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Relative Equilibria in Curved Restricted 4-body Problems

This paper is dedicated to the late Florin Diacu, our friend, colleague and teacher, who brought us much joy and interest in celestial mechanics.

Sawsan Alhowaity, Florin Diacu, and Ernesto Pérez-Chavela

Abstract. We consider the curved 4-body problems on spheres and hyperbolic spheres. After obtaining a criterion for the existence of quadrilateral configurations on the equator of the sphere, we study two restricted 4-body problems, one in which two masses are negligible and another in which only one mass is negligible. In the former, we prove the evidence square-like relative equilibria, whereas in the latter we discuss the existence of kite-shaped relative equilibria.

1 Introduction

The curved N-body problem is a natural extension of the classical Newtonian N-body problem to spaces of constant curvature. The idea of extending the gravitational force between point masses to spaces of constant curvature occurred soon after the discovery of hyperbolic geometry. In the 1830s, independently of each other, Bolyai and Lobachevsky realized that there must be an intimate connection between the laws of physics and the geometry of the universe [1, 23, 24]. Lobachevsky studied the Kepler problem in the three dimensional hyperbolic space by defining a special potential that was an extension of classical Newton potential. Since then, many researchers have studied this problem in two and three dimensional curved spaces. Now we jump ahead in the history of this problem to the developements in recent years.

In the 1990s, the Russian school of celestial mechanics considered both the curved Kepler and the curved 2-body problem, [22, 31]. After understanding that, unlike in the Euclidean case, these problems are not equivalent, the latter failing to be integrable ([31]), the 2-body case was intensively studied by several researchers of this school. In 2005, Cariñena, Rañada, and Santander studied, in a unified way, the two body problem defined on spaces of nonzero constant curvature [2]. This paper was the inspiration for the work of Diacu, Santoprete, and Pérez-Chavela. These authors extended the curved N-body problem for $N > 2$ in a new unified framework, leading to many interesting results [2–19, 28].

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Other researchers developed these ideas further [21,25,26,29–32], and the problem is growing in popularity. In [5], the interested reader can find a nice introduction to this problem with a long historical background.

In this article, we first prove a criterion for the existence of quadrilateral relative equilibria on the equator of the sphere. The main results show that if two masses are negligible and the other two are equal, then square-like relative equilibria exists on spheres, but—surprisingly—not on hyperbolic spheres. The element of surprise arises from the fact that, in the general problem, square-like equilibria exist both on the hyperbolic sphere and on the sphere (except for the case when they are on the equator) [5]. Finally, we prove that if only one mass is negligible and the other three are equal, some kite-shaped relative equilibria exist on spheres, but not on hyperbolic spheres.

After the introduction the paper is organized as follows. In Section 2, we write the equations of motion for our model. In Section 3, we give the formal definition of relative equilibria, showing that for the Principal Axis Theorem, for the case of positive curvature, we can restrict our analysis to the action of just one symmetric group; whereas, for negative curvature we need to analyze three different symmetric groups acting on the corresponding surface. In Sections 4 and 5, we state and prove our main results. Previously we justify why in this paper, in the case of negative curvature we are restricted only to the study of the elliptic relative equilibria. Although the computations are very similar in both cases, we have included both here for the convenience of the reader.

2 Equations of Motion

We consider the motion of four bodies on 2-dimensional surfaces of constant curvature κ. For $\kappa > 0$, we use as a model the spheres of radius $1/\sqrt{\kappa}$. This sphere is embedded in $\mathbb{R}^3$ with the Euclidean metric, and we denote it by $\mathbb{S}^2_\kappa$. For $\kappa = 0$, we take the Euclidean plane $\mathbb{R}^2$, and for $\kappa < 0$ we take the upper part of the hyperboloid $x^2 + y^2 - z^2 = -1/\sqrt{-\kappa}$, embedded in the Minkowski space $\mathbb{R}^{2,1}$, that is $\mathbb{R}^3$ endowed with the Lorenz inner product (for $a, b \in \mathbb{R}^3$, $a \odot b = a_x b_x + a_y b_y - a_z b_z$). This space is known as the *hyperbolic sphere* or the *pseudo sphere*, and it is denoted by $\mathbb{H}^2_\kappa$.

