SOME INEQUALITIES RELATED TO THE WALD-WOLFOWITZ-NOETHER CONDITION*

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1. Introduction. If $\{a_{\nu\alpha} : \alpha = 1, 2, \ldots, N_\nu\}$, with $N_\nu \to \infty$ as $\nu \to \infty$, is a double sequence of real numbers with the property that $\sum_{\alpha=1}^{N_\nu} a_{\nu\alpha} = 0$, then

$$(1.1) \qquad \lim_{\nu\to\infty} \frac{\max_{1\le\alpha\le N_\nu} (a_{\nu\alpha}^2)}{\sum_{\alpha=1}^{N_\nu} a_{\nu\alpha}^2} = 0$$

is known in statistical literature as the Wald-Wolfowitz-Noether condition and it plays an important role in the proofs of certain types of central limit theorems (see e.g., [1], [2]). The purpose of this note is to show that for certain types of numbers ($a_{\nu\alpha}$'s whose construction is described below), the condition (1.1) is always satisfied. For example, consider a sequence $\Lambda_\nu = (\theta_1, \theta_2, \ldots, \theta_\nu)$ of real numbers (not all θ's are equal), and form all possible pairs of differences $a_{ij} = \theta_i - \theta_j$, $i \neq j$. The total number of such differences is $N_\nu = \nu(\nu - 1)$ and if the

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ordered pairs (i, j), with $i \neq j$ and $i, j = 1, 2, \ldots, \nu$, are labelled $\alpha = 1, 2, \ldots, N_\nu$ in some convenient manner, the condition (1.1) takes the form:

$$(1.2) \qquad \lim_{\nu \to \infty} \frac{\max\limits_{1 \leq i, j \leq \nu} (\theta_i - \theta_j)^2}{\sum\limits_{1 \leq i, j \leq \nu} (\theta_i - \theta_j)^2} = 0 .$$

It follows from our results that (1.2) is always satisfied without any restriction on the numbers θ_i. This result was proved and employed by one of us in [2], and although the proof there is quite elementary and does not resemble in any way the proof for the general case here, the result itself is rather elusive. The results of the present note, however, are stronger and more general: we obtain for the left hand side (L.H.S.) of (1.2), with the second power replaced by the absolute p^{th} power, the best possible bound for all p, $0 \leq p < \infty$, which converges to zero as $\nu \to \infty$. Our main result is concerned with finding an upper bound for the L.H.S. of (1.1), with the second power replaced by the absolute p^{th} power and $a_{\nu\alpha}$'s as numbers formed by selecting all possible subsets of size s $(2 \leq s \leq \nu - 1)$ from Λ_ν and using a generalization of the notion of differences, such that the bound is independent of the θ's and converges to zero as $\nu \to \infty$. The authors hope that although the results are of rather elementary nature, they might be useful and of interest to others.

2. <u>The main theorem.</u> Let $\Lambda_\nu = (\theta_1, \theta_2, \ldots, \theta_\nu)$ be a sequence of real numbers, not all equal, and consider all possible subsets, $t_\nu = \binom{\nu}{s}$ in number, of Λ_ν of size $s (2 \leq s \leq \nu - 1)$, labelling them in some convenient manner, $\ell = 1, 2, \ldots, t_\nu$. Let C_ℓ denote the ℓ^{th} subset of size s and $A_\ell = \{i; \theta_i \in C_\ell\}$.

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THEOREM 1. For all p, $0 \leq p < \infty$, and all $s (2 \leq s \leq \nu - 1)$,

$$(2.1) \quad \frac{\max\limits_{1 \leq \ell \leq t_\nu} \max\limits_{i \in A_\ell} |\theta_i - \bar{\theta}_\ell|^p}{\sum\limits_{\ell=1}^{t_\nu} \sum\limits_{i \in A_\ell} |\theta_i - \bar{\theta}_\ell|^p} \leq \frac{\left(\frac{s-1}{s}\right)^p K_p}{\left[\left\{\left(\frac{s-1}{s}\right)^p + \frac{s-1}{s^p}\right\}\binom{\nu-2}{s-1} + \binom{\nu-2}{s-2}\right]}$$

where $\bar{\theta}_\ell = \left(\sum\limits_{i \in A_\ell} \theta_i\right) / s$, and $K_p = 1$ or 2^{p-1} according as $p \leq 1$ or $p \geq 1$.

