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NONRELATIVISTIC LIMIT FOR THE TRAVELLING WAVES OF THE PSEUDORELATIVISTIC HARTREE EQUATION

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Abstract

We consider the pseudorelativistic Hartree equation

$$i\partial_t\psi = (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\psi - (|x|^{-1} * |\psi|^2)\psi \quad \text{with } (t,x) \in \mathbb{R}\times\mathbb{R}^3,$$

which describes the dynamics of pseudorelativistic boson stars in the mean-field limit. We study the travelling waves of the form $\psi(t,x) = e^{it\mu}\varphi_c(x - vt)$, where $v \in \mathbb{R}^3$ denotes the travelling velocity. We prove that φ_c converges strongly to the minimiser φ_∞ of the limit energy $E_\infty(N)$ in $H^1(\mathbb{R}^3)$ as the light speed $c \to \infty$, where $E_\infty(N)$ is the corresponding energy for the limit equation

$$-\frac{1}{2m}\Delta\varphi_\infty + i(v\cdot\nabla)\varphi_\infty - (|x|^{-1} * |\varphi_\infty|^2)\varphi_\infty = -\lambda\varphi_\infty.$$

Since the operator $-\Delta$ is the classical kinetic operator, we call this the nonrelativistic limit. We prove the existence of the minimiser for the limit energy $E_\infty(N)$ by using concentration-compactness arguments.

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1. Introduction and main results

We study the pseudorelativistic Hartree equation

$$i\partial_t\psi = (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\psi - (|x|^{-1} * |\psi|^2)\psi \quad \text{with } (t,x) \in \mathbb{R}\times\mathbb{R}^3. \tag{1.1}$$

In the physical context, the parameter $m > 0$ is the mass of a particle and the symbol $*$ stands for the convolution on $\mathbb{R}^3$. The pseudorelativistic operator $\sqrt{-c^2\Delta + m^2c^4}$ is defined via multiplication in the Fourier space with the symbol $\sqrt{c^2|\xi|^2 + m^2c^4}$ for $\xi \in \mathbb{R}^3$, which describes the kinetic energy of a relativistic particle with mass $m > 0$. The convolution kernel $|x|^{-1}$ represents the Newtonian potential in appropriate physical units.

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A great deal of work has been devoted to the pseudorelativistic Hartree equation. Fröhlich *et al.* [4] proved the existence of travelling solitary waves for (1.1) with $c = 1$ by using concentration-compactness arguments [10, 11] and Lenzman considered local and global well-posedness for Equation (1.1) with $c = 1$ [7]. Lenzmann [8] and Guo and Zeng [5] studied the uniqueness of the ground state for the pseudorelativistic Hartree energy using the nonrelativistic limit of (1.1). For further work on travelling wave solutions of Equation (1.1), we refer the reader to [3, 6, 13].

We focus on travelling solitary waves of the form

$$\psi(t, x) = e^{it\mu}\varphi_c(x - vt), \tag{1.2}$$

with some $\mu \in \mathbb{R}$ and travelling velocity $v \in \mathbb{R}^3$ such that $|v| < 1$. Substituting (1.2) into (1.1) yields

$$(\sqrt{-c^2\Delta + m^2c^4} - mc^2)\varphi_c + i(v \cdot \nabla)\varphi_c - (|x|^{-1} * |\varphi_c|^2)\varphi_c = -\mu\varphi_c, \tag{1.3}$$

which can be viewed as an Euler–Lagrange equation for the minimising problem

$$E_c(N) := \inf\Big\{\mathcal{E}_c(\psi) : \psi \in H^{1/2}(\mathbb{R}^3), \mathcal{N}(\psi) = \int_{\mathbb{R}^3} |\psi(x)|^2\, dx = N\Big\}, \tag{1.4}$$

where

$$\mathcal{E}_c(\psi) := \frac{1}{2}\langle\psi, (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\psi\rangle + \frac{i}{2}\langle\psi, (v \cdot \nabla)\psi\rangle - \frac{1}{4}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\psi|^2\Big)|\psi|^2\, dx,$$

and the space $H^{1/2}(\mathbb{R}^3)$ is defined by $H^{1/2}(\mathbb{R}^3) := \{\psi \in L^2(\mathbb{R}^3) : (1 + |\xi|)^{1/2}\hat{\psi} \in L^2(\mathbb{R}^3)\}$, with the norm

$$\|\psi\|^2_{H^{1/2}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} (1 + |\xi|)|\hat{\psi}(\xi)|^2\, d\xi < \infty.$$

We recall from [4] the following Gagliardo–Nirenberg type inequality: for any $v \in \mathbb{R}^3$ with $|v| < 1$ and $\psi \in H^{1/2}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\psi|^2\Big)|\psi|^2\, dx \le \frac{2}{N_*(v)}\langle\psi, (\sqrt{-\Delta} + iv \cdot \nabla)\psi\rangle\langle\psi, \psi\rangle, \tag{1.5}$$

where $2/N_*(v)$ is the best constant and $N_*(v)$ is given by

$$N_*(v) := \langle Q_v, Q_v\rangle = \|Q_v\|^2_{L^2}.$$

As stated in [4], an optimiser Q_v of (1.5) with $Q_v \in H^{1/2}(\mathbb{R}^3)$ and $Q_v \not\equiv 0$ satisfies

$$\sqrt{-\Delta}Q_v + i(v \cdot \nabla)Q_v - \Big(\frac{1}{|x|} * |Q_v|^2\Big)Q_v = -Q_v.$$

The constant $N_*(v)$ is subject to the bounds $(1 - |v|)N_*(0) \le N_*(v) \le N_*(0) = N_*$, say. From [4], for $|v| < 1$ there exists a critical constant $N_*(v)$ such that travelling waves exist if $0 < N < N_*(v)$ with the light speed $c = 1$. Lenzmann in [8] has given the existence of the ground state for $E_c(N)$ with $v = 0$ for $0 < N < cN_*$. For $v \neq 0$, using similar arguments to [4], it is easy to show the existence of a minimiser of $E_c(N)$. This gives the following existence theorem.

