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Complete Families of Linearly Non-degenerate Rational Curves

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Abstract. We prove that every complete family of linearly non-degenerate rational curves of degree $e > 2$ in $\mathbb{P}^n$ has at most $n-1$ moduli. For $e = 2$ we prove that such a family has at most n moduli. The general method involves exhibiting a map from the base of a family X to the Grassmannian of e-planes in $\mathbb{P}^n$ and analyzing the resulting map on cohomology.

1 Introduction and Main Theorem

The goal of this note is to prove the following theorem.

Theorem 1.1 *If X is the base of a complete family of linearly non-degenerate degree $e \geq 3$ curves in $\mathbb{P}^n$ with maximal moduli, then $\dim X \leq n-1$. If X is the base of such a complete family of non-degenerate degree 2 curves in $\mathbb{P}^n$, then $\dim X \leq n$.*

We first introduce the notation used above. Let Y be a smooth projective variety over $\mathbb{C}$. The Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(Y, \beta)$ parametrizes isomorphism classes of pairs (C, f), where C is a proper, connected, at-worst-nodal, arithmetic genus 0 curve, and f is a stable morphism $f\colon C \to Y$ such that $f_*[C] = \beta \in H_2(Y, \mathbb{Z})$. This is a Deligne–Mumford stack whose coarse moduli space $\overline{M}_{0,0}(Y, \beta)$ is projective. See, for example, [FP].

For the remainder of this paper, we will restrict to the case of degree e curves in $Y = \mathbb{P}^n$. Since $H_2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$, we use the standard notation $e = e \cdot [line]$.

Let $\mathcal{U} \subset \mathcal{M}_{0,0}(\mathbb{P}^n, e)$ be the open substack parametrizing maps $f\colon \mathbb{P}^1 \to \mathbb{P}^n$ that are isomorphisms onto their image such that the span of each image is a $\mathbb{P}^e$. Note that no point in $\mathcal{U}$ admits automorphisms and that $\mathcal{U}$ is isomorphic to an open subscheme in the appropriate Hilbert and Chow schemes. In particular, $\mathcal{U}$ is a quasi-projective variety over $\mathbf{C}$.

Definition 1.2 Suppose X and $\mathcal{C}$ are proper varieties and $\pi\colon \mathcal{C} \to X$ is a proper surjective morphism. We will consider diagrams of the form:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi} & & \\ X. & & \end{array}$$

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In the case where each fiber of π is a $\mathbb{P}^1$, and f, restricted to each fiber, corresponds to a point in $\mathcal{U}$, we will call the diagram a *complete family of linearly non-degenerate degree e curves*. Such a family induces a map $\alpha\colon X \to \mathcal{U}$. If the map is generically finite, that is, if $\dim X = \dim \alpha(X)$, we will call the diagram a *family of maximal moduli*. We will refer to X as the base of the family. Note that $\mathcal{C}$ is the pullback of the universal curve over $\mathcal{U}$, and so we will refer to the map f as ev. The notation $(\mathcal{C}, X, ev, \pi, n, e)$ will denote a complete family of linearly non-degenerate degree e curves in $\mathbb{P}^n$.

One can ask for the largest number of moduli of such a family, that is, the dimension of the base X of a family of maximal moduli. This is also the largest dimension of a proper subvariety of $\mathcal{U}$. A simple argument shows that the number of moduli of a linearly non-degenerate family of degree e curves in $\mathbb{P}^e$ is in fact 0. The bend and break lemma [DEB] gives a strict upper bound on the dimension of complete subvarieties $X \subset \mathcal{M}_{0,0}(\mathbb{P}^n, e)$, namely $2n-2$. When the genus of the curves in question are positive, M. Chang and Z. Ran have shown a similar dimension bound. They proved that if Λ is a closed non-degenerate family of positive genus immersed curves in $\mathbb{P}^n$, then $\dim \Lambda \leq n-2$ [CR]. Theorem 1.1 addresses the situation where the curves are rational and required to be linearly non-degenerate.

