx \Big\} \\ <&\frac{\lambda}{\alpha} M_E(\alpha), \end{aligned}$$

i.e., $\lambda^{-1} M_E(\lambda)$ is decreasing with respect to λ. Thus,

$$M_E(\lambda) < \frac{\lambda}{\alpha} M_E(\alpha) = M_E(\alpha) + \frac{\lambda - \alpha}{\alpha} M_E(\alpha) \leq M_E(\alpha) + M_E(\lambda - \alpha), \quad \forall \alpha \geq \frac{\lambda}{2}.$$

If $\alpha < \frac{\lambda}{2}$, then $\lambda - \alpha > \frac{\lambda}{2}$. Thus, the above inequality implies

$$M_E(\lambda) < M_E(\lambda - \alpha) + M_E(\lambda - (\lambda - \alpha)) = M_E(\alpha) + M_E(\lambda - \alpha).$$

This completes the proof. □

Define

$$\mathfrak{u}_k(x) = |\partial_x u_k|^2.$$

We will use the concentrated compactness principle to establish the compactness.

LEMMA 2.8. *There exists $\{y_k\}_{k=1}^\infty \subset \mathbb{R}$ such that for any $\epsilon > 0$, there exists $r_\epsilon > 0$ satisfying*

$$\int_{U(y_k, r_\epsilon)} \mathfrak{u}_k dx \geq \int_{\mathbb{R}} \mathfrak{u}_k dx - \epsilon,$$

where $U(y_k, r_\epsilon) = \{x \in \mathbb{R} : |x - y_k| < r_\epsilon\}$.

Proof. According to the concentrated compactness principle (see the seminal work of Lions, p.115 ff. in [13]), $\{\mathfrak{u}_k\}_k$ satisfies one of the following three cases:

Case 1. Compactness. There exists $\{y_k\}_{k=1}^{\infty} \subset \mathbb{R}$, such that for any $\epsilon > 0$, there exists $r_\epsilon > 0$ satisfying

$$\int_{U(y_k, r_\epsilon)} \mathfrak{u}_k dx \geq \int_{\mathbb{R}} \mathfrak{u}_k dx - \epsilon,$$

where $U(y_k, r_\epsilon) = \{x \in \mathbb{R} : |x - y_k| < r_\epsilon\}$.
Case 2. Vanishing. For any $r > 0$,

$$\lim_{k \to +\infty} \sup_{y \in \mathbb{R}} \int_{U(y,r)} \mathfrak{u}_k dx = 0.$$

Case 3. Dichotomy. There exists $\alpha \in (0, \lambda)$ such that for $\epsilon > 0$, there exist $r > 0$, $r_k \to +\infty$, $\{y_k\} \subset \mathbb{R}$, and $k_0 \in \mathbb{R}$, such that for any $k \geq k_0$,

$$\max\left\{\left|\int_{|x-y_k|<r} \mathfrak{u}_k dx - \alpha\right|, \left|\int_{|x-y_k|>r_k} \mathfrak{u}_k dx - (\lambda - \alpha)\right|, \left|\int_{r<|x-y_k|<r_k} \mathfrak{u}_k dx\right|\right\} < \epsilon.$$

We claim that $\{\mathfrak{u}_k\}_k(x)$ can only occur in case 1. Indeed, assume that **case 2** holds. Let $\chi(x)$ be a smooth bump function satisfying

$$\chi \in [0,1], \quad \chi \equiv 1 \text{ on } (-1,1), \quad \text{supp}(\chi) \subset (-2,2),$$

then, the Gagliardo–Nirenberg–Sobolev inequality (2.9) implies that $\{u_k\}_{n=0}^{+\infty} \subset H^2$,

$$\begin{aligned} \int_{U(y,1)} |\partial_x u_k|^p \partial_x u_k dx \leq& \int_{\mathbb{R}} |\partial_x u_k(x)\chi(x-y)|^{p+1} dx \\ \leq& C \|\partial_x u_k(\cdot)\chi(\cdot - y)\|_{L^2}^{\frac{p+3}{2}} \|\partial_x (\partial_x u_k(\cdot)\chi(\cdot - y))\|_{L^2}^{\frac{p-1}{2}} \\ \leq& C \|\partial_x u_k(\cdot)\|_{L^2(U(y,2))}^{\frac{p+3}{2}} \end{aligned} \tag{2.16}$$

and

$$\int_{U(y,1)} |\partial_x u_k|^{p+1} dx \leq C \|\partial_x u_k(\cdot)\|_{L^2(U(y,2))}^{\frac{p+3}{2}}. \tag{2.17}$$

Vanishing implies that there exists $k_0 \gg 0$, such that for any $k \geq k_0$,

$$\int_{U(y,2)} \mathfrak{u}_k dx < \epsilon, \quad \forall y \in \mathbb{R}, \forall \epsilon > 0.$$

Selecting $\{y_n\}_{n=0}^{+\infty} \subset \mathbb{R}$ satisfies that $\cup_{n=0}^{+\infty} U(y_n, 1) = \mathbb{R}$, and for any $x \in \mathbb{R}$, there exists $\{n_j\}_{j=0}^{N} \subset \mathbb{N}$, $N < +\infty$ such that

$$x \notin \mathbb{R} \backslash \cup_{j=0}^{N} U(y_{n_j}, 1).$$

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Obviously, $\cup_{n=0}^{+\infty} U(y_n, 2) = \mathbb{R}$. Thus, according to (2.16) and (2.17), we get

$$\begin{aligned}
&\int_{\mathbb{R}} |\partial_x u_k|^p \partial_x u_k dx + \int_{\mathbb{R}} |\partial_x u_k|^{p+1} dx \\
\leq &\int_{\mathbb{R}} |\partial_x u_k|^{p+1} dx \leq \sum_{n=0}^{\infty} \int_{U(y_n,1)} |\partial_x u_k|^{p+1} dx \\
< &C \sum_{n=0}^{\infty} \epsilon^{\frac{p-1}{2}} \|\partial_x u_k(\cdot)\|_{L^2(U(y_n,2))}^2 \leq C\epsilon^{\frac{p-1}{2}} \|\partial_x u_k(\cdot)\|_{L^2}^2 .
\end{aligned}$$

