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BURNSIDE RINGS OF FINITE REPRESENTATION TYPE

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Let G be a finite group. It is proved that the localised Burnside ring $\Omega_p(G)$ is of finite representation type if and only if for each p-perfect subgroup H of G, $\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = \underline{H}\right\}\right| \leqslant 3$, where $\underline{K}$ means the conjugacy class of K.

1. INTRODUCTION

Let G be a finite group and let p be a prime number. Let $\Omega_p(G)$ be the Burnside ring of G localised at p. We are interested in the representation type of the category of $\Omega_p(G)$-lattices (which is Krull-Schmidt by [**1**] 30.18). In this note we characterise in terms of the group G when this category is of finite representation type (in this case we say that $\Omega_p(G)$ is representation-finite).

To state the theorem we need to recall some notation. Let $\mathcal{C}(G)$ be the set of conjugacy classes $\underline{K}$ of subgroups K of G and given any subgroup H of G, let $O^p(H)$ denote the minimal normal subgroup $N \trianglelefteq H$ such that H/N is a p-group. We say that H is p-perfect if $O^p(H) = H$.

THEOREM. *The ring $\Omega_p(G)$ is representation-finite if and only if for each p-perfect subgroup H of G,*

$$\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = \underline{H}\right\}\right| \leqslant 3.$$

We obtain this theorem in Section 2 as an application of the Drozd–Roiter's criterion for finite representation type of commutative orders (see [**4**] and [**1**] 33.14). In Section 3 we obtain some corollaries and give some examples.

For basic background on Burnside rings and orders we refer the reader to [**1, 2, 3**].

2. THE PROOF OF THE THEOREM

Let $\Lambda = \Omega_p(G)$ and Λ' be the unique $\mathbb{Z}_{(p)}$-maximal order in $\Omega_{\mathbb{Q}}(G) := \mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G)$.

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Lemma A. *The following hold.*

(i) $\mathrm{rad}_\Lambda(\Lambda') = \mathrm{rad}_\Lambda(\Lambda)\Lambda' = \mathrm{rad}_{\Lambda'}(\Lambda') = p\Lambda'$.

(ii) $\mathrm{rad}_\Lambda(\Lambda) = p\Lambda' \cap \Lambda$.

(iii) $\mathrm{rad}_\Lambda(\Lambda'/\Lambda) = (p\Lambda' + \Lambda)/\Lambda \cong (p\Lambda')/(p\Lambda' \cap \Lambda) = (p\Lambda')/(\mathrm{rad}_\Lambda(\Lambda))$.

Proof: (i) Since $\mathrm{rad}_\Lambda(\Lambda') = \mathrm{rad}_\Lambda(\Lambda)\Lambda'$ and $\mathrm{rad}_{\Lambda'}(\Lambda') = p\Lambda'$, it is enough to show that $\mathrm{rad}_\Lambda(\Lambda)\Lambda' = p\Lambda'$. There is $m > 0$ such that $\mathrm{rad}_\Lambda(\Lambda)^m \subseteq p\Lambda$, so $(\mathrm{rad}_\Lambda(\Lambda)\Lambda')^m \subseteq p\Lambda' = \mathrm{rad}_{\Lambda'}(\Lambda')$; thus $\mathrm{rad}_\Lambda(\Lambda)\Lambda' \subseteq p\Lambda'$. The other inclusion is obvious.

(ii) Since Λ is a $\mathbb{Z}_{(p)}$-order there is $m > 0$ such that $p^m\Lambda' \subseteq \Lambda$. Therefore, for m large enough, $(p\Lambda' \cap \Lambda)^m \subseteq p\Lambda \subseteq \mathrm{rad}_\Lambda(\Lambda)$; thus $p\Lambda' \cap \Lambda \subseteq \mathrm{rad}_\Lambda(\Lambda)$. Also $\mathrm{rad}_\Lambda(\Lambda) \subseteq \mathrm{rad}_\Lambda(\Lambda)\Lambda' = p\Lambda'$.

(iii) This follows from (i) and (ii). ∎

For the following lemma we need to fix some notation.

Let $\varphi_H : \Omega_p(G) \to \mathbb{Z}_{(p)}$ be the mark of H, that is, if X is a G-set then $\varphi_H(X) = |X^H|$, where X^H is the set of fixed points of X under H.

Let $\varphi : \Omega_p(G) \to \prod_{\underline{H} \in \mathcal{C}(G)} \mathbb{Z}_{(p)}$ be given by $\varphi = (\varphi_H)_{\underline{H} \in \mathcal{C}(G)}$.

Let e_H be the idempotent of $\Omega_{\mathbb{Q}}(G)$ corresponding to H (that is, $\varphi_K(e_H) = 1$ if $\underline{K} = \underline{H}$ and 0 otherwise).

