FORMULÆ FOR SUMS INVOLVING A REDUCED SET OF RESIDUES MODULO *n*

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In this note we prove the following result:

If n is a positive integer >1, *m the square-free part of n, and if*

$$1 = a_1 < a_2 < \ldots < a_{\phi(n)} = n-1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

are the positive integers less than n, relatively prime to n, then

$$\sum_{j=1}^{\phi(n)} ja_j = \frac{\phi(n)}{24}\{8n\phi(n)+6n+2\phi(m)(-1)^{\omega(m)}-2^{\omega(m)}\},$$

where $\omega(m)$ *is the number of prime factors of m.*

To prove this result we consider first the case when n itself is square-free, $n = p_1p_2\ldots p_r$ where the p_i are distinct primes, $i = 1, 2, \ldots, r$. For every integer k, let $M(n, k)$ be the number of integers a_i in the set (1) such that $(a_i+k, n) = 1$. (In the general case when n is not necessarily square-free, $M(n, k)$ is Nagell's totient function. For a discussion of this function see Alder (**1**).) By considering the r congruences

$$x_i+k \equiv 0 \pmod{p_i}$$

where $x_i \not\equiv 0 \pmod{p_i}$, $i = 1, 2, \ldots, r$, it follows easily that

$$M(n, k) = \prod_{\substack{p \mid k \\ p \mid n}} (p-1) \prod_{\substack{p \nmid k \\ p \mid n}} (p-2) = \phi(n) \prod_{\substack{p \nmid k \\ p \mid n}} \frac{p-2}{p-1}.$$

Now let $(k, n) = d_k$; then since n is square-free, $p \mid n$ and $p \nmid k$ if, and only if $p \mid (n/d_k)$, and hence

$$M(n, k) = \phi(n) \prod_{p \mid \frac{n}{d_k}} \frac{p-2}{p-1}.$$

If d is square-free, let $\psi(d) = \prod_{p \mid d} (p-2)$ $(d>1)$, $\psi(1) = 1$. Then ψ is multiplicative on the square-free integers, and

$$M(n, k) = \phi(n)\psi(n/d_k)/\phi(n/d_k) = \phi(d_k)\psi(n/d_k).$$

Now let $s = \phi(n)$ and consider the set of integers

$$a_1, a_2, \ldots, a_s, a_{s+1}, \ldots, a_{2s},$$

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where $a_{s+i} = n+a_i$, $i = 1, 2, \ldots, s$. From the above, if $1 \leqq k \leqq n$, $M(n, k)$ is the number of pairs (a_i, a_j) in this set such that

$$a_i - a_j = k$$

with $j \leqq s$. Hence

$$\sum_{j=1}^{s} \sum_{i=1}^{s} (a_{i+j} - a_j)^2 = \sum_{k=1}^{n} k^2 M(n, k) = \sum_{k=1}^{n} k^2 \phi(d_k) \psi(n/d_k).$$

Now suppose that $d \mid n$; then $d = d_k$ if, and only if $k = d$ (an integer relatively prime to n/d). Thus, if $G(n)$ denotes the sum of the squares of the positive integers less than n, relatively prime to n, then

$$\sum_{k=1}^{n} k^2 M(n, k) = \sum_{d \mid n} d^2 G(n/d) \phi(d) \psi(n/d)$$

$$= n^2 \phi(n) \sum_{d \mid n} \frac{G(d)}{d^2} \frac{\psi(d)}{\phi(d)}.$$

Now it is easy to prove that if m is square-free, then

$$G(m) = \tfrac{1}{6} m \phi(m)(2m + (-1)^{\omega(m)}), \quad m > 1.$$

(See for example, Nagell (**2**) Ex. 35, Ch. I.) Hence

$$\sum_{k=1}^{n} k^2 M(n, k) = \tfrac{1}{6} n^2 \phi(n) \sum_{\substack{d \mid n \\ d > 1}} \frac{\psi(d)}{d} (2d + (-1)^{\omega(d)}) + n^2 \phi(n)$$

$$= \tfrac{1}{3} n^2 \phi(n)(\phi(n) - 1) + \tfrac{1}{6} n^2 \phi(n) \left(\frac{2^{\omega(n)}}{n} - 1 \right) + n^2 \phi(n)$$

$$= \tfrac{1}{6} n^2 \phi(n)(2\phi(n) + 3) + \tfrac{1}{6} n \phi(n) 2^{\omega(n)}.$$

The second of these equalities follows from the fact that in the first sum each of the terms is multiplicative (on the square-free integers).

