RADON-NIKODYM DENSITIES BETWEEN HARMONIC MEASURES ON THE IDEAL BOUNDARY OF AN OPEN RIEMANN SURFACE

MITSURU NAKAI

Dedicated to the memory of Professor TADASI NAKAYAMA

1. Resolutive compactification and harmonic measures. Let R be an open Riemann surface. A compact Hausdorff space R^* containing R as its dense subspace is called a *compactification* of R and the compact set $\Delta = R^* - R$ is called an *ideal boundary* of R. Hereafter we always assume that R does not belong to the class O_G. Given a real-valued function f on Δ, we denote by $\overline{\varphi}_f^{R,R^*}$ (resp. $\underline{\varphi}_f^{R,R^*}$) the totality of lower bounded superharmonic (resp. upper bounded subharmonic) functions s on R satisfying

$$\liminf_{R \ni p \to p^*} s(p) \geq f(p^*) \qquad (\text{resp. } \limsup_{R \ni p \to p^*} s(p) \leq f(p^*))$$

for any point p^* in Δ. If these two families are not empty, then

$$\overline{H}_f^{R,R^*}(p) = \inf\,(s(p)\,;\ s \in \overline{\varphi}_f^{R,R^*}) \text{ and } \underline{H}_f^{R,R^*}(p) = \sup\,(s(p)\,;\ s \in \underline{\varphi}_f^{R,R^*})$$

are harmonic functions on R and $\overline{H}_f^{R,R^*} \geq \underline{H}_f^{R,R^*}$ on R. If these two functions coincide with each other on R, then we denote by H_f^{R,R^*} this common function and call f *resolutive* with respect to R^* (or Δ). We denote by $C(\Delta)$ the totality of bounded real valued continuous functions on Δ. If any function in $C(\Delta)$ is resolutive with respect to Δ, then following Constantinescu and Cornea [1] we say that R^* is a *resolutive compactification* of R. Important examples of resolutive compactifications are Wiener's, Martin's Royden's, Kuramochi's and Kerékjártó-Stoilow's compactifications (see [1]). Hereafter we always consider the resolutive compactification R^* of R.

Fix a point p in R. It is easy to see that $f \to H_f^{R,R^*}(p)$ is a positive linear functional on $C(\Delta)$ and so by Riesz-Markoff-Kakutani's theorem, there exists a positive regular Borel measure μ_p on Δ such that

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$$H_f^{R, R^*}(p) = \int_\Delta f(p^*)\, d\mu_p(p^*).$$

The measure μ_p is called the *harmonic measure* on Δ with the reference point p. We shall investigate the interdependence between each members of the family $(\mu_p;\ p \in R)$ of harmonic measures.

2. Harnack's function. Let k be the *Harnack's function* on $R \times R$, i.e. the function k defined by

$$k(p, p') = \inf(c > 0;\ c^{-1}u(p) \leq u(p') \leq cu(p) \text{ for any } u \in HP(R)).$$

Then $1 \leq k(p, p') < \infty$ for any p and p' in R and $\lim_{p \to p'} k(p, p') = 1$. In fact, let U be a relatively compact simply connected domain in R containing p and p', and ϕ a $1:1$ conformal mapping of U onto $(z;\ |z| < 1)$ with $\phi(p') = 0$. Then by putting $\phi(p) = re^{it}$

$$u(p) = (1/2\pi)\int_0^{2\pi} ((1 - r^2)/(1 - 2r\cos(\theta - t) + r^2))\, u(\phi^{-1}(e^{i\theta}))d\theta$$

for any u in $HP(R)$ and so

$$((1 - r)/(1 + r))\, u(p) \leq u(p') \leq ((1 + r)/(1 - r))\, u(p).$$

Thus $1 \leq k(p, p') \leq (1 + r)/(1 - r) < \infty$ and if $p \to p'$, then $r \to 0$ and so $\lim_{p \to p} k(p, p') = 1$. Moreover it is easy to see that $k(p, p) = 1$, $k(p, p') = k(p', p)$ and $k(p, p'') \leq k(p, p')\, k(p', p'')$ for any p, p' and p'' in R.