Let $e_H^p = \sum e_K$, where the sum runs over all $\underline{K} \in \mathcal{C}(G)$ with $\underline{O^p(K)} = \underline{H}$.

Let $\mathcal{C}_p(G)$ be the set of conjugacy classes of p-perfect subgroups of G. Yoshida has shown that $\{e_H^p : \underline{H} \in \mathcal{C}_p(G)\}$ is a complete set of primitive idempotents of $\Omega_p(G)$ (see [6]; 3.1).

Finally for a finitely generated Λ-module M let $\overline{M} = M/(\mathrm{rad}_\Lambda(M))$, let $\mu_\Lambda(M)$ be the minimal number of generators of M as Λ-module and let $\mathbb{F}_p$ be the field with p elements.

Lemma B. *We have the equality*

$$\mu_\Lambda(\Lambda'/\Lambda) = \sup \left| \left\{ \underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = \underline{H} \right\} \right| - 1,$$

where the supremum is taken over all p-perfect subgroups H of G.

Proof: For any finitely generated Λ-module M we have $\mu_\Lambda(M) = \mu_{\overline{\Lambda}}(\overline{M})$. Thus $\mu_\Lambda(\Lambda'/\Lambda) = \mu_{\overline{\Lambda}}\left(\overline{\Lambda'/\Lambda}\right) = \mu_{\overline{\Lambda}}(\Lambda'/(p\Lambda' + \Lambda))$, by Lemma A. On the other hand $\overline{\Lambda} = \Lambda/(\mathrm{rad}_\Lambda(\Lambda)) = \Lambda/(p\Lambda' \cap \Lambda) \cong (\Lambda + p\Lambda')/(p\Lambda') \subseteq (\Lambda')(p\Lambda')$, again by Lemma A. Then $\mu_\Lambda(\Lambda'/\Lambda) = \mu_{\overline{\Lambda}}(\overline{\Lambda'}/\overline{\Lambda})$. Let

$$P_{H,p} = \mathrm{Ker}\left(\Omega_p(G) \to \mathbb{Z}_{(p)} \to \mathbb{F}_p\right).$$

By [6] 2.2, the set $\{P_{H,p} \mid \underline{H} \in \mathcal{C}_p(G)\}$ consists of the distinct maximal ideals of $\Omega_p(G)$. Then by the Chinese remainder theorem φ induces isomorphisms

$$\overline{\varphi}\colon \overline{\Lambda}' \to \prod_{\underline{H}\in \mathcal{C}(G)} \mathsf{F}_p e_H =: A$$

and

$$\overline{\varphi}\colon \overline{\Lambda} \to \prod_{\underline{H}\in \mathcal{C}_p(G)} \mathsf{F}_p e_H^p =: B.$$

Then $\mu_\Lambda(\Lambda'/\Lambda) = \mu_B(A/B)$. If for $\underline{H} \in \mathcal{C}_p(G)$ we let $B_H = \mathsf{F}_p e_H^p$ and $A_H = \oplus \mathsf{F}_p e_K$, where the sum runs over all $\underline{K} \in \mathcal{C}(G)$ with $\underline{O^p(K)} = \underline{H}$, then we have $\mu_\Lambda(\Lambda'/\Lambda) = \sup \mu_{B_H}(A_H/B_H)$, where the supremum is taken over all $\underline{H} \in \mathcal{C}_p(G)$. On the other hand $\mu_{B_H}(A_H/B_H) = \left|\left\{\underline{K} \in \mathcal{C}(H) : \underline{O^p(K)} = \underline{H}\right\}\right| - 1$, so the result follows. ∎

In the next lemma we use the fact that the minimum integer n such that $ne_H \in \Omega(G)$ is $[N_G(H) : H][H : H']_0$, where $[H : H']_0$ is the product of all the distinct prime factors of $[H : H']$. (See [5], remark after 3.3).

Lemma C. *If $\mu_\Lambda(\Lambda'/\Lambda) \leqslant 2$ then $\mu_\Lambda(\mathrm{rad}_\Lambda(\Lambda'/\Lambda)) \leqslant 1$.*