1.1 Discussion

Question 1.3 What is the best possible result along the lines of Theorem 1.1? For any value $e > 1$, there are certainly examples of complete, linearly non-degenerate r-dimensional families in $\mathbb{P}^{r+e}$. One way to construct such families is to take the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^r \xrightarrow{(e,1)} \mathbb{P}^N,$$

where $N = (e+1)\cdot(r+1) - 1$. Project from a point $p \in \mathbb{P}^N$ not in any $\mathbb{P}^e$ spanned by the image of $\mathbb{P}^1 \times \{q\}$ for every point $q \in \mathbb{P}^r$. This gives an r-dimensional family of non-degenerate degree e curves in $\mathbb{P}^{N-1}$. Continue projecting in this fashion. We can always find a point p to project from as long as $N > r + e$. So we arrive at an r-dimensional family of degree e curves in $\mathbb{P}^{r+e}$.

Question 1.4 Does there exist a complete family with maximal moduli of degree e non-degenerate rational curves in $\mathbb{P}^m$ whose base has dimension greater than $m-e$? Does there exist a complete 2 parameter family of smooth conics in $\mathbb{P}^3$? Does there exist a complete 2 parameter family of smooth cubics in $\mathbb{P}^4$?

Question 1.5 Does there exist a similar bound if the condition of being linearly non-degenerate is removed?

Question 1.6 If the variety swept out by these curves is required to be contained in a smooth hypersurface, does the bound improve? In fact, this question was the original motivation for this work.

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1.2 Outline of Proof

Let $e > 2$ and fix X to be the base of a complete family of linearly non-degenerate degree e curves in $\mathbb{P}^n$ with maximal moduli. Assume that $\dim X \geq n$. Using results from section 2, we will reduce the situation to the case where the universal curve $\mathcal{C}$ over X is the projectivization of a rank 2 vector bundle $\mathcal{E}$ on X. The situation will then be further reduced to the case where we have the following maps.

$$\begin{array}{ccc} \mathcal{C} = \mathbb{P}(\mathcal{E}) & \xrightarrow{ev} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi} & & \\ X & \xrightarrow{\phi} & \mathrm{Gr}(e+1, n+1) \end{array} \tag{1.1}$$

Here ϕ is the generically finite map that associates with each map the e-plane it spans. Using the universal curve $\mathcal{C}$, we will form the following commutative diagram.

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{\gamma} & \mathrm{Fl}(1, \dots, e+1; n+1) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ X & \xrightarrow{\phi} & \mathrm{Gr}(e+1, n+1) \end{array} \tag{1.2}$$

The map γ associates with point of the universal curve (that is, a map $f\colon \mathbb{P}^1 \to \mathbb{P}^n$ and a marked point $p \in \mathbb{P}^1$), the sequence of osculating k-planes to $f(\mathbb{P}^1)$ at $f(p)$. The map between the flag variety and the Grassmannian is the obvious projection.

In Section 3, we will construct an ample line bundle $\mathcal{L}$ on $\mathrm{Fl}(1, \dots, e+1; n+1)$ and give a cohomological argument to show that $c_1(\mathcal{L})^{n+1}$ pulls back to 0 by γ. This will allow us to conclude the proof. In the case $e = 2$, a different computation is needed, but similar ideas apply.

Notation 1.7 Fix the ambient $\mathbb{P}^n$. We will denote by $\mathrm{Fl}(a_1, \dots, a_k; n+1)$ with $a_1 < a_2 < \cdots < a_k$ the flag variety parametrizing vector quotient spaces $\mathbb{C}^{n+1} \to A_k \to A_{k-1} \to \cdots \to A_1$ (all arrows surjective) such that $\dim(A_i) = a_i$. In the special case $\mathrm{Fl}(a; n+1)$ we will write $\mathrm{Gr}(a, n+1)$, the Grassmannian of a dimensional quotients of $\mathbb{C}^{n+1}$. We will follow the convention of [EGAII] and denote the set of hyperplanes in the fibers of $\mathcal{E}$ by $\mathbb{P}(\mathcal{E})$.