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THEOREM 1.1. *Assume that $m > 0$, $v \in \mathbb{R}^3$ and $|v| < 1$. Then there exists a positive constant $N_*(v)$, depending only on v, such that, for $0 < N < cN_*(v)$, the problem (1.4) has a minimiser $\varphi_c \in H^{1/2}(\mathbb{R}^3)$.*

We are interested in the limiting behaviour of minimisers for (1.4) as we pass to the limit $c \to \infty$, which is called the *nonrelativistic limit*. We will show that the minimiser of (1.4) converges strongly in $H^1(\mathbb{R}^3)$ to the minimiser of the problem

$$E_\infty(N) := \inf\Big\{\mathcal{E}_\infty(\psi) : \psi \in H^1(\mathbb{R}^3), \mathcal{N}(\psi) = \int_{\mathbb{R}^3} |\psi(x)|^2\,dx = N\Big\}, \tag{1.6}$$

where $\mathcal{E}_\infty(\psi)$ is given by

$$\mathcal{E}_\infty(\psi) := \frac{1}{4m}\int_{\mathbb{R}^3} |\nabla\psi|^2\,dx + \frac{i}{2}\langle\psi, (v\cdot\nabla)\psi\rangle - \frac{1}{4}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\psi|^2\Big)|\psi|^2\,dx.$$

Any minimiser φ_∞ for (1.6) must satisfy the corresponding Euler–Lagrange equation

$$-\frac{1}{2m}\Delta\varphi_\infty + i(v\cdot\nabla)\varphi_\infty - \Big(\frac{1}{|x|} * |\varphi_\infty|^2\Big)\varphi_\infty = -\lambda\varphi_\infty \tag{1.7}$$

for some Lagrange multiplier $\lambda \in \mathbb{R}$.

We first establish the existence of a minimiser for $E_\infty(N)$.

THEOREM 1.2. *Assume that $v \in \mathbb{R}^3$ and $|v| < 1$, $m > 0$ is sufficiently small and*

$$\frac{1}{2m}\int_{\mathbb{R}^3} |\nabla\psi|^2\,dx + \langle\psi, iv\cdot\nabla\psi\rangle \geq 0 \quad \text{for any } \psi \in H^1(\mathbb{R}^3).$$

Then the problem (1.6) has at least one minimiser.

The next result shows the H^1 convergence for the solution of (1.3) to a solution of the limit equation (1.7) as $c \to \infty$. This is the main theorem of this paper.

THEOREM 1.3. *Under the assumptions of Theorem 1.2, let φ_c be a minimiser of $E_c(N)$ with fixed N satisfying $0 < N < cN_*(v)$. Then, as $c \to \infty$,*

$$\varphi_c \to \varphi_\infty \quad \text{strongly in } H^1(\mathbb{R}^3), \tag{1.8}$$

where φ_∞ is a minimiser of $E_\infty(N)$.

REMARK 1.4. Theorem 1.2 ensures the existence of minimisers of $E_\infty(N)$ in the nonrelativistic limit for small m. Thus, we need the assumption that $m > 0$ is sufficiently small in Theorem 1.3.

Lenzmann in [8] considered the nonrelativistic limit of a solution to (1.3) with $v = 0$. We have to handle an additional term $v \cdot \nabla$, which needs careful analysis. We note that radially symmetric solutions to (1.3) do not exist (see [12]). Since φ_c is not a radial function, we cannot use the method in [8] which invokes Newton's theorem to derive the lower bound for the Lagrange multiplier $-\mu$. Inspired by the work of Choi *et al.* [2], we find a new way to deal with the problem. In a similar way to [2, Lemma 4.3], we obtain the lower bound $H_c \geq B|\xi|$ for the operator $H_c = \sqrt{c^2|\xi|^2 + m^2c^4} - mc^2 + \delta$

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with $\delta > 0$, where $B = \min\{2\delta^{1/2}/(2\sqrt{5}m)^{1/2}, c/2\}$. Based on this inequality and the Gagliardo–Nirenberg inequality, we deduce that φ_c is uniformly bounded in $H^{1/2}(\mathbb{R}^3)$. Then we can derive the upper bound for μ and the uniform boundedness of $\|\varphi_c\|_{H^1(\mathbb{R}^3)}$.

The organisation of the paper is as follows. In Section 2, we consider the nonrelativistic limit and complete the proof of Theorem 1.3. In Section 3, we give the existence result of the limit energy functional by using concentration-compactness arguments.

We use the following notation.