Since $\sup_k \|u_k\|_{H^2} < +\infty$, selecting sufficiently small ϵ yields a contradiction to lemma 2.6. Thus, **case 2** cannot occur. Suppose that **case 3** holds. Dichotomy implies that there exist a subsequence of $\{u_k\}_{k=1}^{+\infty}$ (still denoted as $\{u_k\}_{k=1}^{+\infty}$) and a sequence $\{r_k\}_{k=1}^{+\infty} \subset \mathbb{R}$, satisfying $\lim\limits_{k\to+\infty} r_k = +\infty$ and $\{y_k\}_{k=1}^{+\infty} \subset \mathbb{R}$, such that

$$\begin{aligned}
&\lim_{k\to+\infty} \int_{\mathbb{R}} \left| \partial_x \left[u_k(x)\chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right]\right|^2 dx = \alpha, \\
&\lim_{k\to+\infty} \int_{\mathbb{R}} \left| \partial_x \left[u_k(x)\chi_2\left(\frac{x-y_k}{r_k}\right)\right]\right|^2 dx = \lambda - \alpha, \\
&\int_{\frac{r_k}{2}\leq |x-y_k|<r_k} |\partial_x (u_k(x))|^2 dx \leq \frac{1}{k},
\end{aligned}$$

where $\chi_1(x), \chi_2(x) \in C^\infty(\mathbb{R})$ are smooth cut-off functions satisfying

$$\chi_1(x), \chi_2(x) \in [0,1],\ \forall x \in \mathbb{R}, \quad \chi_1(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| \geq 2, \end{cases} \quad \chi_2(x) = \begin{cases} 1, & |x| \geq 1, \\ 0, & |x| \leq \frac{1}{2}. \end{cases}$$

Select $\{a_k\}_{k=1}^{+\infty}$ and $\{b_k\}_{k=1}^{+\infty} \subset \mathbb{R}$, satisfying

$$\begin{aligned}
&a_k, b_k \to 1, \quad k \to +\infty, \\
&\int_{\mathbb{R}} \left| \partial_x \left[a_k u_k(x)\chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right]\right|^2 dx = \alpha, \quad \forall k \geq 1, \\
&\int_{\mathbb{R}} \left| \partial_x \left[b_k u_k(x)\chi_2\left(\frac{x-y_k}{r_k}\right)\right]\right|^2 dx = \lambda - \alpha, \quad \forall k \geq 1.
\end{aligned}$$

Then,

$$\begin{aligned}
&E[u_k] - E\left[a_k u_k \chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right] - E\left[b_k u_k \chi_2\left(\frac{x-y_k}{r_k}\right)\right] \\
= &-\frac{\alpha_0}{p+1}\int_{\mathbb{R}} |\partial_x u_k|^p \partial_x u_k dx - \frac{\alpha_1}{p+1}\int_{\mathbb{R}} |\partial_x u_k|^{p+1} dx \\
&+ \frac{\beta}{2}\int_{\mathbb{R}} |\partial_x^2 u_k|^2 dx - \frac{\gamma}{2}\int_{\mathbb{R}} |u_k|^2 dx
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{\alpha_0}{p+1}\int_{\mathbb{R}}\left|\partial_x\left(a_k u_k\chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right)\right|^p\partial_x\left(a_k u_k\chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right)dx\\
&+\frac{\alpha_1}{p+1}\int_{\mathbb{R}}\left|\partial_x\left(a_k u_k\chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right)\right|^{p+1}dx\\
&-\frac{\beta}{2}\int_{\mathbb{R}}\left|\partial_x^2\left(a_k u_k\chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right)\right|^2dx+\frac{\gamma}{2}\int_{\mathbb{R}}\left|a_k u_k\chi_1\left(\frac{2(x-y_k)}{r_k}\right)\right|^2dx\\
&+\frac{\alpha_0}{p+1}\int_{\mathbb{R}}\left|\partial_x\left(b_k u_k\chi_2\left(\frac{x-y_k}{r_k}\right)\right)\right|^p\partial_x\left(b_k u_k\chi_2\left(\frac{x-y_k}{r_k}\right)\right)dx\\
&+\frac{\alpha_1}{p+1}\int_{\mathbb{R}}\left|\partial_x\left(b_k u_k\chi_2\left(\frac{x-y_k}{r_k}\right)\right)\right|^{p+1}dx\\
&-\frac{\beta}{2}\int_{\mathbb{R}}\left|\partial_x^2\left(b_k u_k\chi_2\left(\frac{x-y_k}{r_k}\right)\right)\right|^2dx+\frac{\gamma}{2}\int_{\mathbb{R}}\left|b_k u_k\chi_2\left(\frac{x-y_k}{r_k}\right)\right|^2dx\\
=&\int_{\mathbb{R}}\left[1-\chi_1^2\left(\frac{2(x-y_k)}{r_k}\right)-\chi_2^2\left(\frac{x-y_k}{r_k}\right)\right]\left[\frac{\beta}{2}\left|\partial_x^2u_k\right|^2-\frac{\gamma}{2}|u_k|^2\right]dx\\
&+\int_{\mathbb{R}}\left[(1-a_k^2)\chi_1^2\left(\frac{2(x-y_k)}{r_k}\right)+(1-b_k^2)\chi_2^2\left(\frac{x-y_k}{r_k}\right)\right]\left[\frac{\beta}{2}|\partial_x^2u_k|^2-\frac{\gamma}{2}|u_k|^2\right]dx\\
&-\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^p\partial_x u_k\left[1-\chi_1^{p+1}\left(\frac{2(x-y_k)}{r_k}\right)-\chi_2^{p+1}\left(\frac{x-y_k}{r_k}\right)\right]dx+O\left(\frac{1}{r_k}\right)\\
&-\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^p\partial_x u_k\left[\left(1-a_k^{p+1}\right)\chi_1^{p+1}\left(\frac{2(x-y_k)}{r_k}\right)+\left(1-b_k^{p+1}\right)\chi_2^{p+1}\right.\\
&\left.\times\left(\frac{x-y_k}{r_k}\right)\right]dx\\
&-\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^{p+1}\left[1-\chi_1^{p+1}\left(\frac{2(x-y_k)}{r_k}\right)-\chi_2^{p+1}\left(\frac{x-y_k}{r_k}\right)\right]dx\\
&-\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^{p+1}\left[\left(1-a_k^{p+1}\right)\chi_1^{4}\left(\frac{2(x-y_k)}{r_k}\right)+\left(1-b_k^{p+1}\right)\right.\\
&\left.\chi_2^{p+1}\left(\frac{x-y_k}{r_k}\right)\right]dx.
\end{aligned}
$$