Proof. We may assume without loss of generality that $|\theta_1 - \theta_\nu| = \max\limits_{1 \leq i < j \leq \nu} |\theta_i - \theta_j|$. First, we observe that

$$\max_{1 \leq \ell \leq t_\nu} \max_{i \in A_\ell} |\theta_i - \theta_\ell|$$

$$\leq \max_{1 \leq \ell \leq t_\nu} \max_{i \in A_\ell} \left\{ \sum_{k \in A_\ell - \{i\}} \frac{|\theta_i - \theta_k|}{s} \right\}$$

$$\leq \max_{1 \leq \ell \leq t_\nu} \left\{ \left(\frac{s-1}{s}\right) |\theta_1 - \theta_\nu| \right\}$$

$$= \left(\frac{s-1}{s}\right) |\theta_1 - \theta_\nu| ,$$

so that

$$(2.2) \quad \max_{1 \leq \ell \leq t_\nu} \max_{i \in A_\ell} |\theta_i - \bar{\theta}_\ell|^p \leq \left(\frac{s-1}{s}\right)^p |\theta_1 - \theta_\nu|^p .$$

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Next, let $B_i = \{\ell : i \in A_\ell\}$ and denote by $D_{1\nu}$ the class of all subsets of Λ_ν of $(s - 1)$ elements chosen out of $\Lambda_\nu - \{\theta_1, \theta_\nu\}$, labelling these subsets by $r = 1, 2, \ldots, t^*_\nu$ where $t^*_\nu = \binom{\nu-2}{s-1}$. We now observe that

$$\sum_{\ell=1}^{t_\nu} \sum_{i \in A_\ell} |\theta_i - \bar{\theta}_\ell|^p$$

$$\geq \sum_{\ell \in B_1 B_\nu^c} + \sum_{\ell \in B_1^c B_\nu} + \sum_{\ell \in B_1 B_\nu} \left\{ \sum_{i \in A_\ell} |\theta_i - \bar{\theta}_\ell|^p \right\}$$

$$\geq \sum_{\ell \in B_1 B_\nu^c} |\theta_1 - \bar{\theta}_\ell|^p + \sum_{\ell \in B_1 B_\nu^c} \sum_{i \in A_\ell - \{1\}} |\theta_i - \bar{\theta}_\ell|^p$$

$$+ \sum_{\ell \in B_1^c B_\nu} |\theta_\nu - \theta_\ell|^p + \sum_{\ell \in B_1^c B_\nu} \sum_{i \in A_\ell - \{\nu\}} |\theta_i - \bar{\theta}_\ell|^p$$

$$+ \sum_{\ell \in B_1 B_\nu} \left\{ |\theta_1 - \bar{\theta}_\ell|^p + |\theta_\nu - \theta_\ell|^p \right\} .$$

The last inequality follows by ignoring certain terms. Now let C^*_γ denote the γ^{th} subset in $D_{1\nu}$ and $A^*_\gamma = \{i : \theta_i \in C^*_\gamma\}$. Then letting $\bar{\theta}^*_\gamma = \left(\sum_{i \in A^*_\gamma} \theta_i \right) / s - 1$ and regrouping the terms, we note that the last expression is

$$= \sum_{\gamma=1}^{t^*_\nu} \left| \theta_1 - \frac{(s-1)\bar{\theta}^*_\gamma + \theta_1}{s} \right|^p + \sum_{\gamma=1}^{t^*_\nu} \left| \theta_\nu - \frac{(s-1)\bar{\theta}^*_\gamma + \theta_\nu}{s} \right|^p$$

$$+\sum_{\gamma=1}^{t_\nu^*}\sum_{i\in A_\gamma^*}\left|\theta_i-\frac{(s-1)\theta_\gamma^*+\theta_1}{s}\right|^p+\sum_{\gamma=1}^{t_\nu^*}\sum_{i\in A_\gamma^*}\left|\theta_i-\frac{(s-1)\theta_\gamma^*+\theta_\nu}{s}\right|^p$$

$$+\sum_{\ell\in B_1B_\nu}\{|\theta_1-\bar\theta_\ell|^p+|\theta_\nu-\bar\theta_\ell|^p\}$$