- $\rightharpoonup$ denotes weak convergence.
- $\langle\, , \rangle$ denotes the L^2 inner product.
- $f * h$ denotes the convolution on $\mathbb{R}^3$.
- $\hat{f}$ denotes the Fourier transform of the function f (see [9]).
- The value of the positive constant C is allowed to change from line to line and also in the same formula.
- $X \lesssim Y$ ($X \gtrsim Y$) denotes $X \leq CY$ (respectively, $X \geq CY$) for some appropriate positive constant C.
- $v \cdot \nabla = \sum_{k=1}^{3} v_k \partial_{x_k}$, where $v \in \mathbb{R}^3$ is some fixed vector.

2. The nonrelativistic limit

Before considering the nonrelativistic limit, we prove some preliminary lemmas.

LEMMA 2.1. *Let $H_c = \sqrt{c^2|\xi|^2 + m^2c^4} - mc^2 + \delta$ with $\delta > 0$ independent of c. Then $H_c \geq B|\xi|$, where $B = \min\{2\delta^{1/2}/(2\sqrt{5}m)^{1/2}, c/2\}$ is a constant.*

PROOF. Factorising out mc^2 from the square root, we write

$$H_c = mc^2\Big(\sqrt{1 + \Big|\frac{\xi}{mc}\Big|^2} - 1\Big) + \delta = mc^2 f\Big(\Big|\frac{\xi}{mc}\Big|^2\Big) + \delta,$$

where $f(t) = \sqrt{1+t} - 1$. By a Taylor expansion, if $0 \leq t \leq 4$, then there is some $t_* \in [0, 4]$ such that

$$f(t) = \sqrt{1+t} - 1 = f(0) + f'(t_*)t = \frac{t}{2\sqrt{1+t_*}} \geq \frac{t}{2\sqrt{5}}.$$

Hence, if $|\xi| \leq 2mc$, then

$$\begin{aligned} H_c &= mc^2\Big(\sqrt{1 + \Big|\frac{\xi}{mc}\Big|^2} - 1\Big) + \delta = mc^2 f\Big(\Big|\frac{\xi}{mc}\Big|^2\Big) + \delta \\ &\geq mc^2 \frac{|\xi/mc|^2}{2\sqrt{5}} + \delta = \frac{|\xi|^2}{2\sqrt{5}m} + \delta \geq \frac{2\delta^{1/2}}{(2\sqrt{5}m)^{1/2}}|\xi|, \end{aligned}$$

using the fact that $a^2 + b^2 \geq 2ab$ for the last inequality.

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On the other hand, if $|\xi| \geq 2mc$, then

$$H_c = c|\xi|\sqrt{1 + \left|\frac{mc}{\xi}\right|^2} - mc^2 + \delta \geq c|\xi| - mc^2 + \delta \geq c|\xi| - \frac{c|\xi|}{2} + \delta \geq \frac{c}{2}|\xi|.$$

This establishes Lemma 2.1. □

LEMMA 2.2. *If φ_c is a minimiser of $E_c(N)$, then $\{\varphi_c\}$ is uniformly bounded in $H^{1/2}(\mathbb{R}^3)$.*

PROOF. By (1.5),

$$\begin{aligned}
&2\mathcal{E}_c(\varphi_c)\\
&= \langle \varphi_c, (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\varphi_c\rangle + i\langle \varphi_c, (v\cdot\nabla)\varphi_c\rangle - \frac{1}{2}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\varphi_c|^2\Big)|\varphi_c|^2\,dx\\
&\geq \langle \varphi_c, (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\varphi_c\rangle + i\langle \varphi_c, (v\cdot\nabla)\varphi_c\rangle - \frac{N}{N_*(v)}\langle \varphi_c, (\sqrt{-\Delta} + iv\cdot\nabla)\varphi_c\rangle\\
&= \int_{\mathbb{R}^3}(\sqrt{c^2|\xi|^2 + m^2c^4} - mc^2)|\hat{\varphi}_c(\xi)|^2\,d\xi - \int_{\mathbb{R}^3}(v\cdot\xi)|\hat{\varphi}_c(\xi)|^2\,d\xi\\
&\quad - \frac{N}{N_*(v)}\int_{\mathbb{R}^3}(|\xi| - v\cdot\xi)|\hat{\varphi}_c(\xi)|^2\,d\xi.
\end{aligned}$$

It follows from Lemma 2.1 that

$$\begin{aligned}
&2\mathcal{E}_c(\varphi_c) + \delta N\\
&\geq B\int_{\mathbb{R}^3}|\xi||\hat{\varphi}_c(\xi)|^2\,d\xi - \int_{\mathbb{R}^3}(v\cdot\xi)|\hat{\varphi}_c(\xi)|^2\,d\xi - \frac{N}{N_*(v)}\int_{\mathbb{R}^3}(|\xi| - v\cdot\xi)|\hat{\varphi}_c(\xi)|^2\,d\xi.
\end{aligned}$$

For $c > 1$ sufficiently large, $B = \min\{2\delta^{1/2}/(2\sqrt{5}m)^{1/2}, c/2\} = 2\delta^{1/2}/(2\sqrt{5}m)^{1/2}$.

