the median triangle of any self-conjugate can be proved to be circumscribed to the parabola.

If the envelope be a rectangular hyperbola, $(x'y')$ must be on the circle whose equation is

$$x^2 + 2xy\cos C + y^2 + 2(c\cos B - 2b\cos C)x \\ + 2(c\cos A - 2a\cos C)y + 2a^2 + 2b^2 - c^2 = 0.$$

If $(x'y')$ be on the median through A, the envelope passes through A. Hence the envelope is only a circumscribed conic when $(x'y')$ is the centroid.

If $(x'y')$ be on CA, CB or BA, the envelope passes through $(x'y')$.

If $(x'y')$ be A, B, or C, the envelope is two coincident lines, the corresponding medians.

If the envelope be a circle, the point is one of the in-centres or ex-centres of A″B″C″.

If $(x'y')$ be the centre of the envelope, it must be either A, B, C, or the centroid.

Fifth Meeting, March 13, 1891.

J. S. Mackay, Esq., M.A., LL.D., ex-President, in the Chair.

On the Use of Dimensional Equations in Physics.

By William Peddie, D.Sc.

Though every quantity, whatever be its nature, has magnitude, no quantity can be said to be large or small *absolutely*. When we speak of the size of any body we mean its size relatively to the size of some other body with which we compare it. A yard is large if we compare it with an inch; it is small when compared with a mile. In the former case the number which represents it is more than 60,000 times larger than the number by which it is represented in the latter case. A mere number is therefore useless as regards the statement of magnitude, except when accompanied by a clear indication of what the thing measured is compared with. The quantity in terms of which the comparison is made is called the *unit*, and the number which tells how often this unit is contained in a given quantity is called the *numeric* of that quantity.

All dynamical quantities may be made to depend upon *three* units only. These are the units of *mass*, of *length*, and of *time*. Thus the unit of speed, being measured by the distance traversed in a certain time, depends upon the unit of length directly, and upon the unit of time inversely. Hence, by doubling the unit of length we double the speed unit, and therefore halve the numeric of any given speed; whereas, by doubling the unit of time, we halve the speed unit, and therefore double the numeric of a given speed. Again, acceleration, being measured by the increase of speed in a certain time, depends upon the unit of speed directly and upon the unit of time inversely; that is, it depends directly upon the unit of length and inversely upon the square of the unit of time. The manner in which the fundamental units are involved in any quantity determines the *dimensions* of that quantity. If M, L, and T, represent the units of mass, length, and time, the dimensions of speed and acceleration are indicated by the symbols $[LT^{-1}]$ and $[LT^{-2}]$ respectively, and the dimensions of energy are $[ML^2T^{-2}]$, and so on.

We may write the expression for Newton's Second Law in the form

$$f = m\frac{l}{t^2},$$

where l is the distance which the mass m moves over from rest, in the time t, under the action of the force f; and it is understood that unit force is that force which, acting upon unit mass, produces unit acceleration. And we may further regard this equation as a dimensional equation; in which case the sign of equality merely means that the dimensions of the quantity on the left-hand side of the equation are identical with those of the expression on the right-hand side of the equation. But, from a dimensional equation, we cannot make any deduction regarding the absolute magnitude of any of the quantities which are involved, for the equation simply asserts *proportionality* of magnitude between its various terms. Still, by a suitable definition of units, we can pass from the dimensional, to the ordinary, equation. Thus, in the above equation, we may define unit force as the force which, acting on unit mass for unit time, causes the unit of mass to move over unit distance from rest; or we might adopt the definition given at the commencement of this paragraph.

The idea of dimensions is of great importance in physics. It affords a useful check on the accuracy of algebraical work; for the dimensions of all the terms in a physical equation must be the same. But its use is not limited to this extent. For example, we may write the equation

$$f = m\frac{l}{t^2},$$

in the form

$$\alpha f = \beta m \frac{\gamma l}{\frac{\beta\gamma t^2}{\alpha}},$$

from which we see that if, *in two similar material systems, the forces, masses, and lengths, are in the ratios* $\alpha/1$, $\beta/1$, $\gamma/1$, *respectively, and if the systems begin to move in precisely similar manners, the motions will continue to be similar provided that we compare them after the lapse of intervals of time which are in the ratio of* $\sqrt{\frac{\beta\gamma}{\alpha}}$ *to unity in the two systems.*

This principle, which was proved (otherwise) by Newton, has been called the *Principle of Dynamical Similarity.* The above proof is due to Bertrand.