2 Reductions

We first prove some general lemmas. In the following section we will apply these to the case of a complete family of linearly non-degenerate degree e curves.

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Proposition 2.1 *Suppose that $\pi\colon \mathcal{C} \to X$ is a proper surjective morphism of complete varieties where each fiber of π is abstractly isomorphic to $\mathbb{P}^1$. Then there exists a surjective, generically finite map $f\colon X' \to X$ such that in the fiber square*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & \mathcal{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X, \end{array}$$

π' realizes $\mathcal{C}'$ as the projectivization of a rank 2 vector bundle $\mathcal{E}$ on X'. That is, $\mathcal{C}' = \mathbb{P}(\mathcal{E})$.

Proof Let $i\colon \nu \to X$ denote the inclusion of the generic point into X. Let $\mathcal{C}_\nu$ be the generic fiber. That is, there is a fibered square

$$\begin{array}{ccc} \mathcal{C}_\nu & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \nu & \xrightarrow{i} & X. \end{array}$$

Let y be a closed point of $\mathcal{C}_\nu$, and let $X' = \overline{y}$ in $\mathcal{C}$. Note that X' is irreducible and proper, and that $\pi(X') = X$. The restricted map $f = \pi|_{X'}\colon X' \to X$ is proper and has only one point in the generic fiber, so is generically finite.

Consider then, the fibered square that defines $\mathcal{C}'$:

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{f'} & \mathcal{C} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

Note that X' maps to $\mathcal{C}$ by construction, so (by the universal property of fiber products) there is a section of π'. That is, there is a map $\sigma\colon X' \to \mathcal{C}'$ such that $\pi' \circ \sigma = id_{X'}$. The existence of the section will allow us to conclude that $\mathcal{C}' \cong \mathbb{P}(\mathcal{E})$ by a standard argument. For example, the argument used in [HAR, V.2 Proposition 2.2] applies word for word. ■

In the case where a projective bundle over X admits a map to $\mathbb{P}^n$, we are able to adjust the bundle (using another finite base change) to control the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Proposition 2.2 *Suppose that $\mathcal{E}$ is a rank 2 vector bundle on a variety X, and let $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ be the natural map. Suppose in addition that $\mathbb{P}(\mathcal{E})$ admits a map to $\mathbb{P}^n$*

that is degree e on each fiber. Then there exists a finite, surjective map $f\colon X' \to X$ such that in the fiber product diagram

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{E}_{X'}) & \xrightarrow{f'} & \mathbb{P}(\mathcal{E}) & \xrightarrow{ev} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \\ X' & \xrightarrow{f} & X & & \end{array}$$

we have that $\pi'_* ev'^* \mathcal{O}(1) = \mathrm{Sym}^e(\mathcal{E}_{X'})$, where $ev' = ev \circ f'$.

Proof First we remark that $ev^*\mathcal{O}(1)$ is a line bundle that is degree e on each fiber of π. Thus $ev^*\mathcal{O}(1) = \mathcal{O}(e) \otimes \pi^*(N)$ for some line bundle N on X. This follows by the description of the Picard group of a projective bundle [HAR]. Then $\pi_* ev^*\mathcal{O}(1) = \mathrm{Sym}^e(\mathcal{E}) \otimes \mathcal{N}$. If there is a line bundle $\mathcal{L}$ on X such that $\mathcal{L}^e \simeq \mathcal{N}$, then it is an easy exercise to show that $\mathrm{Sym}^e(\mathcal{E}) \otimes N \simeq \mathrm{Sym}^e(\mathcal{E} \otimes \mathcal{L})$, and it is well known ([HAR]) that $\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$. Finally, [BG, Lemma 2.1] implies that there exists a finite surjective map $\tau\colon X' \to X$ and a line bundle $\mathcal{L}$ on X' such that $\mathcal{L}^{\otimes e} \simeq \tau^*\mathcal{N}$. ■

3 Proof

Before looking at the general case, we first prove a stronger (though well-known) result than the main theorem would imply when $n = e$.