Case I: Fix N with $0 < N < cN_*$ and suppose that $0 < N < N_*(v) < cN_*(v)$. Let $\delta = \sqrt{5}m/2$. Then $B = 1$ and

$$\begin{aligned}
2\mathcal{E}_c(\varphi_c) + \frac{\sqrt{5}m}{2}N &\geq \Big(1 - \frac{N}{N_*(v)}\Big)\int_{\mathbb{R}^3}(|\xi| - v\cdot\xi)|\hat{\varphi}_c(\xi)|^2\,d\xi\\
&\geq \Big(1 - \frac{N}{N_*(v)}\Big)(1 - |v|)\int_{\mathbb{R}^3}|\xi|\,|\hat{\varphi}_c(\xi)|^2\,d\xi. \qquad (2.1)
\end{aligned}$$

In the last inequality, we use the fact that $|\xi| - v\cdot\xi \geq (1 - |v|)|\xi|$. Since φ_c is a minimiser of $\mathcal{E}_c(\psi)$, the operator inequality $\sqrt{-c^2\Delta + m^2c^4} - mc^2 \leq -\Delta/2m$ yields

$$\mathcal{E}_c(\varphi_c) \leq \mathcal{E}_c(\varphi_\infty) \leq \mathcal{E}_\infty(\varphi_\infty). \qquad (2.2)$$

Combining (2.1) with (2.2) and noting that $\mathcal{E}_\infty(\varphi_\infty) < 0$ gives

$$\begin{aligned}
\Big(1 - \frac{N}{N_*(v)}\Big)(1 - |v|)\int_{\mathbb{R}^3}|\xi|\,|\hat{\varphi}_c(\xi)|^2\,d\xi &\leq 2\mathcal{E}_c(\varphi_c) + \frac{\sqrt{5}m}{2}N\\
&\leq 2\mathcal{E}_\infty(\varphi_\infty) + \frac{\sqrt{5}m}{2}N \leq \frac{\sqrt{5}m}{2}N. \qquad (2.3)
\end{aligned}$$

Since $|v| < 1$, we have $(1 - N/N_*(v))(1 - |v|) > 0$.

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Case II: Fix N with $0 < N < cN_*$ and suppose that $0 < N_*(v) \leq N < cN_*(v)$. We can take $\delta = 8\sqrt{5}m(N/N_*(v))^2$ in Lemma 2.1. Then $B = 4N/N_*(v)$ and, as in (2.1),

$$\begin{aligned}
&2\mathcal{E}_c(\varphi_c) + 8\sqrt{5}m\Big(\frac{N}{N_*(v)}\Big)^2 N \\
&\quad\geq \frac{4N}{N_*(v)}\int_{\mathbb{R}^3}|\xi|\,|\hat{\varphi}_c(\xi)|^2\,d\xi - \int_{\mathbb{R}^3}|v||\xi|\,|\hat{\varphi}_c(\xi)|^2\,d\xi - \frac{N}{N_*(v)}\int_{\mathbb{R}^3}(|\xi| - v\cdot\xi)|\hat{\varphi}_c(\xi)|^2\,d\xi \\
&\quad\geq \frac{N}{N_*(v)}(3-2|v|)\int_{\mathbb{R}^3}|\xi|\,|\hat{\varphi}_c(\xi)|^2\,d\xi.
\end{aligned}$$

Since $|v| < 1$, we have $(N/N_*(v))(3-2|v|) > 0$ and, as in (2.3), we obtain

$$\frac{N}{N_*(v)}(3-2|v|)\int_{\mathbb{R}^3}|\xi|\,|\hat{\varphi}_c(\xi)|^2\,d\xi \leq \frac{8\sqrt{5}mN^3}{N_*^2(v)}. \tag{2.4}$$

By combining (2.3) and (2.4), we conclude that there exists a constant $C_1 > 0$, which is independent of c, such that

$$\int_{\mathbb{R}^3}|\xi|\,|\hat{\varphi}_c(\xi)|^2\,d\xi \leq C_1.$$

This completes the proof of Lemma 2.2. □

LEMMA 2.3. *If $m > 0$, $v \in \mathbb{R}^3$ and $|v| < 1$, then $E_\infty(N) < 0$.*

PROOF. Fix $\psi(x) \in H^1(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3}|\psi|^2\,dx = N$. Let $\psi^\lambda(x) = \lambda^{3/2}\psi(\lambda x)$ with $\lambda > 0$. Then $\|\psi^\lambda\|_{L^2}^2 = \|\psi\|_{L^2}^2 = N$. By the definition of $\mathcal{E}_\infty(\psi)$,

$$\mathcal{E}_\infty(\psi^\lambda(x)) = \frac{\lambda^2}{2}\Big\langle\psi, \frac{-\Delta}{2m}\psi\Big\rangle + \frac{\lambda i}{2}\langle\psi, (v\cdot\nabla)\psi\rangle - \frac{\lambda}{4}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\psi|^2\Big)|\psi|^2\,dx.$$

Case I: If $i\langle\psi, (v\cdot\nabla)\psi\rangle < 0$ and λ is small enough, then, clearly, $\mathcal{E}_\infty(\psi^\lambda) < 0$.

Case II: If $i\langle\psi, (v\cdot\nabla)\psi\rangle \geq 0$, then

$$\mathcal{E}_\infty(\psi^\lambda(-x)) = \frac{\lambda^2}{2}\Big\langle\psi, \frac{-\Delta}{2m}\psi\Big\rangle - \frac{\lambda i}{2}\langle\psi, (v\cdot\nabla)\psi\rangle - \frac{\lambda}{4}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\psi|^2\Big)|\psi|^2\,dx.$$

If λ is small enough, then $\mathcal{E}_\infty(\psi^\lambda(-x)) < 0$.