3. Harmonic kernel. Let p and q belong to R. By the definition of $k(p, q)$, we see that

$$k(p, q)^{-1} d\mu_q \leq d\mu_p \leq k(p, q) d\mu_q. \tag{1}$$

Thus *measures $\mu_p (p \in R)$ are absolutely continuous with respect to each other* and so the μ_p-integrability and the μ_p-nullity do not depend on the special choice of p in R. We denote by $(d\mu_q/d\mu_p)(p^*)$ the *Radon-Nikodym density* of μ_q with respect to μ_p.

We fix a point o in R. Then we can easily see that the function $p \to \int_\Delta f(p^*) d\mu_p(p^*)$ is harmonic on R if f is μ_o-integrable on Δ. The main assertion in this note is the following

THEOREM. *There exists a function $P_o(p, p^*)$ on $R \times \Delta$ such that*

(a) $P_o(p, p^*) = (d\mu_p/d\mu_o)(p^*)$ *(μ_o-almost everywhere) on Δ as the function*

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of p^* for any fixed p in R;

(b) *$P_o(p, p^*)$ is harmonic on R as the function of p for any fixed p^* in Δ;*

(c) *$k(o, p)^{-1} \leq P_o(p, p^*) \leq k(o, p)$ for any (p, p^*) in $R \times \Delta$;*

(d) *$P_o(p, p^*)$ is Borel measurable on $R \times \Delta$ as the function of (p, p^*).*

Needless to say, such a function $P_o(p, p^*)$ is not unique in the proper sense, but unique in the following sense: if $\tilde{P}_o(p, p^*)$ is another function on $R \times \Delta$ satisfying the above four conditions, then $P_o(p, p^*) = \tilde{P}_o(p, p^*)$ $\tilde{\mu}_o$-almost everywhere on $R \times \Delta$. Here $\tilde{\mu}_o$ is the product measure $\tilde{\mu} \times \mu_o$, where $\tilde{\mu}$ is a measure on R which is equivalent to the Lebesgue measure in each parameter neighborhood of R. Such a $P_o(p, p^*)$ may be called a *harmonic kernel* (or Poisson type kernel) on $R \times \Delta$ with the reference point o. For any Borel function f, μ_o-integrable on Δ, we then have

$$H_f^{R, R^*}(p) = \int_\Delta P_o(p, p^*) f(p^*) \, d\mu_o(p^*).$$

The harmonicity of the function $p \to P_o(p, p^*)$ increases the usefulness of the above integral representation.

4. Proof of Theorem. Let $\tilde{P}(p, p^*)$ be an arbitrary but fixed function on $R \times \Delta$ such that $\tilde{P}(p, p^*) = (d\mu_p/d\mu_o)(p^*)$ (μ_o-almost everywhere) on Δ as the function of p^* for any fixed p in R. We may assume that $\tilde{P}(o, p^*) \equiv 1$ on Δ. Since R is separable, there exists a countable dense subset D of R with $o \in D$.

For any p and q in D, by (1), we see that

$$k(p, q)^{-1}(d\mu_q/d\mu_o)(p^*) \leq (d\mu_p/d\mu_o)(p^*) \leq k(p, q)(d\mu_q/d\mu_o)(p^*)$$

μ_o-almost everywhere on Δ as the function of p^*. Hence there exists a Borel set $E(p, q)$ in Δ such that

$$\mu_o(E(p, q)) = 0$$

and

$$k(p, q)^{-1}\tilde{P}(q, p^*) \leq \tilde{P}(p, p^*) \leq k(p, q)\tilde{P}(q, p^*)$$

for any p^* in $\Delta - E(p, q)$. Let

$$E = \cup_{p, q \in D} E(p, q).$$

Since D is countable, $\mu_o(E) = 0$. Hence

$$k(p, q)^{-1}\tilde{P}(q, p^*) \leq \tilde{P}(p, p^*) \leq k(p, q)\tilde{P}(q, p^*)$$

for any p and q in D and p^* in $\Delta - E$. In particular, since $\tilde{P}(o, p^*) = 1$ on Δ,