Proposition 3.1 *If $n = e$, and $(\mathcal{C}, X, ev, \pi, n, n)$ is a family of maximal moduli as in Definition 1.2, then $\dim X = 0$. That is, there is no complete curve contained in $\mathcal{U} \subset \mathcal{M}_{0,0}(\mathbb{P}^n, n)$.*

Proof The space of rational normal curves in projective space is well known to be $\mathbf{PGL}_{n+1}/\mathbf{PGL}_2$. By Matsushima's criterion, the quotient of a reductive affine group scheme by a reductive subgroup is affine [B]. As no affine variety contains a positive dimensional complete subvariety, the proposition follows. Note that there has been recent success in determining the effective cone of this moduli space (see [CHS]). ■

We are now ready to prove the main theorem.

Proof of Theorem 1.1 Fix $(\mathcal{C}, X, ev, \pi, n, e)$ to be a family of maximal moduli as in Definition 1.2 with $2 < e < n$. By way of contradiction, assume that $\dim X \geq n$. By taking an irreducible proper subvariety of X and restricting the family, we may assume that $\dim X = n$.

For any point $x \in X$, denote by $\phi(x)$ the linear e-plane spanned by the image of the map corresponding to x. That is, $\phi(x) = \mathrm{Span}(ev(\pi^{-1}(x)))$. The map $\phi\colon X \to \mathrm{Gr}(e+1, n+1)$ is well defined because each curve corresponding to a point in X is linearly non-degenerate. This morphism factors through $\alpha\colon X \to \mathcal{U}$ (notation as in Definition 1.2), and so is generically finite by Proposition 3.1.

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Applying Proposition 2.1 and then Proposition 2.2, we may assume that there is a generically finite, surjective map $f\colon X' \to X$ such that we have a fiber product diagram

$$\begin{array}{ccccc} \mathbb{P}(\mathcal{E}) & \xrightarrow{f'} & \mathcal{C} & \xrightarrow{ev} & \mathbb{P}^n \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} & & \\ X' & \xrightarrow{f} & X & & \end{array}$$

where $\mathcal{E}$ is a rank two vector bundle on X' and $\pi'_*(f' \circ ev)^*\mathcal{O}(1) = \mathrm{Sym}^e(\mathcal{E})$. The collection $(\mathbb{P}(\mathcal{E}), X', f' \circ ev, \pi', n, e)$ is still a family of linearly non-degenerate degree e curves with maximal moduli, and $\dim X' = n$. The composed map $f \circ \phi$ is a generically finite map from X' to the Grassmannian. To simplify notation, we rename this new family $(\mathbb{P}(\mathcal{E}), X, ev, \pi, n, e)$.

We construct the universal section. Let $Y = \mathbb{P}(\mathcal{E})$ and consider the fiber product diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}_Y) & \longrightarrow & \mathbb{P}(\mathcal{E}) \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ Y & \longrightarrow & X. \end{array}$$

We have a natural section $\sigma\colon Y \to \mathbb{P}(\mathcal{E}_Y)$ given by the diagonal map. This section corresponds to a surjection $\mathcal{E}_Y \to \mathcal{L}$, where $\mathcal{L} = \sigma^*\mathcal{O}_{\mathbb{P}(\mathcal{E}_Y)}(1)$. Let $\mathcal{L}_1 = \mathcal{L}$, and let $\mathcal{L}_2$ be the line bundle such that

$$0 \to \mathcal{L}_2 \to \mathcal{E}_Y \to \mathcal{L}_1 \to 0.$$

This sequence induces a filtration on $\mathrm{Sym}^e(\mathcal{E})$:

$$\mathrm{Sym}^e(\mathcal{E}_Y) = F^0 \supset F^1 \supset \cdots F^e \supset F^{e+1} = 0$$