By proposition 2.3, we get

$$
\begin{aligned}
&\int_{\mathbb{R}}\left[1-\chi_1^2\left(\frac{2(x-y_k)}{r_k}\right)-\chi_2^2\left(\frac{x-y_k}{r_k}\right)\right]\left[\frac{\beta}{2}\left|\partial_x^2u_k\right|^2-\frac{\gamma}{2}|u_k|^2\right]dx\\
=&\int_{\frac{r_k}{2}<|x-y_k|<r_k}\left[1-\chi_1^2\left(\frac{2(x-y_k)}{r_k}\right)-\chi_2^2\left(\frac{x-y_k}{r_k}\right)\right]\left[\frac{\beta}{2}\left|\partial_x^2u_k\right|^2-\frac{\gamma}{2}|u_k|^2\right]dx\\
\to&0,\quad k\to+\infty.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&-\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^p\partial_x u_k\left[1-\chi_1^{p+1}\left(\frac{2(x-y_k)}{r_k}\right)-\chi_2^{p+1}\left(\frac{x-y_k}{r_k}\right)\right]dx\to0,\\
&k\to+\infty,
\end{aligned}
$$

and

$$-\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^{p+1}\left[1-\chi_1^{p+1}\left(\frac{2(x-y_k)}{r_k}\right)-\chi_2^{p+1}\left(\frac{x-y_k}{r_k}\right)\right]dx \to 0,$$
$$k\to+\infty.$$

Since $a_k, b_k \to 1$, we obtain

$$\int_{\mathbb{R}}\left[(1-a_k^2)\chi_1^2\left(\frac{2(x-y_k)}{r_k}\right)+(1-b_k^2)\chi_2^2\left(\frac{x-y_k}{r_k}\right)\right]\left[\frac{\beta}{2}|\partial_x^2 u_k|^2-\frac{\gamma}{2}|u_k|^2\right]dx$$
$$\to 0, \quad k\to+\infty,$$

$$-\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^p\partial_x u_k\left[\left(1-a_k^{p+1}\right)\chi_1^{p+1}\left(\frac{2(x-y_k)}{r_k}\right)+\left(1-b_k^{p+1}\right)\right.$$
$$\left.\chi_2^{p+1}\left(\frac{x-y_k}{r_k}\right)\right]dx\to 0, \quad k\to+\infty,$$

and

$$-\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|\partial_x u_k|^{p+1}\left[\left(1-a_k^{p+1}\right)\chi_1^{p+1}\left(\frac{2(x-y_k)}{r_k}\right)\right.$$
$$\left.+\left(1-b_k^{p+1}\right)\chi_2^{p+1}\left(\frac{x-y_k}{r_k}\right)\right]dx\to 0, \quad k\to+\infty.$$

Therefore,

$$E[u_k]\geq M_E(\alpha)+M_E(\lambda-\alpha)+o_k(1).$$

Taking $k\to+\infty$ in the above equation, we get $M_E(\lambda)\geq M_E(\alpha)+M_E(\lambda-\alpha)$. This contradicts the strict subadditivity of lemma 2.7. Thus, we exclude **case 2**. This completes the proof. □

Next, we use lemma 2.8 to prove the existence of minimizers, which leads to the existence of constrained solitary waves.

Proof of theorem 1.2. According to lemma 2.5, theorem 1.2 is deduced by the following proposition. □

PROPOSITION 2.9. *There exists a solution for the minimization problem (2.7).*

Proof. Let $z_k(x) = u_k(x-y_k)$. Using (2.9) and the Young inequality, we get

$$\begin{aligned}E[z_k] &= -\frac{\alpha_0}{p+1}\int_{\mathbb{R}}|z_k'|^p z_k' dx-\frac{\alpha_1}{p+1}\int_{\mathbb{R}}|z_k'|^{p+1}dx+\frac{\beta}{2}\int_{\mathbb{R}}|z_k''|^2dx-\frac{\gamma}{2}\int_{\mathbb{R}}|z_k|^2dx\\ &\geq -\frac{|\alpha_0|+|\alpha_1|}{p+1}\|z_k'\|_{L^{p+1}}^{p+1}+\frac{\beta}{2}\|z_k''\|_{L^2}^2-\frac{\gamma}{2}\|z_k\|_{L^2}^2\end{aligned}$$

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$$\geq -C\|z_k'\|_{L^2}^{\left(1-\beta_{p+1}\right)(p+1)}\|z_k''\|_{L^2}^{\beta_{p+1}(p+1)} + \frac{\beta}{2}\|z_k''\|_{L^2}^2 - \frac{\gamma}{2}\|z_k\|_{L^2}^2$$
$$\geq -C_\epsilon \lambda^{\frac{\left(1-\beta_{p+1}\right)(p+1)}{2-\beta_{p+1}(p+1)}} + \left(\frac{\beta}{2}-\epsilon\right)\|z_k''\|_{L^2}^2 - \frac{\gamma}{2}\|z_k\|_{L^2}^2,$$

where $0 < \epsilon < \frac{\beta}{2}$ and $1 < p < 5$. This implies that $\{z_k\}_{k=1}^{+\infty} \subset H^2$ is bounded. Thus, there exists a subsequence of $\{z_k\}_{k=1}^{+\infty}$ (still denoted by $\{z_k\}_{k=1}^{+\infty}$) such that $z_k \rightharpoonup z$ in H^2. By lemma 2.8, there exists $r_\epsilon > 0$ such that