Combining Cases *I* and *II* gives $E_\infty(N) < 0$. This completes the proof. □

LEMMA 2.4. *Let φ_c be a minimiser of $E_c(N)$ satisfying the assumptions of Theorem 1.1 and let μ be the associated Lagrange multiplier to φ_c. Then there exists a constant $K > 0$ such that $|\mu| \leq K$, where the constant $K > 0$ is independent of $c > 0$.*

PROOF. First, we claim that $\mu > 0$. The minimiser φ_c of $E_c(N)$ satisfies the Euler–Lagrange equation (1.3). Multiplying by φ_c and integrating gives

$$-\mu N = 2\mathcal{E}_c(\varphi_c) - \frac{1}{2}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\varphi_c|^2\Big)|\varphi_c|^2\,dx.$$

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We recall the operator inequality

$$\sqrt{-c^2\Delta + m^2c^4} \le -\frac{1}{2m}\Delta + mc^2,$$

which follows directly in the Fourier domain and we note that $\sqrt{1+t} \le t/2 + 1$ for all $t \ge 0$. Since φ_c is a minimiser of $E_c(N)$, we have $\mathcal{E}_c(\varphi_c) \le \mathcal{E}_c(\varphi_\infty) \le \mathcal{E}_\infty(\varphi_\infty)$. Consequently, by Lemma 2.3,

$$-\mu N = 2\mathcal{E}_c(\varphi_c) - \frac{1}{2}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\varphi_c|^2\Big)|\varphi_c|^2\,dx \le 2\mathcal{E}_c(\varphi_c) \le 2\mathcal{E}_\infty(\varphi_\infty) < 0.$$

This implies that $\mu > 0$.

Next, we prove the upper bound for μ. By (1.3),

$$-\mu N = \langle\varphi_c, (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\varphi_c\rangle + i\langle\varphi_c, (v\cdot\nabla)\varphi_c\rangle - \int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\varphi_c|^2\Big)|\varphi_c|^2\,dx.$$

Since $\sqrt{-c^2\Delta + m^2c^4} - mc^2 > 0$, by (1.5),

$$\begin{aligned}-\mu N &\ge i\langle\varphi_c, (v\cdot\nabla)\varphi_c\rangle - \int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |\varphi_c|^2\Big)|\varphi_c|^2\,dx\\ &\ge i\langle\varphi_c, (v\cdot\nabla)\varphi_c\rangle - \frac{2N}{N_*(v)}\langle\varphi_c, (\sqrt{-\Delta} + iv\cdot\nabla)\varphi_c\rangle.\end{aligned}$$

Therefore,

$$\mu N \le \frac{2N}{N_*(v)}\langle\varphi_c, (\sqrt{-\Delta} + iv\cdot\nabla)\varphi_c\rangle - i\langle\varphi_c, (v\cdot\nabla)\varphi_c\rangle.$$

By a Fourier transform and Plancherel's theorem [9, Theorem 5.3], using similar arguments to those in the proof of [4, Lemma A.4],

$$i\langle\varphi_c, (v\cdot\nabla)\varphi_c\rangle = -\int_{\mathbb{R}^3}(v\cdot\xi)|\hat{\varphi}_c(\xi)|^2\,d\xi.$$

Since $\sqrt{-\Delta} + iv\cdot\nabla \le \sqrt{-\Delta}$, this yields

$$\mu N \le \frac{2N}{N_*(v)}\langle\varphi_c, \sqrt{-\Delta}\varphi_c\rangle + |v|\langle\varphi_c, \sqrt{-\Delta}\varphi_c\rangle \lesssim \|\varphi_c\|_{H^{1/2}(\mathbb{R}^3)},$$

where we use the fact that $v\cdot\xi \le |v||\xi|$. Since φ_c is uniformly bounded in $H^{1/2}(\mathbb{R}^3)$, we can find a constant $K > 0$ such that $\mu < K$.

This completes the proof of Lemma 2.4. □

LEMMA 2.5. *If φ_c is a minimiser of $E_c(N)$, then there exists a constant $M > 0$ independent of c such that $\|\varphi_c\|_{H^1(\mathbb{R}^3)} \le M$.*

PROOF. Since $\|\varphi_c\|_{L^2}^2 = N$, we only need to derive a uniform bound for $\|\nabla\varphi_c\|_{L^2}$. It follows from (1.3) that