Now, we will arrange these objects in $\mathbb{R}^3$, maintaining the different metric for the sphere and the pseudo sphere, such that they all have a common point at which lie all the north poles of the spheres and the vertices of the hyperbolic spheres, to all of which the plane $\mathbb{R}^2$ is tangent. We fix the origin of the new coordinate system at this point. In other words, we translate the origin to the north pole of the sphere and the pseudo sphere, and abusing notation, we keep the same notation for these objects. Then we can write

$$\mathbb{S}^2_\kappa := \left\{ (x, y, z) \mid \kappa(x^2 + y^2 + z^2) + 2\kappa^{\frac{1}{2}} z = 0 \right\} \text{ for } \kappa > 0,$$

$$\mathbb{H}^2_\kappa := \left\{ (x, y, z) \mid \kappa(x^2 + y^2 - z^2) + 2|\kappa|^{\frac{1}{2}} z = 0, \quad z \geq 0 \right\} \text{ for } \kappa < 0.$$

Now consider four point masses, $m_i > 0$, $i = 1, 2, 3, 4$, whose position vectors, velocities, and accelerations are given by

$$\mathbf{r}_i = (x_i, y_i, z_i), \quad \dot{\mathbf{r}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i), \quad \ddot{\mathbf{r}}_i = (\ddot{x}_i, \ddot{y}_i, \ddot{z}_i), \quad i = 1, 2, 3, 4.$$

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Then, as shown in [9], the equations of motion take the form

$$
\begin{aligned}
\ddot{x}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\big[\,x_j - \big(1 - \frac{\kappa r_{ij}^2}{2}\big)\,x_i\big]}{\big(1 - \frac{\kappa r_{ij}^2}{4}\big)^{3/2} r_{ij}^3} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)x_i,\\
\ddot{y}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\big[\,y_j - \big(1 - \frac{\kappa r_{ij}^2}{2}\big)\,y_i\big]}{\big(1 - \frac{\kappa r_{ij}^2}{4}\big)^{3/2} r_{ij}^3} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)y_i,\\
\ddot{z}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\big[\,z_j - \big(1 - \frac{\kappa r_{ij}^2}{2}\big)\,z_i\big]}{\big(1 - \frac{\kappa r_{ij}^2}{4}\big)^{3/2} r_{ij}^3} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)(\kappa z_i + \sigma|\kappa|^{1/2}), \quad i = 1, 2, 3, 4,
\end{aligned}
\tag{2.1}
$$

where $\sigma = 1$ for $\kappa \geq 0$, $\sigma = -1$ for $\kappa < 0$, and

$$
r_{ij} := \begin{cases} [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2} & \text{for } \kappa \geq 0,\\ [(x_i - x_j)^2 + (y_i - y_j)^2 - (z_i - z_j)^2]^{1/2} & \text{for } \kappa < 0, \end{cases}
$$

for $i, j \in \{1, 2, 3, 4\}$. The above system has eight constraints, namely,

$$
\begin{aligned}
\kappa(x_1^2 + y_i^2 + \sigma z_i^2) + 2|\kappa|^{1/2} z_i &= 0,\\
\kappa \mathbf{r}_i \cdot \dot{\mathbf{r}}_i + |\kappa|^{1/2} \dot{z}_i &= 0, \quad i = 1, 2, 3, 4.
\end{aligned}
$$

If satisfied at an initial instant, these constraints are satisfied for all time, because the sets $\mathbb{S}_\kappa^2, \mathbb{R}^2$, and $\mathbb{H}_\kappa^2$ are invariant for the equations of motion, [5]. Notice that for $\kappa = 0$, we recover the classical Newtonian equations of the 4-body problem on the Euclidean plane, namely,

$$
\ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{N} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3},
$$

where $\mathbf{r}_i = (x_i, y_i, 0)$, $i = 1, 2, 3, 4$.