such that $F^p/F^{p+1} \simeq \mathcal{L}_2^p \otimes \mathcal{L}_1^{e-p}$ ([HAR, II.5]). Note that Y corresponds to curves parametrized by X and a point on that curve. We have a natural map from $Y \to \mathrm{Gr}(e+1, n+1)$ by composition, and the data of the F^ps induce a map from $\gamma\colon Y \to \mathrm{Fl}(1, \dots, e+1; n+1)$. Informally, the information of "the point" on the curve induces a linear filtration of the $\mathbb{P}^e$ spanned by the curve. The linear spaces in between the point and the entire $\mathbb{P}^e$ are the osculating k-planes, $k = 1, \dots, e$. We can see this by working locally where the map is defined by $t \to (1, t, t^2, \dots, t^e, 0, \dots, 0)$. All the maps in diagrams (1.1) and (1.2) have been constructed.

On $\mathrm{Fl}(1, \dots, e+1; n+1)$ we have the natural sequence of universal quotient bundles.

$$\mathcal{O}^{n+1} \to \mathcal{Q}_{e+1} \to \cdots \to \mathcal{Q}_1 \to 0.$$

Recall the previously defined map: $\gamma\colon \mathbb{P}(\mathcal{E}) \to \mathrm{Fl}(1, \dots, e+1; n+1)$. The proof hinges on the following construction.

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Proposition 3.2 *There exists an ample line bundle on the flag variety* $\mathrm{Fl}(1,\ldots,e+1;$ $n+1)$ *whose first Chern class* $D \in H^2(\mathrm{Fl},\mathbb{Z})$ *satisfies* $\gamma^*(D^{n+1}) = 0$.

Assuming the proposition for the moment, we always have ([FUL]) that $D^{\dim Y} \cdot \gamma(Y) > 0$, because γ is generically finite and D is ample. Since $\dim Y = n+1$, we can rewrite this as $(D|_{\gamma(Y)})^{n+1} > 0$. Applying Lemma 3.3, we see that $\gamma^*(D^{n+1}) > 0$, which contradicts Proposition 3.2. Hence we can conclude that $\dim \mathbb{P}(\mathcal{E}) < n+1$ and so $\dim X < n$. The theorem follows. ■

It remains to prove Proposition 3.2.

Proof For $p = 0,\ldots,e$, let $x_p = c_1(\ker \mathcal{Q}_{p+1} \to \mathcal{Q}_p)$. By construction of γ we have $\gamma^* x_p = c_1(F_p/F_{p+1}) = pc_1(\mathcal{L}_2) + (e-p)c_1(\mathcal{L}_1)$.

Consider the projection map $pr\colon \mathrm{Fl}(1,\ldots,n;n+1) \to \mathrm{Fl}(1,\ldots,e+1;n+1)$ and the injective map it induces on cohomology (always with rational coefficients)

$$pr^*\colon H^*(\mathrm{Fl}(1,\ldots,e+1;n+1)) \to H^*(\mathrm{Fl}(1,\ldots,n;n+1)).$$

It is well known that $H^*(\mathrm{Fl}(1,\ldots,n;n+1)) = \mathbb{Q}[x_0,\ldots,x_n]/\mathcal{I}$, where $\mathcal{I}$ is the ideal of symmetric polynomials in the x_is [FUL]. By a slight abuse of notation, denote $pr^*(x_i)$ again by x_i.

In the cohomology ring of full flags, we claim that $x_p^{n+1} = 0$ for each p. To see this, note that in this ring, the identity

$$T^{n+1} = (T - x_1)\cdot(T - x_2)\cdots(T - x_n)$$

holds, since on the right-hand side each coefficient of T^k with $k < n+1$ is a symmetric polynomial. Taking $T = x_p$ proves the identity. Then since pr^* is injective, we must also have that $x_p^{n+1} = 0$ in the cohomology ring of partial flags, so

$$(pc_1(\mathcal{L}_2) + (e-p)c_1(\mathcal{L}_1))^{n+1} = 0 \text{ for each } p = 0,\ldots,e.$$