$$\int_{(U(0,r_\epsilon))^c} |\partial_x z_k|^2 dx < \epsilon. \tag{2.18}$$

By the Rellich–Kondrachov compact embedding $H^1(U(0,r_\epsilon)) \hookrightarrow L^2(U(0,r_\epsilon))$, there exists a subsequence of $\{z_k\}_{k=1}^{+\infty}$ (still denoted by $\{z_k\}_{k=1}^{+\infty}$) satisfying $\partial_x z_k \to \partial_x z$ in $L^2(U(0,r_\epsilon))$. Selecting $\epsilon = \frac{1}{n}$, letting $n \to +\infty$, and using (2.18), there exists a subsequence $\{z_k\}_{k=1}^{+\infty}$ satisfying $\partial_x z_k \to \partial_x z$ in L^2. In addition, using $H^1 \subset L^\infty$ and

$$||x|^a x - |y|^a y| \leq C|x-y|\left(|x|^a + |y|^a\right), \quad \forall x, y \in \mathbb{R},$$

we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}} |\partial_x z_k|^p \partial_x z_k dx - \int_{\mathbb{R}} |\partial_x z|^p \partial_x z dx\right| \\
\leq& C\int_{\mathbb{R}} |\partial_x z_k - \partial_x z| \left(|\partial_x z_k|^p + |\partial_x z|^p\right) dx \\
\leq& C\|\partial_x z_k - \partial_x z\|_{L^2}\left(\|\partial_x z_k\|_{L^2} + \|\partial_x z\|_{L^2}\right)^p
\end{aligned}$$

and

$$\left|\int_{\mathbb{R}} |\partial_x z_k|^{p+1} dx - \int_{\mathbb{R}} |\partial_x z|^{p+1} dx\right| \leq C\|\partial_x z_k - \partial_x z\|_{L^2}\left(\|\partial_x z_k\|_{L^2} + \|\partial_x z\|_{L^2}\right)^p.$$

Thus,

$$\begin{aligned}
\lim_{k\to+\infty}\int_{\mathbb{R}} |\partial_x z_k|^p \partial_x z_k dx &= \int_{\mathbb{R}} |\partial_x z|^p \partial_x z dx, \\
\lim_{k\to+\infty}\int_{\mathbb{R}} |\partial_x z_k|^{p+1} dx &= \int_{\mathbb{R}} |\partial_x z|^{p+1} dx.
\end{aligned} \tag{2.19}$$

Based on the lower semi-continuity of the norm and (2.19), we get

$$M_E(\lambda) = \lim_{k\to+\infty} E[z_k] \geq E[z].$$

Thus, $E[z] = M_E(\lambda)$, which means that z is a minimizer. This completes the proof of theorem 1.2. □

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3. Spectral stability

In this section, we consider the stability of the constrained solitary waves constructed in § 2.

3.1. Instability index and spectral stability

According to § 2, in order to study the spectral stability, we need to discuss the existence of nontrivial solution (ν, z) to the eigenvalue problem (1.10). We will use the instability index theory, which is a powerful tool for studying the spectral stability (see [4–7, 20]). We will introduce some basic results of instability index and establish a sufficient condition for spectral stability of the constrained solitary waves. Here, we adopt the theory of [11]. Consider a general linear Hamiltonian system $\partial_t u = JLu$, where J is anti-self-dual in the sense of $J^* = -J$ and L is a bounded symmetric operator in the Hilbert space satisfying $L^* = L$, such that $\langle Lu, v\rangle$ is a bounded symmetric bilinear form. For our problem, $J = \partial_x$, i.e., we consider the eigenvalue problem

$$\partial_x \mathcal{L} z = \nu z, \tag{3.1}$$

where $\mathcal{L} : X \to X^*$ is a bounded symmetry operator, $\dim(\mathrm{Ker}[\mathcal{L}]) < +\infty$, and

$$X = X_- \oplus \mathrm{Ker}[\mathcal{L}] \oplus X_+, \quad \dim(X_-) < +\infty.$$

Here, $\mathcal{L}_-|_{X_-} \leq -\delta$, $\mathcal{L}_+|_{X_+} \geq \delta$ for some $\delta > 0$, and X is a real Hilbert space. Denote $n^-(\mathcal{L}) := \dim(X_-)$ by the Morse index. Let $E_0 = \{u \in X : (\partial_x \mathcal{L})^k u = 0,\ k \in \mathbb{Z}^+\}$, then $\mathrm{Ker}[\mathcal{L}] \subset E_0$. Let $E_0 = \mathrm{Ker}[\mathcal{L}] \oplus \tilde{E}_0$, $Z \subset \tilde{E}_0$ satisfying $\langle \mathcal{L}z, z\rangle < 0$, $\forall z \in Z$, and $k_0^{\leq 0} = \max(\dim(Z))$. Let the number of solutions of (1.8) be k_c. According to Theorem 2.3 in [11], we have $k_c \leq n^-(\mathcal{L}) - k_0^{\leq 0}$. In particular, if $n^-(\mathcal{L}) = 1$ and $k_0^{\leq 0} \geq 1$, then the problem (1.8) is spectrally stable. For the eigenvalue problem (1.10), we select $X = H^1 \cap \dot{H}^{-1}$.

Next, we derive the Vakhitov–Kolokolov stability criterion. Suppose that Υ is sufficiently smooth satisfying $\Upsilon' \in \mathrm{Ker}[\mathcal{L}]$ and $\Upsilon \perp \mathrm{Ker}[\mathcal{L}]$. Since

$$(\partial_x \mathcal{L})^2(\mathcal{L}^{-1}\Upsilon) = (\partial_x \mathcal{L})\Upsilon' = \partial_x(\mathcal{L}\Upsilon') = 0, \quad (\partial_x \mathcal{L})(\mathcal{L}^{-1}\Upsilon) = \Upsilon',$$

we have $\mathcal{L}^{-1}\Upsilon \in \mathrm{Ker}[(\partial_x \mathcal{L})^2] \setminus \mathrm{Ker}[\partial_x \mathcal{L}] \subset \tilde{E}_0$. If $\langle \mathcal{L}(\mathcal{L}^{-1}\Upsilon), \mathcal{L}^{-1}\Upsilon\rangle < 0$, we get $k_0^{\leq 0}(\mathcal{L}) \geq 1$. This combined with $n^-(\mathcal{L}) = 1$ gives the spectral stability. Moreover, $\left\langle \mathcal{L}(\mathcal{L}^{-1}\Upsilon), \mathcal{L}^{-1}\Upsilon\right\rangle = \left\langle \mathcal{L}^{-1}\Upsilon, \Upsilon\right\rangle$. Note that if $\phi = \phi_\lambda$ is the minimizer of the minimization problem (1.7), then the eigenvalue problem (1.10) satisfies $\mathcal{L}\phi' = 0$. In fact, we have