Using the following results, which are easily verified,

$$\sum_{j=1}^{s} \sum_{i=1}^{s} (a_{i+j} - a_j)^2 = \sum_{j=1}^{s} \sum_{i=1}^{s-j} (a_{i+j} - a_j)^2 + \sum_{j=1}^{s} \sum_{i=s-j+1}^{s} (a_{i+j} - a_j)^2$$

$$= \sum_{j=1}^{s} \sum_{i=j+1}^{s} (a_i - a_j)^2 + \sum_{j=1}^{s} \sum_{i=1}^{j} (a_{s+i} - a_j)^2$$

$$= \sum_{j=1}^{s} \sum_{i=1}^{s} (a_i - a_j)^2 + \frac{n^2}{2} s(s+1) + 2n \sum_{j=1}^{s} \sum_{i=1}^{j} (a_i - a_j),$$

$$\sum_{j=1}^{s} \sum_{i=1}^{s} (a_i - a_j)^2 = \frac{ns^2}{6} (n + 2(-1)^{\omega(n)}),$$

$$\sum_{j=1}^{s} \sum_{i=1}^{j} (a_i - a_j) = \frac{n}{2} s(s+1) - 2 \sum_{j=1}^{s} j a_j,$$

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we deduce that

$$\sum_{k=1}^{n} k^2 M(n, k) = \tfrac{1}{6}ns^2\,(n+2(-1)^{\omega(n)}) + \tfrac{3}{2}n^2 s(s+1) - 4n \sum_{j=1}^{s} ja_j.$$

Thus

$$4n \sum_{j=1}^{s} ja_j = \tfrac{3}{2}n^2 s(s+1) + \tfrac{1}{6}ns^2\,(n+2(-1)^{\omega(n)}) - \tfrac{1}{6}ns2^{\omega(n)} - \tfrac{1}{6}n^2 s(2s+3),$$

which, after simplification, yields

$$\sum_{j=1}^{\phi(n)} ja_j = \frac{\phi(n)}{24}\{8n\phi(n) + 6n + 2\phi(n)(-1)^{\omega(n)} - 2^{\omega(n)}\}.$$

Now let n be an arbitrary integer >1 and let m be its square-free part. If $1 = b_1 < b_2 < \ldots < b_{\phi(m)} = m-1$ are the positive integers less than m relatively prime to m, then the integers $b_j + lm$, $j = 1, 2, \ldots, \phi(m)$, $l = 0, 1, \ldots, \frac{n}{m} - 1$, are all the positive integers less than n relatively prime to n. Thus if these integers are denoted by $a_1 < a_2 < \ldots < a_{\phi(n)}$, then

$$\sum_{j=1}^{\phi(n)} ja_j = \sum_{j=1}^{\phi(m)} \sum_{l=0}^{\frac{n}{m}-1} (j + l\phi(m))(b_j + lm).$$

It is now an easy matter to deduce from this the result stated earlier for arbitrary integers >1.

Corollary 1. *In the same notation,*

$$\sum_{j=1}^{\phi(n)} j^2 a_j = \frac{\phi(n)}{24}(\phi(n)+1)\{6n\phi(n) + 2n + 2\phi(m)(-1)^{\omega(m)} - 2^{\omega(m)}\}.$$

Corollary 2.

$$\sum_{j=1}^{\phi(n)} ja_j^2 = \frac{\phi(n)}{24}\{6n^2\phi(n) + 4n^2 + 2m(-1)^{\omega(m)}(2\phi(n)+1) - n2^{\omega(m)}\}.$$

The proofs are straightforward and follow from the identity

$$\sum_{j=1}^{\phi(n)} j^\alpha a_j^\beta = \sum_{j=1}^{\phi(n)} (\phi(n)+1-j)^\alpha (n-a_j)^\beta;$$

$\alpha = 2$, $\beta = 1$ gives corollary 1, and $\alpha = 1$, $\beta = 2$, together with the fact that

$$G(n) = \tfrac{1}{6}\phi(n)(2n^2 + m(-1)^{\omega(m)})$$

for arbitrary integers $n > 1$, yields corollary 2.

REFERENCES

(1) H. L. Alder, A generalisation of the Euler ϕ-function, *Amer. Math. Monthly*, **65** (1958), 690-692.

(2) T. Nagell, *Introduction to Number Theory* (Uppsala, 1950).

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