(2) $$k(p, o)^{-1} \leq \tilde{P}(p, p^*) \leq k(p, o)$$

for any p in D and p^* in $\Delta - E$. Thus

(3) $$|\tilde{P}(p, p^*) - \tilde{P}(q, p^*)| \leq k(p, o) \max(k(p, q) - 1, 1 - k(p, q)^{-1})$$

for any p and q in D and p^* in $\Delta - E$. We saw in Section 2 that

$$1 \leq k(p, q) \leq k(p, p_0)\, k(p, q_0), \qquad 1 \leq k(p, o) \leq k(o, p_0)\, k(p, p_0)$$

and

$$\lim_{D \ni p \to p_0} k(p, p_0) = \lim_{D \ni q \to p_0} k(q, p_0) = 1$$

for any p_0 in R. From these and (3), it follows that

$$\lim_{D \ni p, q \to p_0} |\tilde{P}(p, p^*) - \tilde{P}(q, p^*)| = 0,$$

or equivalently that

$$\lim_{D \ni p \to p_0} \tilde{P}(p, p^*)$$

exists for any p_0 in R and if p_0 belongs to D, then

$$\lim_{D \ni p \to p_0} \tilde{P}(p, p^*) = \tilde{P}(p_0, p^*).$$

Hence if we set

$$P(p, p^*) = \lim_{D \ni p' \to p} \tilde{P}(p', p^*)$$

in $R \times (\Delta - E)$, then the function $p \to P(p, p^*)$ is continuous on R for fixed p^* in $\Delta - E$. For arbitrary point p in R, take a sequence $(p_n)_{n=1}^{\infty}$ of points in D with $p_n \to p$. Then for any function f in $C(\Delta)$, by using (2), the definition of $P(p, p^*)$ and Lebesgue's convergence theorem,

$$\begin{aligned}
\int_{\Delta} f(p^*)\, d\mu_p(p^*) &= H_f^{R, R^*}(p) \\
&= \lim_{n \to \infty} H_f^{R, R^*}(p_n) \\
&= \lim_{n \to \infty} \int_{\Delta - E} \tilde{P}(p_n, p^*) f(p^*)\, d\mu_o(p^*) \\
&= \int_{\Delta - E} \lim_{n \to \infty} \tilde{P}(p_n, p^*) f(p^*)\, d\mu_o(p^*) \\
&= \int_{\Delta - E} P(p, p^*) f(p^*)\, d\mu_o(p^*).
\end{aligned}$$

This shows that $d\mu_p(p^*) = P(p, p^*)\, d\mu_o(p^*)$. Hence $P(p, p^*) = (d\mu_p / d\mu_o)(p^*)$.

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μ_o-almost everwhere.

Let ϕ be an analytic mapping of the open unit disc $(z;\ |z|<1)$ onto R. Now we prove that the function $p \to P(p, p^*)$ is harmonic on R for almost every fixed p^* in Δ. For the aim, we have only to show that the function $z \to P(\phi(z), p^*)$ is harmonic on $(z;\ |z|<1)$ for almost every fixed p^* in Δ, since $p \to P(p, p^*)$ is continuous on R for any fixed p^* in $\Delta - E$. Since $p^* \to P(\phi(z), p^*)$ is Borel measurable on Δ for any fixed z in $(z;\ |z|<1)$ and $z \to P(\phi(z), p^*)$ is continuous on R for any fixed p^* in $\Delta - E$, it is easy to see that the function $(z, p^*) \to P(\phi(z), p^*)$ is Borel measurable on $R \times \Delta$.