To simplify notation, in what follows we write $z = c_1(\mathcal{L}_1)$ and $y = c_1(\mathcal{L}_2)$. For relevant facts about the cohomology ring of the flag variety, see Appendix A. For any $D = \lambda_0 x_0 + \cdots + \lambda_e x_e$ we have

$$\begin{aligned}\gamma^*(D) = \gamma^*(\lambda_0\cdot x_0 + \cdots + \lambda_e \cdot x_e) &= \sum_{p=0}^{e} \lambda_p \cdot (py + (e-p)z)\\ &= (\lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots + e\lambda_e)y + (e\lambda_0 + (e-1)\lambda_1 + \cdots + \lambda_{e-1})z\end{aligned}$$

Let A be the coefficient of y, and let B be the coefficient of z. If we can choose $\lambda_0,\ldots,\lambda_e$ so that $\gamma^*(D) = Ay + Bz$ is a $\mathbb{Q}$ multiple of one of the $(py + (e-p)z)$, then for some rational number m we have

$$\gamma^*(D^{n+1}) = (m(py + (e-p)z))^{n+1} = 0.$$

It remains to show that D can be chosen with these properties. See Appendix A for a description of the ample cone of the flag variety. To arrange this choice of D, set

$$\lambda_0 = \frac{1}{e}, \lambda_1 = \frac{1}{e-1}, \dots, \lambda_i = \frac{1}{e-i}, \dots, \lambda_{e-1} = 1.$$

Then, obviously, we have that $B = e$. We will prove that λ_e can be chosen to satisfy

$$\lambda_e > \lambda_{e-1} = 1 \quad \text{and} \quad \tfrac{A}{B} = e - 1.$$

This is equivalent to

$$e\lambda_e = e(e-1) - \Sigma_{i=1}^{e-1} \frac{i}{e-i}, \quad \lambda_e = (e-1) - \Sigma_{i=1}^{e-1} \frac{i}{e(e-i)}$$

Using partial fractions and simplifying, we get

$$\lambda_e = e - \Sigma_{i=0}^{e-1} \frac{1}{e-i}.$$

It is then easy to show this is strictly larger than 1 as long as $e \geq 3$. Therefore, D can be chosen with the required positivity property, and the proof is complete when $e \geq 3$. A simple calculation shows this method cannot work when $e = 2$. To show a slightly weaker result in that case, we need another method. ■

We include the statement of the projection formula used in the proof above.

Lemma 3.3 ([DEB]) *Let $\pi\colon V \to W$ be a surjective morphism between proper varieties. Let $D_1, \dots, D_r$ be Cartier divisors on W with $r \geq \dim(V)$. Then the projection formula holds,* i.e.,

$$\pi^* D_1 \cdots \pi^* D_r = \deg(\pi)(D_1 \cdots D_r).$$

4 The Proof for Conics

In this section we prove the dimension bound for complete families of smooth conics with maximal moduli. Note that for conics (and cubics), being linearly non-degenerate is equivalent to having smooth images.

Theorem 4.1 *If $(\mathcal{C}, X, ev, \pi, 2, n)$ is a family of linearly non-degenerate conics in $\mathbb{P}^n$ with maximal moduli, then $\dim X \leq n$.*

Proof Exactly as in the case $e > 2$, we apply Proposition 2.1 and then Proposition 2.2 to reduce to the case where the family has the form

$$\begin{array}{ccc} \mathcal{C} = \mathbb{P}(\mathcal{E}) & \xrightarrow{ev} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi} & & \\ X & & \end{array}$$

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where $\mathcal{E}$ is a rank two vector bundle on X and $\pi_* ev^* \mathcal{O}(1) = \mathrm{Sym}^2(\mathcal{E})$. As in the higher degree case, we have a generically finite map $\phi\colon X \to \mathrm{Gr}(3, n+1)$. On the Grassmannian $\mathrm{Gr}(3, n+1)$, we have the tautological exact sequence

$$0 \to \mathcal{S} \to \mathcal{O} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the tautological rank 3 quotient bundle. Applying [BG, Lemma 2.1] again, and pulling back the family one more time, we may further assume that $\phi^*(\mathcal{Q}) = \mathrm{Sym}^2(\mathcal{E})$.