Many physical problems can be solved with extreme simplicity by its means For example, we can determine the speed of propagation of a wave along a stretched cord. Let T be the tension of the cord, and let m be its mass per unit of length. The pressure towards the centre of curvature at any part of the wave T/ρ, where ρ is the radius of curvature and is proportional to the linear dimension l of the wave. Hence the pressure per similar length in each of two such waves is T. And the mass per similar length is ml. Hence the expression for the Second Law of Motion gives

$$\frac{l^2}{t^2} = \frac{T}{m}.$$

But $l/t = v$, the speed of propagation of the wave. Hence the speed is directly as the square root of the tension, and is inversely as the square root of the mass per unit of length.

This result may be directly applied to the problem of the vibrations of a stretched string. For, λ being the length of a wave and the other quantities having the same meanings as above, we get

$$\frac{\lambda}{v} = \frac{\lambda\sqrt{m}}{\sqrt{T}}.$$

https://doi.org/10.1017/S0013091500030789 Published online by Cambridge University Press

Now λ/v is the time of a complete vibration, which is therefore directly proportional to the length of a loop of the string and to the square root of the mass per unit length of the cord, and is inversely proportional to the square root of its tension.

Again, in the case of a gas, of density ρ and under pressure p, the dimensional equation takes the form

$$\frac{pl^2}{l}=\frac{\rho l^2 l}{t^2},$$

from which we conclude that the period of vibration when a disturbance of given wave-length travels through the gas is

$$t=l\sqrt{\frac{\rho}{p}},$$

and that the speed of propagation of that disturbance is

$$v=\sqrt{\frac{p}{\rho}},$$

the proper choice of units being made.

In an oscillatory wave, propagated by gravity, the force is proportioned to the intensity of gravity g, and to the mass of the wave, which is proportional to the density ρ of the liquid and to the square of the length. Hence the fundamental equation takes the form

$$l^2\rho g=\frac{\rho l^2 l}{t^2},$$

from which we get

$$t=\sqrt{\frac{l}{g}}$$

and

$$v=\frac{l}{t}=\sqrt{lg}.$$

This shows that the speed of propagation is proportional conjointly to the square roots of the wave length and of gravity, and is independent of the density of the liquid.

In a ripple, propagated by surface tension, the pressure per unit area of the surface of the ripple is proportional directly to the surface tension T and inversely to the radius of curvature. Hence the pressure per similar area is proportional to T and to the square of the length of the wave. Also the mass is proportional to the density and to the cube of the wave length. Hence we get

$$t = l\sqrt{\frac{\rho l}{\mathrm{T}}},$$

$$v = \frac{l}{t} = \sqrt{\frac{\mathrm{T}}{\rho l}},$$

which shows that the speed of propagation of a ripple is proportional to the square root of the surface tension directly and is inversely proportional to the square roots of the density and the wave length conjointly. Hence we see that, as is well known to be the case, ripples run faster the shorter they are, while gravity waves run faster the longer they are.

As a single additional example of the application of the Second Law, consider the small vibrations of a liquid sphere. Let the radius of the sphere be a, while the density of the liquid is ρ, and its surface tension is T. The force which acts, per unit of area, to restore the spherical form after a slight disturbance is $2\mathrm{T}/a$, and hence the force per similar area is $2\mathrm{T}a$. The mass is proportional to ρa^3, and the ranges of similar disturbances are proportional to a. Hence we get

$$t = a^{\frac{3}{2}}\sqrt{\frac{\rho}{\mathrm{T}}},$$

which shows that the periods of similar vibrations are proportional to the power 3/2 of the radius.