Lemma 3.1. See [22] *If the solution $\phi = \phi_\lambda$ satisfies*

$$n^-(\mathcal{L}_+) = 1, \quad \phi \perp \mathrm{Ker}(\mathcal{L}_+), \quad \left\langle \mathcal{L}_+^{-1}\phi, \phi\right\rangle < 0,$$

then ϕ is spectrally stable, i.e., the eigenvalue problem (1.7) has no nontrivial solution. Furthermore, $\sigma(\partial_x \mathcal{L}_+) \subset \mathrm{i}\mathbb{R}$.

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To verify the conditions of lemma 3.1, we introduce the following lemma.

LEMMA 3.2. See [22] *Let L be a self-adjoint operator on a Hilbert space X satisfies $L|_{\{\phi_0\}^\perp} \geq 0$, where ϕ_0 satisfying $\|\phi_0\|_{L^2} = 1$ and $\phi_0 \perp \mathrm{Ker}[L]$. If $\langle L\phi_0, \phi_0\rangle \leq 0$, then $\langle L^{-1}\phi_0, \phi_0\rangle \leq 0$.*

3.2. Weak non-degeneracy and spectral stability

In this section, we prove the weak non-degeneracy and spectral stability of constrained solitary waves, which gives a proof of theorem 1.3.

First, we consider the number of negative eigenvalues of the linear operator.

PROPOSITION 3.3. *Suppose $\phi = \phi_\lambda$ is a minimizer of the constrained minimization problem (2.8), ω satisfies (A.4). Then, the linearized operator $\mathcal{L}_+ := -(\omega-\alpha)\,\mathrm{Id} + \alpha_0 p|\phi|^{p-2}\phi + \alpha_1 p|\phi|^{p-1} + \beta\partial_x^2 - \gamma\partial_x^{-2}$ satisfies*

$$\mathcal{L}_+|_{\{\phi\}^\perp} \geq 0.$$

Furthermore, $\mathcal{L}_+$ has a unique negative eigenvalue.

Proof. For v_δ defined by (A.3), we have

$$\begin{aligned}
\mathcal{E}[v_\delta] =& M_{\mathcal{E}}(\lambda) + \delta\Big\{ \int_{\mathbb{R}} \left[\alpha_0|\phi|^p\psi + \alpha_1|\phi|^{p-1}\phi\psi - \beta\phi'\psi' + \gamma\partial_x^{-1}\phi\partial_x^{-1}\psi\right] dx \\
&\qquad - \frac{1}{\lambda}\left[\int_{\mathbb{R}} \left(\alpha_0|\phi|^{p-1}\phi + \alpha_1|\phi|^{p+1} - \beta|\phi'|^2 + \gamma|\partial_x^{-1}\phi|^2\right) dx\right] \int_{\mathbb{R}} \phi\psi dx \Big\} \\
&+ \frac{\delta^2}{2}\Big\{ \int_{\mathbb{R}} \left[\alpha_0 p|\phi|^{p-2}\phi|\psi|^2 + \alpha_1 p|\phi|^{p-1}|\psi|^2 - \beta|\psi'|^2 + \gamma|\partial_x^{-1}\psi|^2\right] dx \\
&\qquad - \frac{1}{\lambda}\left[\int_{\mathbb{R}} \left(\alpha_0|\phi|^p\phi + \alpha_1|\phi|^{p+1} - \beta|\phi'|^2 + \gamma|\partial_x^{-1}\phi|^2\right) dx\right] \int_{\mathbb{R}} |\psi|^2 dx \Big\} \\
&+ \delta^2(p+1)\frac{1}{\lambda}\left(\int_{\mathbb{R}} \phi\psi dx\right) \int_{\mathbb{R}} \left[\alpha_0|\phi|^p\psi + \alpha_1|\phi|^{p-1}\phi\psi - \beta\phi'\psi'\right. \\
&\quad \left. + \gamma\partial_x^{-1}\phi\partial_x^{-1}\psi\right] dx \\
&+ \delta^2\frac{1}{\lambda^2}\left(\int_{\mathbb{R}} \phi\psi dx\right)^2 \int_{\mathbb{R}} \left[-\frac{p+3}{2}\alpha_0|\phi|^p\phi - \frac{p+3}{2}\alpha_1|\phi|^{p+1} + \beta|\phi'|^2\right. \\
&\quad \left. + \gamma|\partial_x^{-1}\phi|^2\right] dx \\
&+ O\left(\delta^3\right).
\end{aligned}$$

Since ϕ is a minimizer of problem (2.8) and w satisfies (A.4), the terms of δ^2 must be non-negative. Thus, if we choose ψ satisfying $\psi \perp \phi$ and $\|\psi\|_{L^2} = 1$, then

$$\begin{aligned}
&\int_{\mathbb{R}} \left[\alpha_0 p|\phi|^{p-2}\phi|\psi|^2 + \alpha_1 p|\phi|^{p-1}|\psi|^2 - \beta|\psi'|^2 + \gamma|\partial_x^{-1}\psi|^2\right] dx \\
&- \frac{1}{\lambda}\left[\int_{\mathbb{R}} \left(\alpha_0|\phi|^p\phi + \alpha_1|\phi|^{p+1} - \beta|\phi'|^2 + \gamma|\partial_x^{-1}\phi|^2\right) dx\right] \int_{\mathbb{R}} |\psi|^2 dx \geq 0,
\end{aligned}$$