Now we proceed with a Chern class computation. First, we compute the Chern polynomial

$$c_t(\mathrm{Sym}^2(\mathcal{E})) = 1 + 3c_1(\mathcal{E})t + (2c_1(\mathcal{E})^2 + 4c_2(\mathcal{E}))t^2 + 4c_1(\mathcal{E})c_2(\mathcal{E})t^3.$$

If we let $A = 3c_1(\mathcal{E})$, $B = 2c_1(\mathcal{E})^2 + 4c_2(\mathcal{E})$, and $C = 4c_1(\mathcal{E})c_2(\mathcal{E})$, an easy computation shows that

$$9AB - 27C - 2A^3 = 0.$$

Write $\widetilde{A} = c_1(\mathcal{Q})$, $\widetilde{B} = c_2(\mathcal{Q})$, and $\widetilde{C} = c_3(\mathcal{Q})$. These classes pull back under ϕ in the following way:

$$A = c_1(\mathrm{Sym}^2(\mathcal{E})) = c_1(\phi^*(\mathcal{Q})) = \phi^*(c_1(\mathcal{Q})) = \phi^*(\widetilde{A}).$$

Here, we have used the properties of ϕ and the functoriality of Chern classes. Similarly, $B = \phi^*(\widetilde{B})$ and $C = \phi^*(\widetilde{C})$. By the functoriality of Chern classes and the above relationships, we have

$$\phi^*(9\widetilde{A}\widetilde{B} - 27\widetilde{C} - 2\widetilde{A}^3) = 0.$$

Let $\xi = 9\widetilde{A}\widetilde{B} - 27\widetilde{C} - 2\widetilde{A}^3$. It becomes convenient to rewrite ξ in terms of the Chern roots of $\mathcal{Q}$. If $\alpha_1, \alpha_2, \alpha_3$ are the Chern roots of $\mathcal{Q}$, then we calculate

$$\begin{aligned}
\widetilde{A} &= \alpha_1 + \alpha_2 + \alpha_3 \\
\widetilde{B} &= \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 \\
\widetilde{C} &= \alpha_1\alpha_2\alpha_3 \\
\xi &= (\alpha_1 + \alpha_2 - 2\alpha_3)(\alpha_2 + \alpha_3 - 2\alpha_1)(\alpha_1 + \alpha_3 - 2\alpha_2)
\end{aligned}$$

Now let $f = \phi_*[X] \in H^*(\mathrm{Gr}(3, n+1), \mathbb{Q})$, where $[X]$ is the fundamental class of X. The projection formula then gives $\xi \cdot f = 0$.

Since $c_1(\mathcal{Q})$ is positive, $c_1(\phi^*\mathcal{Q})$ is positive by Lemma 3.3, and we get the desired bound on $\dim X$ by showing that $c_1(\phi^*\mathcal{Q})^{n+1} = 0$. Since we have already shown that $\phi^*(\xi) = 0$, it would suffice to show that $c_1(\mathcal{Q})^{n+1}$ is divisible by ξ in $H^*(\mathrm{Gr}(3, n+1))$. Instead, we show that this relationship holds in the cohomology ring of full flags and argue that this is enough to conclude the proof.

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Claim. ξ divides $(\alpha_1 + \alpha_2 + \alpha_3)^{n+1}$ in $H^*(\mathrm{Fl}, \mathbb{Q})$, where Fl denotes the space of full flags.