Lastly, as an example of the use of dimensional equations apart from the consideration of the Second Law of Motion, consider the conduction of heat downwards through the earth's crust, for which we have the well-known equation

$$c\frac{dv}{dt} = -\frac{d}{dx}\left(k\frac{dv}{dx}\right),$$

where c is the thermal capacity, k is the thermal conductivity (which we shall assume to be constant), v is temperature, t is time, and x is distance measured downwards from the surface. The quantities dv, dt, dx, being of the same dimensions as v, t, and x, respectively, we get, as a dimensional equation

$$\frac{c}{t} = \frac{k}{x^2}.$$

This gives
$$x = \sqrt{\frac{kt}{c}},$$
which means that if the times are altered in any fixed proportion p (say), the lengths must be altered in proportion of the square root of p in order that the flow of heat may take place under *similar* conditions in the altered circumstances. In other words, *the distances at which similar effects are felt (for example, the distance below the surface at which the periodic variations of surface temperature cease to be felt) are proportional to the square root of the period.*

Again,
$$\frac{x}{t} = \sqrt{\frac{k}{ct}}.$$
Here we may suppose that x represents the length of a wave, while t represents the periodic time, so that the fraction on the right-hand side of the equation is proportional to the rate at which the wave of heat travels downwards. We see therefore that this rate is directly proportional to the square root of the conductivity, and is inversely proportional to the square roots of the thermal capacity and the periodic time conjointly.

It follows from this that, when the period is constant, *the date at which the maximum temperature reaches any given depth is later than the date at which it left the surface in direct proportion to the depth.*

[The law which regulates the diminution of the range of temperature with increase of depth cannot be obtained from the original equation in the way in which we have obtained the two laws just enunciated, for the temperature is not involved in the dimensional equation which we deduce from it.

But we may write the original equation in the form
$$c\,\frac{dv}{dx}\cdot\frac{dx}{dt} = -\frac{d}{dx}\left(k\,\frac{dv}{dx}\right);$$
and we may suppose that dx/dt is the speed with which the heat wave travels downwards, in which case the equation becomes
$$\sqrt{\frac{ck}{\mathrm{T}}}\cdot\frac{dv}{dx} = -\frac{d}{dx}\left(k\,\frac{dv}{dx}\right),$$
where T is the periodic time, and dv/dx now represents the rate at which (say) the maximum temperature changes as the wave passes down.

https://doi.org/10.1017/S0013091500030789 Published online by Cambridge University Press

This equation asserts that the rate of diminution of the rate of change of temperature with depth is proportional to the rate of change itself. In other words, the rate of change diminishes in geometrical progression as the depth increases in arithmetical progression. And its rate of diminution is $\sqrt{c}/\sqrt{k\mathrm{T}}$, k being assumed to be constant. But, since the rate of diminution of the rate of alteration of the range is proportional to the rate of alteration itself, it follows that the rate of alteration bears the same ratio to the range. Hence *the range diminishes in geometrical progression as the depth increases in arithmetical progression*, the rate of diminution being directly as the square root of the thermal capacity, and inversely as the square roots of the conductivity and the periodic time conjointly.]

The above examples will serve to illustrate the extreme ease with which the consideration of dimensional equations leads to the solutions of problems which are usually attacked by the aid of recondite methods alone.

Some relations between the orthic and the median triangles.

By A. J. Pressland, M.A.

Figure 15.

Let ABC be the triangle, X, Y, Z the feet of the altitudes, H the orthocentre, A′,B′,C′ the mid points of the sides.

Let ZY meet A′B′ in D, A′C′ in D′, B′C′ in R:
ZX meet B′C′ in E, A′B′ in E′, A′C′ in S:
XY meet C′A′ in F, B′C′ in F′, A′B′ in T.

§ 1. The following triangles are similar to ABC; AYZ, XBZ, XYC.

B′YD, C′D′Z are similar to AYZ and have parallel sides.
EC′Z, XA′E′ are similar to XBZ and have parallel sides.
XFA′, F′YB′ are similar to XYC and have parallel sides.

Y is the internal centre of similitude of the circles AYZ, B′YD.
Z „ „ „ „ „ „ AYZ, C′D′Z.
Z „ external „ „ „ „ XBZ, EC′Z.
X „ internal „ „ „ „ XBZ, XA′E′.
X „ external „ „ „ „ XYC, XFA′.
Y „ „ „ „ „ „ XYC, F′YB′.

https://doi.org/10.1017/S0013091500030789 Published online by Cambridge University Press