3 Relative Equilibria

It is well known that in the curved N-body problem, the total energy and the angular momentum are first integrals, but the linear momentum is no longer a constant of motion, which is a big difference from the Euclidean case [15]. The goal of this paper is to describe particular solutions for this problem, the simplest ones, called relative equilibria. The formal definition follows.

Definition 3.1 *Relative equilibria* are solutions of the curved N-body problem in which the mutual distances among the particles remain constant for all time $t \in \mathbb{R}$. That is, the particles move like a rigid body.

So in order to study relative equilibria, we must to analyze all isometries for both the sphere $\mathbb{S}_\kappa^2$ and the pseudo-sphere $\mathbb{H}_\kappa^2$. According to the above definition, the relative equilibria will be the solutions of the equations of motion that are invariant under the action of the isometry groups for the respective surfaces of positive and negative curvature.

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3.1 Relative Equilibria for Positive κ

This is the simplest case, because we know that all isometries in $\mathbb{R}^3$ are rotations, and The Principal Axis Theorem states that any rotation in $\mathbb{R}^3$, is around any fixed axis [20]. So in this case, without loss of generality, we can assume that the rotation is around the z-axis, and we have that a relative equilibrium is a solution of the equations of motion that is invariant under the action of the isometry given by the rotation matrix

$$A(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

3.2 Relative Equilibria for Negative κ

Let $\mathrm{Lor}(\mathbb{H}^2_\kappa, \odot)$ be the group of all orthogonal transformations of determinant 1 that maintains the upper part of the hyperboloid invariant (the Lorentz group formed by all isometries of $\mathbb{H}^2_\kappa$) (see [13, 15, 28] for more details). Applying the corresponding Principal Axis Theorem [27] to $\mathrm{Lor}(\mathbb{H}^2_\kappa, \odot)$, which states that any 1–parameter subgroup of $\mathrm{Lor}(\mathbb{L}^2, \odot)$ can be written, in a proper basis, as

$$A(t) = P \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1},$$

or

$$A(t) = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} P^{-1},$$

or

$$A(t) = P \begin{pmatrix} 1 & -t & t \\ t & 1 - t^2/2 & t^2/2 \\ t & -t^2/2 & 1 + t^2/2 \end{pmatrix} P^{-1},$$

where $P \in \mathrm{Lor}(\mathbb{H}^2_\kappa, \odot)$. Then any isometry of $\mathrm{Lor}(\mathbb{H}^2_\kappa, \odot)$ can be written as a composition of some of the above three transformations, called elliptic, hyperbolic, and parabolic, respectively. So in this case, the relative equilibria on the pseudo sphere are the solutions of the equations of motion that are invariant under some isometry of $\mathrm{Lor}(\mathbb{H}^2_\kappa, \odot)$.

4 The Case of Positive Curvature

By the previous discussion, in this case we have to find the initial conditions which lead to the solutions that are invariant under the action of the isometry given by the rotation matrix defined by equation (3.1).

First, we introduce new coordinates (φ, ω), which were originally used in [9] for the case $N = 3$, to detect relative equilibria on and near the equator of $\mathbb{S}^2_\kappa$, where φ measures the angle from the x-axis in the xy-plane, while ω is the height on the

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vertical z-axis. In these new coordinates, the 8 constraints for the original equations of motion (2.1) become

$$x_i^2 + y_i^2 + \omega_i^2 + 2\kappa^{-1/2}\omega_i = 0, \quad i = 1,2,3,4. \tag{4.1}$$