Proof: From Lemma A, $\mathrm{rad}_\Lambda(\Lambda'/\Lambda) = (p\Lambda')/(\mathrm{rad}_\Lambda(\Lambda))$ and from Lemma B, if $\mu_\Lambda(\Lambda'/\Lambda) \leqslant 2$ then $p^3 \nmid |G|$. Therefore $p^2\Lambda' \subseteq \Lambda$ and, in fact, $p^2\Lambda' \subseteq \mathrm{rad}_\Lambda(\Lambda)$. If $p\Lambda' \subseteq \Lambda$, we are done. Thus we assume that p^2 divides $|G|$. Hence $(p\Lambda')/(\mathrm{rad}_\Lambda(\Lambda))$ is a $\overline{\Lambda}$-module. Therefore $\mu_{\overline{\Lambda}}(\mathrm{rad}_\Lambda(\Lambda'/\Lambda)) = \sup \mu_{B_H}(((p\Lambda')/\mathrm{rad}_\Lambda(\Lambda))e_H^p)$, with B_H as in the proof of Lemma B and the supremum taken over all $\underline{H} \in \mathcal{C}_p(G)$. Given $\underline{H} \in \mathcal{C}_p(G)$, there are two cases:

If $\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = \underline{H}\right\}\right| \leqslant 2$, then $p\Lambda' e_H^p \subseteq \mathrm{rad}_\Lambda(\Lambda)$, so $\mu_{B_H}((p\Lambda')/(\mathrm{rad}_\Lambda(\Lambda))e_H^p) = 0$.

If $\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = \underline{H}\right\}\right| = 3$ then $e_H^p = e_{H_0} + e_{H_1} + e_{H_2}$ with $H \trianglelefteq H_i$ and $[H_i : H] = p^i$. Clearly pe_{H_2} and pe_H^p lie in $\mathrm{rad}_\Lambda(\Lambda)$ by the remark, so $pe_{H_0} + pe_{H_1} \in \mathrm{rad}_\Lambda(\Lambda)$. Therefore $(p\Lambda')/(\mathrm{rad}_\Lambda(\Lambda))e_H^p = \overline{\Lambda} pe_{H_0}$. We conclude then that $\mu_{\overline{\Lambda}}(\mathrm{rad}_\Lambda(\Lambda'/\Lambda)) \leqslant 1$. ∎

Now, using Lemmas A and B our Theorem clearly follows from the Drozd–Roiter's criterion.

3. Some consequences and examples

Corollary A. *If $\Omega_p(G)$ is representation-finite then $p^3 \nmid |G|$.*

For p-groups the only p-perfect subgroup is the trivial one, so we obtain the following corollary.

Corollary B. *Let G be a p-group. Then $\Omega_p(G)$ is representation-finite if and only if G is cyclic of order dividing p^2.* ∎

If G is not a p-group we can only say:

Corollary C. *If a p-Sylow subgroup of G is cyclic of order dividing p^2, then $\Omega_p(G)$ is representation-finite.*

Proof: Let $\underline{H} \in \mathcal{C}_p(G)$. If $\underline{K} \in \mathcal{C}(G)$ with $O^p(K) = \underline{H}$ then we may assume $H \leqslant K \leqslant N_G(H)$. Let $H \leqslant H_p \leqslant N_G(H)$ be such that H_p/H is a p-Sylow subgroup of $(N_G(H))/H$. If $p^2 \nmid [H_p : H]$ then clearly $\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = \underline{H}\right\}\right| \leqslant 2$. If $p^2 \mid [H : H_p]$, then H_p/H is isomorphic to a p-Sylow subgroup of G, hence it is cyclic and therefore

$$\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = \underline{H}\right\}\right| = 3.$$

∎

The converse of Corollary C is false, as the following example shows.

Example A. Let $G = A_4$, the alternating group of degree 4, and $p = 2$. Then $\Omega_p(G)$ is representation-finite and a 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z})$.

Having in mind our Theorem and Example A one might naively suspect that if $p^3 \nmid |G|$ and all the subgroups of the same order of a p-Sylow subgroup of G are conjugate (that is $\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^p(K)} = 1\right\}\right| \leqslant 3$), then $\Omega_p(G)$ is representation-finite. However, the following example shows that this is not true.

Example B. Let $P_1 = \langle x \rangle$, $P_2 = \langle y \rangle$ be cyclic groups of order 2, $Q_1 = \langle w \rangle$, $Q_2 = \langle z \rangle$ cyclic groups of order 5 and $T = \langle a \rangle$ a cyclic group of order 3. Let T act on $P_1 \times P_2 \times Q_1 \times Q_2$ by $axa^{-1} = y$, $aya^{-1} = xy$, $awa^{-1} = w^3z$ and $aza^{-1} = w^2z$. (This is justified since $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ is an element of $GL(2, \mathbb{Z}/5\mathbb{Z})$ of order 3.) Then if $p = 3$ and G is the semidirect product $(P_1 \times P_2 \times Q_1 \times Q_2) \rtimes T$, we have the desired example. (Indeed $\left|\left\{\underline{K} \in \mathcal{C}(G) : \underline{O^2(K)} = \underline{Q}_1\right\}\right| \geqslant 4$).

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