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$$
\begin{aligned}
&c^2\|\nabla\varphi_c\|_{L^2}^2 + m^2c^4\|\varphi_c\|_{L^2}^2 \\
&\quad = \langle\sqrt{-c^2\Delta + m^2c^4}\varphi_c, \sqrt{-c^2\Delta + m^2c^4}\varphi_c\rangle \\
&\quad = \langle(-\mu + mc^2 + |x|^{-1} * |\varphi_c|^2 - iv\cdot\nabla)\varphi_c, (-\mu + mc^2 + |x|^{-1} * |\varphi_c|^2 - iv\cdot\nabla)\varphi_c\rangle \\
&\quad = \mu^2N - 2\mu mc^2N - 2\mu\langle\varphi_c, (|x|^{-1} * |\varphi_c|^2)\varphi_c\rangle + 2\mu\langle\varphi_c, iv\cdot\nabla\varphi_c\rangle + m^2c^4N \\
&\qquad + 2mc^2\langle\varphi_c, (|x|^{-1} * |\varphi_c|^2)\varphi_c\rangle - 2mc^2\langle\varphi_c, iv\cdot\nabla\varphi_c\rangle - \langle v.\nabla\varphi_c, v\cdot\nabla\varphi_c\rangle \\
&\qquad + \langle(|x|^{-1} * |\varphi_c|^2)\varphi_c, (|x|^{-1} * |\varphi_c|^2)\varphi_c\rangle - 2\langle(|x|^{-1} * |\varphi_c|^2)\varphi_c, iv\cdot\nabla\varphi_c\rangle.
\end{aligned}
$$

To bound the terms on the right, we note that Kato's inequality $|x|^{-1} \le |\nabla|$ implies that

$$\|\,|x|^{-1} * |\varphi_c|^2\|_{L^\infty} \lesssim \langle\varphi_c, |\nabla|\varphi_c\rangle \lesssim \|\varphi_c\|_{L^2}\|\nabla\varphi_c\|_{L^2}. \tag{2.5}$$

On the other hand, since $v\cdot\nabla \le |v||\nabla|$ and $|v| < 1$,

$$|\langle\varphi_c, iv\cdot\nabla\varphi_c\rangle| \le \langle\varphi_c, |\nabla|\varphi_c\rangle \le \|\varphi_c\|_{L^2}\|\nabla\varphi_c\|_{L^2}. \tag{2.6}$$

From (2.5) and (2.6),

$$
\begin{aligned}
c^2\|\nabla\varphi_c\|_{L^2}^2 &\le \mu^2N + 2\mu N^{1/2}\|\nabla\varphi_c\|_{L^2} + 2mc^2N^{3/2}\|\nabla\varphi_c\|_{L^2} \\
&\quad + 2mc^2N^{1/2}\|\nabla\varphi_c\|_{L^2} + N^2\|\nabla\varphi_c\|_{L^2}^2 + 2N\|\nabla\varphi_c\|_{L^2}^2.
\end{aligned}
$$

From Lemma 2.4, μ is uniformly bounded. As $c \to \infty$, N is fixed and m is sufficiently small, we conclude that there exists a constant $M > 0$ such that $\|\nabla\varphi_c\|_{L^2} \le M$. By choosing $M > 0$ possibly larger, we arrive at the bound in the lemma. □

PROOF OF THEOREM 1.3. First, we claim that $\{\varphi_c\}$ is a minimising sequence of $E_\infty(N)$. Since φ_c is a ground state of $E_c(N)$,

$$
\begin{aligned}
0 \le E_\infty(N) - E_c(N) &\le \mathcal{E}_\infty(\varphi_c) - \mathcal{E}_c(\varphi_c) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \bar{\varphi}_c\Big(\frac{-\Delta}{2m} - (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\Big)\varphi_c\,dx.
\end{aligned} \tag{2.7}
$$

From the proof of [1, Lemma 6.1],

$$\lim_{c\to\infty}\Big\langle f, \Big(\sqrt{-c^2\Delta + m^2c^4} - mc^2 + \frac{1}{2m}\Delta\Big)\varphi_c\Big\rangle = 0 \quad \text{for all } f \in H^1(\mathbb{R}^3). \tag{2.8}$$

This is easy to verify for a test function $f \in C_0^\infty(\mathbb{R}^3)$ by taking the Fourier transform and observing that

$$\sqrt{c^2\xi^2 + m^2c^4} - mc^2 - \frac{\xi^2}{2m} \to 0 \quad \text{for every } \xi \in \mathbb{R}^3 \text{ as } c \to \infty.$$

By a simple density argument, (2.8) extends to all $f \in H^1(\mathbb{R}^3)$. Therefore,

$$\lim_{c\to\infty}\int_{\mathbb{R}^3} \bar{\varphi}_c\Big[\frac{-\Delta}{2m} - (\sqrt{-c^2\Delta + m^2c^4} - mc^2)\Big]\varphi_c\,dx = 0. \tag{2.9}$$

From (2.7) and (2.9), we conclude that, as $c \to \infty$,

$$E_c(N) \to E_\infty(N) \quad \text{and} \quad \mathcal{E}_\infty(\varphi_c) \to E_\infty(N).$$

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Hence, $\{\varphi_c\}$ is a minimising sequence of $E_\infty(N)$. Combining this with the existence of a minimiser for $E_\infty(N)$ gives (1.8) and completes the proof of Theorem 1.3. □

3. The existence of a minimiser for $E_\infty(N)$

In this section, we prove the existence of a minimiser for the limit energy $E_\infty(N)$.