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i.e., $\langle \mathcal{L}_+\psi, \psi\rangle \geq 0$. Thus, we get $\mathcal{L}_+|_{\{\phi\}^\perp} \geq 0$, which means that the second smallest eigenvalue of $\mathcal{L}_+$ must be non-negative, i.e., $n(\mathcal{L}_+) \leq 1$. In addition, by lemma 2.6 and (A.2), we have

$$\begin{aligned} \langle \mathcal{L}_+\phi, \phi\rangle &= \left\langle \left(-(\omega-\alpha)\operatorname{Id} + \alpha_0 p|\phi|^{p-2}\phi + \alpha_1 p|\phi|^{p-1} + \beta\partial_x^2 - \gamma\partial_x^{-2}\right)\phi, \phi\right\rangle \\ &= \left\langle -(p-1)\left(\alpha_0|\phi|^{p-2}\phi + \alpha_1|\phi|^{p-1}\right)\phi, \phi\right\rangle < 0. \end{aligned} \tag{3.2}$$

This implies that there exist negative eigenvalues for $\mathcal{L}_+$. Therefore, there exists a unique negative eigenvalue for $\mathcal{L}_+$. □

Proof of theorem 1.3. First, we show that $\phi = \phi_\lambda$ satisfies weak non-degeneracy, i.e., $\phi \perp \operatorname{Ker}[\mathcal{L}_+]$. Here, ϕ is the minimizer of the minimization problem (1.7) and $\mathcal{L}_+$ is the linearized operator of equation (1.1), i.e., we consider $\mathcal{L}_+ = -(\omega - \alpha)\operatorname{Id} + \alpha_0 p|\phi|^{p-1} + \alpha_1 p|\phi|^{p-2}\phi + \beta\partial_x^2 - \gamma\partial_x^{-2}$. Considering that the minimizer of the minimization problem (1.6) and the linearized operators corresponding to the equation (1.3) are analogous. For any $\Upsilon \in \operatorname{Ker}[\mathcal{L}_+]$ satisfying

$$\|\Upsilon\|_{L^2} = 1.$$

According to proposition 3.3, we get

$$\mathcal{L}_+|_{\{\phi\}^\perp} \geq 0. \tag{3.3}$$

By

$$\left(\Upsilon - \frac{1}{\lambda}\langle\Upsilon, \phi\rangle\phi\right) \perp \phi, \quad \|\phi\|_{L^2}^2 = \lambda,$$

and (3.2), we have

$$0 \leq \left\langle \mathcal{L}_+\left[\Upsilon - \frac{1}{\lambda}\langle\Upsilon, \phi\rangle\phi\right], \Upsilon - \frac{1}{\lambda}\langle\Upsilon, \phi\rangle\phi\right\rangle = \frac{1}{\lambda^2}\langle\Upsilon, \phi\rangle^2\langle\mathcal{L}_+\phi, \phi\rangle \leq 0.$$

Thus, $\langle\Upsilon, \phi\rangle = 0$. This proves that ϕ has weak non-degeneracy.

Second, we prove the spectral stability. According to lemma 3.2, we select $L = \mathcal{L}_+$ and $\phi_0 = \frac{1}{\sqrt{\lambda}}\phi$. By (3.2) and (3.3), we obtain that lemma 3.2 implies $\langle\mathcal{L}_+^{-1}\phi, \phi\rangle \leq 0$. Since $\langle\mathcal{L}_+^{-1}\phi, \phi\rangle \neq 0$, we get $\langle\mathcal{L}_+^{-1}\phi, \phi\rangle < 0$. Thus, using lemma 3.1, we obtain that ϕ is spectrally stable. This proves theorem 1.3. □

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Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Availability of data and materials

Not applicable. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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Appendix A. Derivation of Euler–Lagrange equations

In this appendix, we derive the Euler–Lagrange equations corresponding to the minimization problems (1.6) and (1.7).

PROPOSITION A.1. *There exists $\omega \in \mathbb{R}$, such that the solutions to the constrained minimization problems (1.6) and (1.7), respectively, satisfy the Euler–Lagrange equations*

$$(\alpha-\omega)\phi'' + \alpha_0 \left(|\phi'|^p\right)' + \alpha_1 \left(|\phi'|^{p-1}\phi'\right)' + \beta\phi'''' - \gamma\phi = 0 \tag{A.1}$$

and

$$(\alpha-\omega)\phi + \alpha_0|\phi|^p + \alpha_1|\phi|^{p-1}\phi + \beta\phi'' - \gamma\partial_x^{-2}\phi = 0. \tag{A.2}$$

Proof. Let

$$u_\delta = \sqrt{\lambda}\frac{\phi+\delta\psi}{\|\phi'+\delta\psi'\|_{L^2}},$$

where ψ is a test function. Obviously, $\|\partial_x u_\delta\|_{L^2} = \lambda$ and

$$\begin{aligned} E[u_\delta] =& -\frac{\alpha_0}{p+1}\int_{\mathbb{R}} |\partial_x u_\delta|^p \partial_x u_\delta dx - \frac{\alpha_1}{p+1}\int_{\mathbb{R}} |\partial_x u_\delta|^{p+1} dx \\ &+ \frac{\beta}{2}\int_{\mathbb{R}} |\partial_x^2 u_\delta|^2 dx - \frac{\gamma}{2}\int_{\mathbb{R}} |u_\delta|^2 dx \\ =& M_E(\lambda) + \delta\Big\{ \int_{\mathbb{R}} \left(-\alpha_0|\phi'|^p\psi' - \alpha_1|\phi'|^{p-1}\phi'\psi' + \beta\phi''\psi'' - \gamma\phi\psi\right) dx \\ &- \frac{1}{\lambda}\left[\int_{\mathbb{R}} \left(-\alpha_0|\phi'|^p\phi' - \alpha_1|\phi'|^{p+1} + \beta|\phi''|^2 - \gamma|\phi|^2\right) dx\right]\int_{\mathbb{R}} \phi'\psi' dx \Big\} + O(\delta). \end{aligned}$$

Since $E[u_\delta] \geq M_E(\lambda)$, $\forall \delta \in \mathbb{R}$, we choose w satisfying

$$\alpha - \omega = \frac{1}{\lambda}\int_{\mathbb{R}} \left(-\alpha_0|\phi'|^p\phi' - \alpha_1|\phi'|^{p+1} + \beta|\phi''|^2 - \gamma|\phi|^2\right) dx,$$

then

$$\Big\langle (\alpha-\omega)\phi'' + \alpha_0\left(|\phi'|^p\right)' + \alpha_1\left(|\phi'|^{p-1}\phi'\right)' + \beta\phi'''' - \gamma\phi, \psi \Big\rangle = 0, \quad \forall\psi,$$

This implies that ϕ is a distribution solution of (A.1).