With the notation,

$$\Omega_i = x_i^2 + y_i^2 = -\kappa^{-1/2}\omega_i(\kappa^{1/2}\omega_i + 2) \geq 0, \quad \omega_i \in [-2\kappa^{-1/2}, 0], \quad i = 1,2,3,4,$$

where equality occurs when the body is at the North or the South Pole of the sphere, the (φ, ω)-coordinates are given by the transformations

$$x_i = \Omega_i^{1/2}\cos\varphi_i, \quad y_i = \Omega_i^{1/2}\sin\varphi_i.$$

Thus, the equations of motion (2.1) take the form

$$\begin{aligned}
\ddot{\varphi}_i &= \Omega_i^{-1/2}\sum_{j=1, j\neq i}^{N} \frac{m_j\Omega_j^{1/2}\sin(\varphi_j - \varphi_i)}{\rho_{ij}^3(1 - \frac{\kappa\rho_{ij}^2}{4})^{3/2}} - \frac{\dot{\varphi}_i\dot{\Omega}_i}{\Omega_i},\\
\ddot{\omega}_i &= \Omega_i^{-1/2}\sum_{j=1, j\neq i}^{N} \frac{m_j\left[\omega_j + \omega_i + \frac{\kappa\rho_{ij}^2}{2}(\omega_i + \kappa^{-1/2})\right]}{\rho_{ij}^3\left(1 - \frac{\kappa\rho_{ij}^2}{4}\right)^{3/2}}\\
&\quad - (\kappa\omega_i + \kappa^{1/2})\left(\frac{\dot{\Omega}_i^2}{4\Omega_i} + \dot{\varphi}_i{}^2\Omega_i + \dot{\omega}_i^2\right),
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
&\dot{\Omega}_i = -2\kappa^{-1/2}\dot{\omega}_i(\kappa^{1/2}\omega_i + 1),\\
&\rho_{ij}^2 = \Omega_i + \Omega_j - 2\Omega_i^{1/2}\Omega_j^{1/2}\cos(\varphi_i - \varphi_j) + (\omega_i - \omega_j)^2, \quad i, j = 1,2,3,4, \ i \neq j.
\end{aligned}$$

4.1 Relative Equilibria on the Equator

If we restrict the motion of the four bodies to the equator of $\mathbb{S}_\kappa^2$, then

$$\omega_i = -\kappa^{-1/2}, \quad \dot{\omega}_i = 0, \quad \Omega_i = \kappa^{-1}, \quad i = 1,2,3,4,$$

and the equations of motion (4.2) take the simple form

$$\ddot{\varphi}_i = \kappa^{3/2}\sum_{j=1, j\neq i}^{4} \frac{m_j\sin(\varphi_j - \varphi_i)}{|\sin(\varphi_j - \varphi_i)|^3}, \quad i = 1,2,3,4.$$

For the relative equilibria, the angular velocity is the same constant for all masses, so we denote this velocity by $\alpha \neq 0$ and take

$$\varphi_1 = \alpha t + a_1, \quad \varphi_2 = \alpha t + a_2, \quad \varphi_3 = \alpha t + a_3, \quad \varphi_4 = \alpha t + a_4,$$

where a_1, a_2, a_3, a_4 are real constants, so

$$\ddot{\varphi}_i = 0, \quad i = 1,2,3,4.$$

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Using the notation

$$s_1 := \frac{\kappa^{3/2}\sin(\varphi_1-\varphi_2)}{|\sin(\varphi_1-\varphi_2)|^3},\quad s_2 := \frac{\kappa^{3/2}\sin(\varphi_2-\varphi_3)}{|\sin(\varphi_2-\varphi_3)|^3},\quad s_3 := \frac{\kappa^{3/2}\sin(\varphi_3-\varphi_1)}{|\sin(\varphi_3-\varphi_1)|^3},$$
$$s_4 := \frac{\kappa^{3/2}\sin(\varphi_4-\varphi_1)}{|\sin(\varphi_4-\varphi_1)|^3},\quad s_5 := \frac{\kappa^{3/2}\sin(\varphi_2-\varphi_4)}{|\sin(\varphi_2-\varphi_4)|^3},\quad s_6 := \frac{\kappa^{3/2}\sin(\varphi_3-\varphi_4)}{|\sin(\varphi_3-\varphi_4)|^3}.$$

The first theorem can be expressed as follows.