LEMMA 3.1. *If $\{\varphi_c\}$ is a minimising sequence for $E_\infty(N)$, then $E_\infty(N)$ is a continuous function of N.*

PROOF. Let $\{\varphi_c\}$ be a minimising sequence for $E_\infty(N)$ such that $\lim_{c\to\infty} \mathcal{E}_\infty(\varphi_c) = E_\infty(N)$ with $\|\varphi_c\|_{L^2}^2 = N$. For any $N_1 > 0$,

$$\begin{aligned}
E_\infty(N_1) &\le \mathcal{E}_\infty\Big(\sqrt{\frac{N_1}{N}}\varphi_c\Big) \quad \text{since } \Big\|\sqrt{\frac{N_1}{N}}\varphi_c\Big\|_{L^2}^2 = N_1 \\
&= \frac{1}{4m}\frac{N_1}{N}\int_{\mathbb{R}^3}|\nabla\varphi_c|^2\,dx + \frac{N_1}{2N}\langle\varphi_c, iv\cdot\nabla\varphi_c\rangle - \frac{1}{4}\Big(\frac{N_1}{N}\Big)^2\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|\varphi_c|^2\Big)|\varphi_c|^2\,dx \\
&= \mathcal{E}_\infty(\varphi_c) + \frac{1}{4m}\Big(\frac{N_1}{N}-1\Big)\int_{\mathbb{R}^3}|\nabla\varphi_c|^2\,dx + \frac{1}{2}\Big(\frac{N_1}{N}-1\Big)\langle\varphi_c, iv\cdot\nabla\varphi_c\rangle \\
&\quad - \frac{1}{4}\Big[\Big(\frac{N_1}{N}\Big)^2 - 1\Big]\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|\varphi_c|^2\Big)|\varphi_c|^2\,dx.
\end{aligned}$$

Since $\{\varphi_c\}$ is uniformly bounded in $H^1(\mathbb{R}^3)$, the two integrals and $|\langle\varphi_c, iv\cdot\nabla\varphi_c\rangle|$ can be bounded by a constant $C > 0$ which is independent of the light speed c. Thus,

$$E_\infty(N_1) - E_\infty(N) \le C\Big|\frac{N_1}{N} - 1\Big|. \tag{3.1}$$

By similar arguments,

$$E_\infty(N) - E_\infty(N_1) \le C\Big|\frac{N}{N_1} - 1\Big|. \tag{3.2}$$

From (3.1) and (3.2), it follows that $E_\infty(N_1) \to E_\infty(N)$ as $N_1 \to N$. This completes the proof of Lemma 3.1. □

LEMMA 3.2. *For $m > 0$ sufficiently small, we have the strict binding inequality*

$$E_\infty(N) < E_\infty(\alpha) + E_\infty(N-\alpha) \tag{3.3}$$

for $0 < \alpha < N$.

PROOF. For any $\varepsilon > 0$, there exists $Q \in H^1(\mathbb{R}^3)$ with $\|Q\|_{L^2}^2 = \lambda < N$ such that $E_\infty(\lambda) \le \mathcal{E}_\infty(Q) \le E_\infty(\lambda) + \varepsilon$. Choose $\theta > 1$ such that $\theta\lambda \le N$. Then

$$\begin{aligned}
E_\infty(\theta\lambda) \le \mathcal{E}_\infty(\sqrt{\theta}Q) &= \frac{\theta}{4m}\int_{\mathbb{R}^3}|\nabla Q|^2\,dx + \frac{\theta}{2}\langle Q, iv\cdot\nabla Q\rangle - \frac{\theta^2}{4}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|Q|^2\Big)|Q|^2\,dx \\
&= \frac{1}{2}(\theta-\theta^2)\Big[\frac{1}{2m}\int_{\mathbb{R}^3}|\nabla Q|^2\,dx + \langle Q, iv\cdot\nabla Q\rangle\Big] + \theta^2\mathcal{E}_\infty(Q).
\end{aligned}$$

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For $m > 0$ sufficiently small,

$$\frac{1}{2m}\int_{\mathbb{R}^3} |\nabla Q|^2\,dx + \langle Q, iv\cdot\nabla Q\rangle \geq 0. \tag{3.4}$$

Since $\theta > 1$, we have $E_\infty(\theta\lambda) \leq \theta^2\mathcal{E}_\infty(Q)$ and, in addition,

$$E_\infty(\theta\lambda) \leq \theta^2(E_\infty(\lambda) + \varepsilon). \tag{3.5}$$

Next, we claim that

$$E_\infty(N) < \frac{N}{\alpha}E_\infty(\alpha) \quad \text{for } 0 < \alpha < N. \tag{3.6}$$

Indeed, if $E_\infty(\alpha) \geq 0$, (3.6) obviously holds since $E_\infty(N) < 0$. If $E_\infty(\alpha) < 0$, taking $\theta = N/\alpha$, $\alpha = \lambda$ and $\varepsilon < (\theta^{-1} - 1)E_\infty(\alpha)$ in (3.5) gives (3.6). In the same way, replacing α with $N - \alpha$ gives

$$E_\infty(N) < \frac{N}{N-\alpha}E_\infty(N-\alpha). \tag{3.7}$$

Combining (3.6) and (3.7) yields (3.3) and completes the proof of Lemma 3.2. □

By Lemma 2.5, the minimising sequence $\{\varphi_c\}$ is uniformly bounded in $H^1(\mathbb{R}^3)$. Consequently, there exists a subsequence $\{\varphi_{c_k}\}$ such that $\varphi_{c_k} \rightharpoonup \varphi_\infty$. We now apply the concentration-compactness lemma.