Consider the fiber square

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\phi'} & \mathrm{Fl} \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\phi} & \mathrm{Gr}(3, n+1). \end{array}$$

We have presentations for the cohomology rings

$$H^*(\mathrm{Gr}, \mathbb{Q}) = \mathbb{Q}[\alpha_1, \alpha_2, \alpha_3]/I,$$
$$H^*(\mathrm{Fl}, \mathbb{Q}) = \mathbb{Q}[\alpha_1, \dots, \alpha_{n+1}]/(\mathrm{Symm}),$$

where Symm is the ideal generated by the elementary symmetric functions, and the injective map p^* satisfies $p^*(\alpha_i) = \alpha_i$ for $i = 1, 2, 3$. In $H^*(\mathrm{Fl}, \mathbb{Q})$ we have

$$T^{n+1} = (T - \alpha_1) \cdots (T - \alpha_{n+1}),$$

as before. Evaluate the two sides of the equation at $T = \frac{\alpha_1+\alpha_2+\alpha_3}{3}$ to find

$$\begin{aligned}(\alpha_1 + \alpha_2 + \alpha_3)^{n+1} &= \Big(\frac{\alpha_2 + \alpha_3 - 2\alpha_1}{3}\Big)\Big(\frac{\alpha_1 + \alpha_3 - 2\alpha_2}{3}\Big)\Big(\frac{\alpha_1 + \alpha_2 - 2\alpha_3}{3}\Big)g'(\alpha) \\ &= \xi \cdot g(\alpha)\end{aligned}$$

for some polynomials g' and g, which proves the claim. To finish the proof, note that the fibers of p are projective varieties, that is, effective cycles, and so the same is true of p'. By [FUL], we have

$$(p')^*\phi^*(c_1(\mathcal{Q}))^{n+1} = (\phi')^* p^*(c_1(\mathcal{Q}))^{n+1}$$

The left-hand side of the equation gives an effective cycle on $\widetilde{X}$, in particular, a non-zero cohomology class. On the right side, however, we get

$$\begin{aligned}(\phi')^* p^*(c_1(\mathcal{Q}))^{n+1} &= (\phi')^*(\alpha_1 + \alpha_2 + \alpha_3)^{n+1} = (\phi')^*(\xi \cdot g(\alpha)) = (\phi')^*(p^*\xi \cdot g(\alpha)) \\ &= (\phi')^* p^*\xi \cdot (\phi')^* g(\alpha) = (p')^*\phi^*\xi \cdot (\phi')^* g(\alpha) \\ &= 0 \cdot (\phi')^* g(\alpha) = 0\end{aligned}$$

This gives a contradiction, so we conclude that $\dim(X) \leq n$. ■

A Appendix: Divisors on the Flag Variety

In this appendix we include some notes on the ample cone of the flag variety $F = \mathrm{Fl}(1, \dots, e+1; n+1)$. Let w_i be the $\mathbb{P}^1$ constructed by letting the i-th flag vary while leaving the others constant. These $e+1$ lines freely generate the homology group $H_2(F)$. They are also generators of the effective cone of curves. The $e+1$ Chern classes $x_p = c_1(\ker(\mathcal{Q}_{p+1} \to \mathcal{Q}_p))$ generate $H^2(F)$, and we check that the intersection matrix $\langle x_i, w_j \rangle$ is given by

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix}$$

with 1's on the diagonal and -1's on the lower diagonal. The ample cone of F is given by combinations of the x_i's that evaluate positively, that is, by $\mathbb{Q}$ divisors $\lambda_0 x_0 + \dots + \lambda_e x_e$, where $0 < \lambda_1 < \lambda_2 < \dots < \lambda_e$.

In fact, it is well known that for varieties of the type $F = G/B$, the Picard group of F is isomorphic to the character group of F, often denoted $X(T)$, where T is a maximal torus. Any character can be written as a linear combination of the fundamental weights $\lambda = \sum a_i t_i$, and a character is called *dominant* if all $a_i \geq 0$ and *regular* if all a_i are non-zero. The ample divisors correspond exactly to the dominant and regular characters (see [LG]). In our case, the full flag variety corresponds to G/B for $G = SL(n+1)$. The simple roots correspond to $s_i = \alpha_i - \alpha_{i+1}$ for $0 \leq i \leq n$. Suppose $L = \lambda_0 x_0 + \dots + \lambda_n x_n$, where the x_i are as above. Then L corresponds to the weight $\lambda_0 s_0 + \dots + \lambda_n s_n$, which is dominant if and only if L is ample, if and only if $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$. The case of the partial flag variety then follows immediately from this one.

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