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Similarly, let

$$v_\delta = \sqrt{\lambda}\frac{\phi + \delta\psi}{\|\phi + \delta\psi\|_{L^2}}, \tag{A.3}$$

then, $\|v_\delta\|_{L^2} = \lambda$ and

$$\begin{aligned}\mathcal{E}[v_\delta] =& -\frac{\alpha_0}{p+1}\int_{\mathbb{R}} |v_\delta|^p v_\delta dx - \frac{\alpha_1}{p+1}\int_{\mathbb{R}} |v_\delta|^{p+1} dx \\ &+ \frac{\beta}{2}\int_{\mathbb{R}} |\partial_x v_\delta|^2 dx - \frac{\gamma}{2}\int_{\mathbb{R}} |\partial_x^{-1} v_\delta|^2 dx \\ =& M_{\mathcal{E}}(\lambda) + \delta\Big\{ \int_{\mathbb{R}} \left[\alpha_0|\phi|^p\psi + \alpha_1|\phi|^{p-1}\phi\psi - \beta\phi'\psi' + \gamma\partial_x^{-1}\phi\partial_x^{-1}\psi\right] dx \\ &- \frac{1}{\lambda}\left[\int_{\mathbb{R}} \left(\alpha_0|\phi|^p\phi + \alpha_1|\phi|^{p+1} - \beta|\phi'|^2 + \gamma|\partial_x^{-1}\phi|^2\right) dx\right] \int_{\mathbb{R}} \phi\psi dx\Big\} + O\left(\delta^2\right).\end{aligned}$$

Since $\mathcal{E}[v_\delta] \geq M_{\mathcal{E}}(\lambda)$, $\forall\delta \in \mathbb{R}$, we choose w satisfying

$$\alpha - \omega = \frac{1}{\lambda}\int_{\mathbb{R}} \left(\alpha_0|\phi|^p\phi + \alpha_1|\phi|^{p+1} - \beta|\phi'|^2 + \gamma|\partial_x^{-1}\phi|^2\right) dx, \tag{A.4}$$

then

$$\left\langle(\alpha - \omega)\phi + \alpha_0|\phi|^p + \alpha_1|\phi|^{p-1}\phi + \beta\phi'' - \gamma\partial_x^{-2}\phi, \psi\right\rangle = 0, \quad \forall\psi,$$

This implies that ϕ is a distribution solution of (A.2). □

Appendix B. Pohozaev identity

We establish the following Pohozaev identity.

LEMMA B.1. *Suppose that $\phi \in H^2$ is a weak solution of (1.5), then*

$$\begin{aligned}\int_{\mathbb{R}} |\phi''|^2 dx =& \frac{(2p-1)\alpha_0}{2(p+1)\beta}\int_{\mathbb{R}} |\phi'|^p\phi' dx + \frac{(2p-1)\alpha_1}{2(p+1)\beta}\int_{\mathbb{R}} |\phi'|^{p+1} dx \\ &- \frac{\gamma}{\beta}\int_{\mathbb{R}} |\phi|^2 dx, \\ \frac{2(\omega-\alpha)}{3}\int_{\mathbb{R}} |\phi'|^2 dx =& \frac{(3-2p)\alpha_0}{3(p+1)}\int_{\mathbb{R}} |\phi'|^2\phi' dx + \frac{(3-2p)\alpha_1}{3(p+1)}\int_{\mathbb{R}} |\phi'|^4 dx \\ &+ \frac{4\gamma}{3}\int_{\mathbb{R}} |\phi|^2 dx.\end{aligned} \tag{A.1}$$

Proof. Multiplying ϕ at both sides of (1.5) and integrating the result over $\mathbb{R}$, based on proposition 2.3, we get

$$\begin{aligned}&(\omega - \alpha)\int_{\mathbb{R}} |\phi'|^2 dx + \beta\int_R |\phi''|^2 dx \\ =& \frac{\alpha_0}{p+1}\int_{\mathbb{R}} |\phi'|^p\phi' dx + \frac{\alpha_1}{p+1}\int_{\mathbb{R}} |\phi'|^{p+1} dx + \gamma\int_{\mathbb{R}} |\phi|^2 dx.\end{aligned} \tag{A.2}$$

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In addition, note that

$$(\alpha-\omega)\int_{\mathbb{R}}\phi''x\phi'dx=\frac{\omega-\alpha}{2}\int_{\mathbb{R}}|\phi'|^2dx,$$

$$\begin{aligned}\alpha_0\int_{\mathbb{R}}(|\phi'|^p)'\,x\phi'dx=&-\alpha_0\int_{\mathbb{R}}|\phi'|^p\phi'dx-\frac{\alpha_0}{p+1}\int_{\mathbb{R}}(|\phi'|^p\phi')'\,xdx\\ =&-\frac{p\alpha_0}{p+1}\int_{\mathbb{R}}|\phi'|^p\phi'dx,\end{aligned}$$

$$\begin{aligned}\alpha_1\int_{\mathbb{R}}(|\phi'|^p\phi')'\,\phi'xdx=&-\alpha_1\int_{\mathbb{R}}|\phi'|^{p+1}dx-\frac{\alpha_1}{p+1}\int_{\mathbb{R}}\left(|\phi'|^{p+1}\right)'xdx\\ =&-\frac{p\alpha_1}{p+1}\int_{\mathbb{R}}|\phi'|^{p+1}dx,\end{aligned}$$

$$\begin{aligned}\beta\int_{\mathbb{R}}\phi''''\phi'xdx=&\beta\int_{\mathbb{R}}|\phi''|^2dx-\frac{\beta}{2}\int_{\mathbb{R}}\left(|\phi''|^2\right)'xdx\\ =&\frac{3\beta}{2}\int_{\mathbb{R}}|\phi''|^2dx,\end{aligned}$$

$$-\gamma\int_{\mathbb{R}}\phi\phi'xdx=\frac{\gamma}{2}\int_{\mathbb{R}}|\phi|^2dx.$$