LEMMA 3.3. *Let $\{\varphi_c\}$ be a bounded sequence in $H^1(\mathbb{R}^3)$ satisfying $\|\varphi_c\|_{L^2}^2 = N$. Then, there exists a subsequence $\{\varphi_{c_k}\}$ satisfying one of the following three possibilities.*

(i) *Compactness: there exists a sequence $\{y_k\}$ in $\mathbb{R}^3$ such that, for every $\bar{\varepsilon} > 0$, there exists R, $0 < R < \infty$, with*

$$\int_{|x-y_k|<R} |\varphi_{c_k}|^2\,dx \geq N - \bar{\varepsilon}.$$

(ii) *Vanishing: for all $R > 0$,*

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^3}\int_{|x-y|<R} |\varphi_{c_k}(x)|^2\,dx = 0.$$

(iii) *Dichotomy: there exists $\alpha \in (0, N)$ such that, for every $\bar{\varepsilon} > 0$, there exist two bounded sequences $\{\varphi_k^1\}$ and $\{\varphi_k^2\}$ in $H^1(\mathbb{R}^3)$ and $k_0 \geq 0$ such that, for all $k \geq k_0$,*

$$\|\varphi_{c_k} - (\varphi_k^1 + \varphi_k^2)\|_p \leq \delta_p(\bar{\varepsilon}) \quad \text{for } 2 \leq p < 6,$$

with $\delta_p(\bar{\varepsilon}) \to 0$ as $\bar{\varepsilon} \to 0$, and, as $k \to \infty$, $\operatorname{dist}(\operatorname{supp}\varphi_k^1, \operatorname{supp}\varphi_k^2) \to \infty$,

$$\left|\int_{\mathbb{R}^3} |\varphi_k^1|^2\,dx - \alpha\right| \leq \bar{\varepsilon} \quad \text{and} \quad \left|\int_{\mathbb{R}^3} |\varphi_k^2|^2\,dx - (N-\alpha)\right| \leq \bar{\varepsilon}.$$

Invoking Lemma 3.3, we obtain a suitable subsequence φ_{c_k} with $\varphi_{c_k} \rightharpoonup \varphi_\infty$, which satisfies either (i), (ii) or (iii). We rule out (ii) and (iii) as follows.

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Vanishing does not occur. If vanishing occurs, it follows from [4, Lemma A.1] that

$$\lim_{k\to\infty}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|\varphi_{c_k}|^2\Big)|\varphi_{c_k}|^2\,dx = 0.$$

A similar statement can be found in [10, 11] in the context of other variational problems. By (3.4), we deduce that

$$E_\infty(N) = \lim_{k\to\infty}\mathcal{E}_\infty(\varphi_{c_k}) = \lim_{k\to\infty}\Big(\frac{1}{4m}\int_{\mathbb{R}^3}|\nabla\varphi_{c_k}|^2\,dx + \frac{1}{2}\langle\varphi_c, iv\cdot\nabla\varphi_c\rangle\Big) \geq 0,$$

which contradicts $E_\infty(N) < 0$. Thus, vanishing does not occur.

Dichotomy does not occur. If (iii) is true for φ_{c_k}, by the same arguments as in [4],

$$E_\infty(N) \geq E_\infty(\alpha) + E_\infty(N-\alpha)$$

for $0 < \alpha < N$. This contradicts the strict binding inequality. Thus, dichotomy does not occur. Therefore, we have compactness.

Proof of Theorem 1.2. From the above arguments, we have shown that there exists a subsequence φ_{c_k} such that Lemma 3.3(i) holds for some sequence $\{y_k\}$ in $\mathbb{R}^3$. We now define the sequence

$$\tilde{\varphi}_k := \varphi_{c_k}(\cdot + y_k).$$

Since $\{\tilde{\varphi}_k\}$ is uniformly bounded in $H^1(\mathbb{R}^3)$, we can pass to a subsequence, still denoted by $\{\tilde{\varphi}_k\}$, such that $\{\tilde{\varphi}_k\}$ converges weakly in $H^1(\mathbb{R}^3)$ to some $\varphi_\infty \in H^1(\mathbb{R}^3)$ as $k \to \infty$. Moreover, thanks to the Rellich-type theorem for $H^1(\mathbb{R}^3)$ (see [9, Theorem 8.6]), $\tilde{\varphi}_k \to \varphi_\infty$ strongly in $L^p_{loc}(\mathbb{R}^3)$ as $k \to \infty$ for $2 \leq p < 6$. Since

$$\int_{|x|<R}|\tilde{\varphi}_k|^2\,dx \geq N - \bar{\varepsilon},$$

for every $\bar{\varepsilon} > 0$ and suitable $R = R(\bar{\varepsilon}) < \infty$, we conclude that $\tilde{\varphi}_k \to \varphi_\infty$ strongly in $L^p(\mathbb{R}^3)$ as $k \to \infty$ for $2 \leq p < 6$. By the same arguments as in [4],

$$\lim_{k\to\infty}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|\tilde{\varphi}_k|^2\Big)|\tilde{\varphi}_k|^2\,dx = \int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|\varphi_\infty|^2\Big)|\varphi_\infty|^2\,dx.$$

By weak lower semicontinuity, we conclude that

$$E_\infty(N) \leq \mathcal{E}_\infty(\varphi_\infty) \leq \liminf_{k\to\infty}\mathcal{E}_\infty(\tilde{\varphi}_k) = E_\infty(N).$$

This implies that φ_∞ is a minimiser of $E_\infty(N)$. □

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