Theorem 4.1 *A necessary condition that the quadrilateral inscribed in the equator of $\mathbb{S}^2_\kappa$, with the four masses $m_1, m_2, m_3, m_4 > 0$ at its vertices, forms a relative equilibrium is that $s_1 s_6 + s_3 s_5 = s_2 s_4$.*

Proof We obtain from the equations of motion corresponding to $\ddot{\varphi}_i$ that

$$-m_2 s_1 + m_3 s_3 + m_4 s_4 = 0, \qquad -m_1 s_3 + m_2 s_2 - m_4 s_6 = 0,$$
$$m_1 s_1 - m_3 s_2 - m_4 s_5 = 0, \qquad -m_1 s_4 + m_2 s_5 + m_3 s_6 = 0.$$

To have other solutions of the masses than $m_1 = m_2 = m_3 = m_4 = 0$, the determinant of the above system must vanish, which is equivalent to $s_1 s_6 + s_3 s_5 = s_2 s_4$. This remark completes the proof. ■

4.2 Equivalent Equations of Motion

In this subsection, we obtain another form of equations of motion in which the action of the isometry groups that define the relative equilibria is conserved. Let us now introduce some equivalent equations of motion that are suitable for the kind of solutions we are seeking. First, by eliminating ω_i from the constraints given by equation (4.1), we get

$$\kappa(x_i^2 + y_i^2) + (|\kappa|^{1/2} z_i + 1)^2 = 1,$$

and solving explicitly for z_i, we obtain

$$z_i = |\kappa|^{-1/2}\Big[\sqrt{1-\kappa(x_i^2+y_i^2)} - 1\Big].$$

The idea here is to eliminate the four equations involving z_1, z_2, z_3, z_4, but they still appear in the terms r_{ij}^2 in the form $\sigma(z_i - z_j)^2$ as

$$\sigma(z_i - z_j)^2 = \frac{\kappa(x_i^2 + y_i^2 - x_j^2 - y_j^2)^2}{\Big[\sqrt{1-\kappa(x_i^2+y_i^2)} + \sqrt{1-\kappa(x_j^2+y_j^2)}\Big]^2}.$$

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The case of physical interest is when κ is not far from zero, so the above expression exist even for small $\kappa > 0$ under this assumption. Then the equations of motion become

$$
\begin{aligned}
\ddot{x}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\left[x_j - \left(1 - \frac{\kappa\rho_{ij}^2}{2}\right)x_i\right]}{\left(1 - \frac{\kappa\rho_{ij}^2}{4}\right)^{3/2}\rho_{ij}^3} - \kappa({\dot{x}_i}^2 + {\dot{y}_i}^2 + \kappa B_i)x_i \\
\ddot{y}_i &= \sum_{j=1, j\neq i}^{N} \frac{m_j\left[y_j - \left(1 - \frac{\kappa\rho_{ij}^2}{2}\right)y_i\right]}{\left(1 - \frac{\kappa\rho_{ij}^2}{4}\right)^{3/2}\rho_{ij}^3} - \kappa({\dot{x}_i}^2 + {\dot{y}_i}^2 + \kappa B_i)y_i,
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
\rho_{ij}^2 &= (x_i - x_j)^2 + (y_i - y_j)^2 + \frac{\kappa(A_i - A_j)^2}{(\sqrt{1-\kappa A_i} + \sqrt{1-\kappa A_j})^2}, \\
A_i &= x_i^2 + y_i^2, \\
B_i &= \frac{(x_i\dot{x}_i + y_i\dot{y}_i)^2}{1 - \kappa(x_i^2 + y_i^2)}, \qquad i = 1, 2, 3, 4.
\end{aligned}
$$

It is obvious that for $\kappa = 0$ we recover the classical Newtonian equations of motion of the planar 4-body problem. Also, since the relative equilibria for Newtonian equations are invariant under the action of the rotation matrix given by (3.1), and the other terms in equation (4.3) depend essentially on mutual distances and its derivatives, then the corresponding relative equilibria for the new system is conserved by (3.1).