$$=\left(\frac{s-1}{s}\right)^p\sum_{\gamma=1}^{t_\nu^*}\{|\theta_1-\bar\theta_\gamma^*|^p+|\theta_\nu-\bar\theta_\gamma^*|^p\}$$

$$+\sum_{\gamma=1}^{t_\nu^*}\sum_{i\in A_\gamma^*}\frac{\left\{\left|\theta_1-\left(s\theta_i-(s-1)\bar\theta_\gamma^*\right)\right|^p+\left|\theta_\nu-\left(s\theta_i-(s-1)\bar\theta_\gamma^*\right)\right|^p\right\}}{s^p}$$

$$+\sum_{\ell\in B_1B_\nu}\{|\theta_1-\bar\theta_\ell|^p+|\theta_\nu-\bar\theta_\ell|^p\}$$

$$\geq\left(\frac{s-1}{s}\right)^p\sum_{\gamma=1}^{t_\nu^*}\frac{1}{K_p}|\theta_1-\theta_\nu|^p+\frac{(s-1)}{s^p}\sum_{\gamma=1}^{t_\nu^*}\frac{1}{K_p}|\theta_1-\theta_\nu|^p$$

$$+\sum_{\ell\in B_1B_\nu}\frac{1}{K_p}|\theta_1-\theta_\nu|^p$$

where $K_p = 1$ or 2^{p-1} according to $p \leq 1$ or $p \geq 1$. The last inequality follows from a well known result (see [3], p. 156).

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Consequently,

$$(2.3)\qquad \sum_{\ell=1}^{t_\nu} \sum_{i\in A_\ell} |\theta_i - \bar{\theta}_\ell|^p$$

$$\geq \left[\left\{\left(\frac{s-1}{s}\right)^p + \frac{(s-1)}{s^p}\right\}\binom{\nu-2}{s-1} + \binom{\nu-2}{s-2}\right]\frac{|\theta_1-\theta_\nu|^p}{K_p}.$$

The proof of (2.1) is then complete on account of (2.2) and (2.3).

From Theorem 1 the result stated in the previous section follows immediately.

COROLLARY <u>For all</u> p, $0 \leq p < \infty$, <u>and all</u> s $(2 \leq s < \nu)$,

$$(2.4)\qquad \lim_{\nu\to\infty} \frac{\max\limits_{1\leq \ell \leq t_\nu} \max\limits_{i\in A_\ell} |\theta_i - \bar{\theta}_\ell|^p}{\sum\limits_{\ell=1}^{t_\nu} \sum\limits_{i\in A_\ell} |\theta_i - \bar{\theta}_\ell|^p} = 0.$$

We remark that the result of Theorem 1 will remain valid if $\Lambda_\nu = (\theta_1, \theta_2, \ldots, \theta_\nu)$ is replaced by a double sequence $\Lambda_\nu^* = (\theta_{\nu 1}, \theta_{\nu 2}, \ldots, \theta_{\nu\nu})$. Accordingly the result of the above corollary will again hold if the θ's are allowed to vary as $\nu \to \infty$.

3. <u>The case $s = 2$.</u> We will show in this section that, for the case $s = 2$, the bound derived in the preceding theorem is sharp for all $1 \leq p < \infty$. In addition, for this case, we also obtain an improved constant bound for all $0 \leq p \leq 1$ and show that it is also attained. For $s > 2$, however, our bounds do not seem to be the best possible. The derivation of the best possible bounds for $s > 3$ and for all p, $0 \leq p < \infty$, we pose as

an interesting open question. Let us denote for convenience, the left-hand and the right-hand sides of (2.1) by $R_\nu(p, s)$ and $B_\nu(p, s)$ respectively.

THEOREM 2. (i) For $1 \le p < \infty$, $R_\nu(p, 2) \le B_\nu(p, 2)$ and the bound is sharp for each p;

(ii) For $0 < p \le 1$, $R_\nu(p, 2) \le B_\nu(1, 2)$ $= \frac{1}{2(\nu - 1)}$ and the bound is sharp for each p.