Let $(z_n)_{n=1}^{\infty}$ be a countable dense subset of $(z;\ |z|<1)$. Fix an arbitrary positive integer n and choose a countable dense subset $(r_m)_{m=1}^{\infty}$ of the open interval $(0, 1-|z_n|)$. Then for any f in $C(\Delta)$, since $\int_{\Delta} P(\phi(z), p^*) f(p^*) d\mu_o(p^*)$ is harmonic in z of $(z;\ |z|<1)$, by Fubini's theorem,

$$\int_{\Delta} P(\phi(z_n), p^*) f(p^*)\, d\mu_o(p^*) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\int_{\Delta} P(\phi(z_n + r_m e^{i\theta}), p^*) f(p^*)\, d\mu_o(p^*)\Big] d\theta$$
$$= \int_{\Delta}\Big[\frac{1}{2\pi}\int_0^{2\pi} P(\phi(z_n + r_m e^{i\theta}), p^*) d\theta\Big] f(p^*) d\mu_o(p^*).$$

Hence there exists a set $F_{n,m}$ in Δ with $\mu_o(F_{n,m}) = 0$ such that for any p^* in $\Delta - F_{n,m}$ it holds that

$$P(\phi(z_n), p^*) = \frac{1}{2\pi}\int_0^{2\pi} P(\phi(z_n + r_m e^{i\theta}), p^*)\, d\theta. \tag{4}$$

Let $F_n = E \cup (\cup_{m=1}^{\infty} F_{n,m})$. Then $\mu_o(F_n) = 0$ and the identity (4) holds for any $m = 1, 2, \ldots$ and p^* in $\Delta - F_n$. By the continuity of $P(\phi(z), p^*)$ in z for any fixed p^* in $\Delta - E$, we conclude that

$$P(\phi(z_n), p^*) = \frac{1}{2\pi}\int_0^{2\pi} P(\phi(z_n + re^{i\theta}), p^*)\, d\theta \tag{5}$$

for any r in $0<r<1-|z_n|$ and p^* in $\Delta - F_n$. Finally let $F = \cup_{n=1}^{\infty} F_n$. Then $\mu_o(F) = 0$ and (5) holds for any $n = 1, 2, \ldots$ and any r in $0<r<1-|z_n|$ and any p^* in $\Delta - F$. By the continuity of $P(\phi(z), p^*)$ in z for any fixed p^* in $\Delta - E$, we conclude that

$$P(\phi(z), p^*) = \frac{1}{2\pi}\int_0^{2\pi} P(\phi(z + re^{i\theta}), p^*) d\theta$$

for any z in the unit disc and r in $0<r<1-|z|$ and p^* in $\Delta - F$, which shows that $z \to P(\phi(z), p^*)$ is harmonic in $(z;\ |z|<1)$ for any fixed p^* in $\Delta - F$.

Thus the function $p \to P(p, p^*)$ is harmonic on R for any fixed p^* in $\Delta - F$ with $\mu_o(F) = 0$. Let

$$P_o(p, p^*) = \begin{cases} P(p, p^*), & \text{for } (p, p^*) \text{ in } R \times (\Delta - F); \\ 1, & \text{for } (p, p^*) \text{ in } R \times F. \end{cases}$$

Then for any fixed p in R, $P_o(p, p^*) = P(p, p^*) = (d\mu_p / d\mu_o)(p^*)$ (μ_o-almost everywhere) on Δ. Thus (a) is satisfied by $P_o(p, p^*)$ thus constructed. It is also clear that $P_o(p, p^*)$ satisfies (b). The condition (c) follows immediately from (b), the definition of $k(o, p)$ and the fact that $P_o(o, p^*) \equiv 1$ for any p^* in Δ (see (2)). The last condition (d) is an easy consequence of (a) and (b).

References

[1] C. Constantinescu-A. Cornea: Ideale Ränder Riemannscher Flächen, Springer-Verlag, 1963.

Mathematical Institute
Nagoya University

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