4.3 The Case of Two Negligible Masses

We now consider the case when two out of the four given masses are negligible, $m_3 = m_4 = 0$. Then the equations of motion become

$$
\begin{cases}
\ddot{x}_1 = \dfrac{m_2\left[x_2 - \left(1 - \frac{\kappa\rho_{12}^2}{2}\right)x_1\right]}{\left(1 - \frac{\kappa\rho_{12}^2}{2}\right)^{3/2}\rho_{12}^3} - \kappa({\dot{x}_1}^2 + {\dot{y}_1}^2 + \kappa B_1)x_1, \\
\ddot{y}_1 = \dfrac{m_2\left[y_2 - \left(1 - \frac{\kappa\rho_{12}^2}{2}\right)y_1\right]}{\left(1 - \frac{\kappa\rho_{12}^2}{4}\right)^{3/2}\rho_{12}^3} - \kappa({\dot{x}_1}^2 + {\dot{y}_1}^2 + \kappa B_1)y_1,
\end{cases}
\tag{4.4}
$$

$$
\begin{cases}
\ddot{x}_2 = \dfrac{m_1\left[x_1 - \left(1 - \frac{\kappa\rho_{12}^2}{2}\right)x_2\right]}{\left(1 - \frac{\kappa\rho_{12}^2}{4}\right)^{3/2}\rho_{12}^3} - \kappa({\dot{x}_2}^2 + {\dot{y}_2}^2 + \kappa B_2)x_2, \\
\ddot{y}_2 = \dfrac{m_1\left[y_1 - \left(1 - \frac{\kappa\rho_{12}^2}{2}\right)y_2\right]}{\left(1 - \frac{\kappa\rho_{12}^2}{4}\right)^{3/2}\rho_{12}^3} - \kappa({\dot{x}_2}^2 + {\dot{y}_2}^2 + \kappa B_2)y_2,
\end{cases}
\tag{4.5}
$$

$$
\begin{cases}
\ddot{x}_3 = \dfrac{m_1\left[x_1 - \left(1 - \frac{\kappa\rho_{13}^2}{2}\right)x_3\right]}{\left(1 - \frac{\kappa\rho_{13}^2}{4}\right)^{3/2}\rho_{13}^3} + \dfrac{m_2\left[x_2 - \left(1 - \frac{\kappa\rho_{32}^2}{2}\right)x_3\right]}{\left(1 - \frac{\kappa\rho_{23}^2}{4}\right)^{3/2}\rho_{23}^3} - \kappa({\dot{x}_3}^2 + {\dot{y}_3}^2 + \kappa B_3)x_3, \\
\ddot{y}_3 = \dfrac{m_1\left[y_1 - \left(1 - \frac{\kappa\rho_{13}^2}{2}\right)y_3\right]}{\left(1 - \frac{\kappa\rho_{13}^2}{4}\right)^{3/2}\rho_{13}^3} + \dfrac{m_2\left[y_2 - \left(1 - \frac{\kappa\rho_{32}^2}{2}\right)y_3\right]}{\left(1 - \frac{\kappa\rho_{32}^2}{2}\right)^{3/2}\rho_{32}^3} - \kappa({\dot{x}_3}^2 + {\dot{y}_3}^2 + \kappa B_3)y_3,
\end{cases}
\tag{4.6}
$$

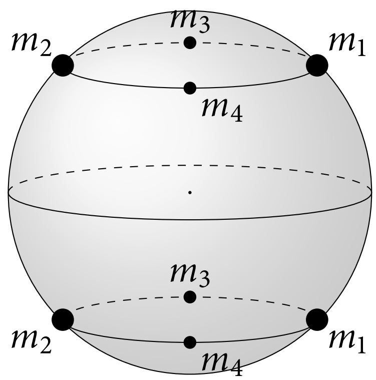

Figure 1: The case of 2 equal masses and 2 negligible masses in the northern or in the southern hemisphere