Multiplying $x\phi'$ at both sides of (1.5) and integrating the result over $\mathbb{R}$, we get

$$\begin{aligned}&\frac{\omega-\alpha}{2}\int_{\mathbb{R}}|\phi'|^2dx+\frac{3\beta}{2}\int_{\mathbb{R}}|\phi''|^2dx\\ =&\frac{p\alpha_0}{p+1}\int_{\mathbb{R}}|\phi'|^p\phi'dx+\frac{p\alpha_1}{p+1}\beta\int_{\mathbb{R}}|\phi'|^{p+1}dx-\frac{\gamma}{2}\int_{\mathbb{R}}|\phi|^2dx.\end{aligned}\tag{A.3}$$

Combining (A.2) and (A.3), we get the Pohozaev identity (A.1). □

Remark B.2. According to lemma B.1, for the weak solution $\phi\in H^1\cap H^{-1}$ of equation $(\alpha-\omega)\phi+\alpha_0|\phi|^p+\alpha_1|\phi|^{p-1}\phi+\beta\phi''-\gamma\partial_x^{-2}\phi=0$, we have

$$\begin{aligned}\int_{\mathbb{R}}|\phi'|^2dx=&\frac{(2p-1)\alpha_0}{2(p+1)\beta}\int_{\mathbb{R}}|\phi|^2\phi dx+\frac{(2p-1)\alpha_1}{2(p+1)\beta}\int_{\mathbb{R}}|\phi|^4dx\\ &+\frac{\gamma}{\beta}\int_{\mathbb{R}}|\partial^{-1}\phi|^2dx,\\ \frac{2(\omega-\alpha)}{3}\int_{\mathbb{R}}|\phi|^2dx=&\frac{5\alpha_0}{18}\int_{\mathbb{R}}|\phi|^2\phi dx+\frac{\alpha_1}{6}\int_{\mathbb{R}}|\phi|^4dx+\frac{4\gamma}{3}\int_{\mathbb{R}}|\partial^{-1}\phi|^2dx.\end{aligned}$$

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Appendix C. Proof of inequality (2.14)

We prove the inequality (2.14), which can be obtained by the following estimate. It is a modified version of one in [22].

Proposition C.1. *The following inequality holds:*

$$\left|\sum_{n=-\infty}^{+\infty}\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx-\frac{1}{8}\int_{\mathbb{R}}\chi^{p+1}(x)dx\right|$$
$$\leq\frac{7(p+1)}{32}\pi\epsilon\sqrt{\beta}\int_{\mathbb{R}}\left|\chi^{3}(y)\chi'(y)\right|dx.$$

Proof. Splitting the interval $(2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta},2\pi(n+1)\epsilon\sqrt{\beta})$ into seven intervals with the same length, i.e.,

$$\left(2\pi n\epsilon\sqrt{\beta}+\frac{m\pi}{4}\epsilon\sqrt{\beta},2\pi n\epsilon\sqrt{\beta}+\frac{(m+1)\pi}{4}\epsilon\sqrt{\beta}\right),\quad m=1,2,\cdots,7.$$

We can calculate

$$\begin{aligned}
&8\sum_{n=-\infty}^{+\infty}\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx\\
=&\sum_{n=-\infty}^{+\infty}\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi(n+1)\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx\\
&+\sum_{n=-\infty}^{+\infty}\left[7\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx-\sum_{m=1}^{7}\int_{2\pi n\epsilon\sqrt{\beta}+\frac{m\pi}{4}\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{(m+1)\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx\right]\\
=&\int_{\mathbb{R}}\chi^{p+1}(x)dx\\
&+\sum_{n=-\infty}^{+\infty}\sum_{m=1}^{7}\left[\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx-\int_{2\pi n\epsilon\sqrt{\beta}+\frac{m\pi}{4}\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{(m+1)\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx\right].
\end{aligned}$$

Thus, according to

$$\begin{aligned}
&\left|\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx-\int_{2\pi n\epsilon\sqrt{\beta}+\frac{m\pi}{4}\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{(m+1)\pi}{4}\epsilon\sqrt{\beta}}\chi^{p+1}(x)dx\right|\\
=&\left|\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}}\left[\chi^{p+1}(x)-\chi^{p+1}\left(x+\frac{m\pi}{4}\epsilon\sqrt{\beta}\right)\right]dx\right|\\
\leq&(p+1)\int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}}\int_{x}^{x+\frac{m\pi}{4}\epsilon\sqrt{\beta}}\left|\chi^{p}(y)\chi'(y)\right|dx
\end{aligned}$$

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$$
\begin{aligned}
&\leq (p+1) \int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}} \int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{(m+1)\pi}{4}\epsilon\sqrt{\beta}} |\chi^p(y)\chi'(y)|\, dx \\
&\leq \frac{p+1}{4}\pi\epsilon\sqrt{\beta} \int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi(n+1)\epsilon\sqrt{\beta}} |\chi^p(y)\chi'(y)|\, dx, \quad m = 1, 2, \cdots, 7,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left| 8 \sum_{n=-\infty}^{+\infty} \int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}} \chi^{p+1}(x)dx - \int_{\mathbb{R}} \chi^{p+1}(x)dx \right| \\
\leq &\sum_{n=-\infty}^{+\infty} \sum_{m=1}^{7} \left| \int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{\pi}{4}\epsilon\sqrt{\beta}} \chi^{p+1}(x)dx - \int_{2\pi n\epsilon\sqrt{\beta}+\frac{m\pi}{4}\epsilon\sqrt{\beta}}^{2\pi n\epsilon\sqrt{\beta}+\frac{(m+1)\pi}{4}\epsilon\sqrt{\beta}} \chi^{p+1}(x)dx \right| \\
\leq &\frac{7(p+1)}{4}\pi\epsilon\sqrt{\beta} \sum_{n=-\infty}^{+\infty} \int_{2\pi n\epsilon\sqrt{\beta}}^{2\pi(n+1)\epsilon\sqrt{\beta}} |\chi^p(y)\chi'(y)|\, dx \\
\leq &\frac{7(p+1)}{4}\pi\epsilon\sqrt{\beta} \int_{\mathbb{R}} |\chi^p(y)\chi'(y)|\, dx.
\end{aligned}
$$

This completes the proof. □

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