Proof. For $s = 2$, it follows from Theorem 1 that for $1 \le p < \infty$, $R_\nu(p, 2) \le B_\nu(p, 2)$. To prove that the bound $B_\nu(p, 2)$ is sharp, we need simply to show that the bound is attained for some sequence Λ_ν. Consider the sequence $\Lambda_\nu^{(1)} = (0, 1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, i.e. one zero, one 1, and $(\nu - 2)\ \frac{1}{2}$'s; then it is easy to show that $R_\nu(p, 2)$ at $\Lambda_\nu^{(1)}$ is equal to $B_\nu(p, 2)$. To prove (ii), we first note that for $0 < p \le 1$,

$$R_\nu(p, 2) = \frac{\max\limits_{i \neq j} |\theta_i - \theta_j|^p}{\sum\limits_{i \neq j} |\theta_i - \theta_j|^p}$$

$$\le \frac{|\theta_1 - \theta_\nu|^p}{2\left\{\sum\limits_{i \neq 1, \nu} |\theta_1 - \theta_i|^p + \sum\limits_{i \neq 1, \nu} |\theta_i - \theta_\nu|^p + |\theta_1 - \theta_\nu|^p\right\}}$$

$$\le \frac{|\theta_1 - \theta_\nu|^p}{2\left[\sum\limits_{i \neq 1, \nu} |\theta_1 - \theta_\nu|^p + |\theta_1 - \theta_\nu|^p\right]}$$

$$= \frac{1}{2(\nu - 1)} = B_\nu^*(p, 2) \qquad \text{(say)}$$

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where $B_\nu^*(p, 2)$ is in fact equal to $B_\nu(1, 2)$. It can easily be shown that $B_\nu(1, 2) \leq B_\nu(p, 2)$; hence the new bound $B_\nu^*(p, 2)$ is a better bound than that given in Theorem 1. Finally, by taking the sequence $\Lambda_\nu^{(2)} = (0, 1, 1, \ldots, 1)$, i.e. one zero and $(\nu - 1)$ 1's, it is easy to see that $B_\nu^*(p, 2)$ is attained for each p, $0 \leq p \leq 1$. The proof is thus complete.

4. Concluding remarks. It is important to observe that in proving Theorem 1, we have employed the technique of finding an upper bound (2.2) for the numerator of $R_\nu(p, s)$ and a lower bound (2.3) for the denominator of $R_\nu(p, s)$; and these upper and lower bounds respectively are sharp for all s, $2 \leq s \leq \nu - 1$, and all p, $1 \leq p < \infty$. (The equality in (2.2) will hold for $\Lambda_\nu^{(2)}$ and similarly equality in (2.3) will hold for $\Lambda_\nu^{(1)}$). However, the ratio of these bounds, $B_\nu(p, s)$, is sharp only for the case $s = 2$. For securing a sharp bound for $s > 2$, the present approach does not seem to be adequate. It is also noteworthy that for $s > 2$, and $0 < p < 1$, the best lower bound for $R_\nu(p, s)$ must depend upon p (unlike the case $s = 2$). Specifically, it can be shown that for sufficiently large ν, $B_\nu(1, s) > B_\nu(p, s)$ for all $0 \leq p < 1$ and all $s > 2$.

We would like to point out also that an analogue of (2.4) exists in general function spaces. Let $\{f_\alpha\}$ be elements of some normed linear space, not all the same, then we can prove that for $1 \leq p < \infty$,

$$(4.1) \qquad \lim_{\nu \to \infty} \frac{\max\limits_{1 \leq i, j \leq \nu} \|f_i - f_j\|^p}{\sum\limits_{1 \leq i, j \leq \nu} \|f_i - f_j\|^p} = 0 \; .$$

Statements of this sort are useful in the study of exterior power spaces and Gram-determinants (see [4] and the references given there). Similarly an integral analogue of (2.4) may be stated as follows: for $1 \leq p < \infty$, and any continuous function

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$f(x, y)$,

$$(4.2) \qquad \lim_{\substack{x \to \infty \\ y \to \infty}} \frac{\max\limits_{\substack{0 \le s \le x \\ 0 \le t \le y}} |f(s,t)|^p}{\int_0^x \int_0^y |f(s,t)|^p \, ds\,dt} = 0 .$$

Although the proofs of (4.1) and (4.2) are different in nature from the proof of Theorem 1 presented above, they can easily be constructed with the aid of Hölder's inequality.

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