$$(4.7)\quad \begin{cases} \ddot{x}_4 = \dfrac{m_1\left[x_1-\left(1-\frac{\kappa\rho_{14}^2}{2}\right)x_4\right]}{\left(1-\frac{\kappa\rho_{14}^2}{4}\right)^{3/2}\rho_{14}^3} + \dfrac{m_4\left[x_4-\left(1-\frac{\kappa\rho_{42}^2}{2}\right)x_2\right]}{\left(1-\frac{\kappa\rho_{42}^2}{4}\right)^{3/2}\rho_{42}^3} - \kappa(\dot{x_4}^2+\dot{y_4}^2+\kappa B_4)x_4, \\ \ddot{y}_4 = \dfrac{m_1\left[y_1-\left(1-\frac{\kappa\rho_{14}^2}{2}\right)y_4\right]}{\left(1-\frac{\kappa\rho_{14}^2}{4}\right)^{3/2}\rho_{14}^3} + \dfrac{m_4\left[y_4-\left(1-\frac{\kappa\rho_{42}^2}{2}\right)y_2\right]}{\left(1-\frac{\kappa\rho_{42}^2}{4}\right)^{3/2}\rho_{42}^3} - \kappa(\dot{x_4}^2+\dot{y_4}^2+\kappa B_4)y_4, \end{cases}$$

where $\rho_{ij}^2 = \rho_{ji}^2,\ i \neq j,$

$$\rho_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + \frac{\kappa(x_i^2 + y_i^2 - x_j^2 - y_j^2)^2}{[\sqrt{1-\kappa(x_i^2+y_i^2)} + \sqrt{1-\kappa(x_j^2+y_j^2)}]^2}.$$

We can now show that when $m_1 = m_2 =: m > 0$ and $m_3 = m_4 = 0$, square-like relative equilibria, *i.e.*, equilateral equiangular quadrilaterals, always exist on $\mathbb{S}_\kappa^2$.

Theorem 4.2 *In the curved 4-body problem, assume that $m_1 = m_2 =: m > 0$ and $m_3 = m_4 = 0$. Then, in $\mathbb{S}_\kappa^2$, there are two circles of radius $0 < r < \kappa^{-1/2}$, parallel with the equator, such that a square configuration inscribed in this circle, with m_1, m_2 at the opposite ends of one diagonal and m_3, m_4 at the opposite ends of the other diagonal, forms a relative equilibrium.*

Proof Observe that the variable r defined below is related with the high from the equator to the plane containing the configuration inscribed on the circle of radius r in positive (northern hemisphere) or negative sense (southern hemisphere). So we can assume without loss of generality that the bodies are in the northern hemisphere.

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Then we must check the existence of a solution of the form

$$
\begin{aligned}
&\mathbf{q} = (q_1, q_2, q_3, q_4) \in \mathbb{S}^2_\kappa, && \mathbf{q_i} = (x_i, y_i), \quad i = 1,2,3,4,\\
&x_1 = r\cos\alpha t, && y_1 = r\sin\alpha t,\\
&x_2 = -r\cos\alpha t, && y_2 = -r\sin\alpha t,\\
&x_3 = r\cos(\alpha t + \pi/2) = -r\sin\alpha t, && y_3 = r\sin(\alpha t + \pi/2) = r\cos\alpha t,\\
&x_4 = -r\cos(\alpha t + \pi/2) = r\sin\alpha t, && y_4 = -r\sin(\alpha t + \pi/2) = -r\cos\alpha t,
\end{aligned}
$$

where

$$x_i^2 + y_i^2 = r^2, \quad \rho^2 = \rho_{13}^2 = \rho_{14}^2 = \rho_{23}^2 = \rho_{24}^2 = 2r^2, \quad \rho_{12}^2 = \rho_{34}^2 = 4r^2.$$

Substituting these expressions into system (4.1), the first four equations lead us to

$$\alpha^2 = \frac{m}{4r^3(1-\kappa r^2)^{3/2}},$$

whereas the last four equations yield

$$\alpha^2 = \frac{2m(1-\frac{\kappa\rho^2}{2})}{\rho^3(1-\frac{\kappa\rho^2}{4})^{3/2}(1-\kappa r^2)}.$$

So, to have a solution, the equation

$$\frac{m}{4r^3(1-\kappa r^2)^{3/2}} = \frac{2m(1-\frac{\kappa\rho^2}{2})}{\rho^3(1-\frac{\kappa\rho^2}{4})^{3/2}(1-\kappa r^2)}$$

must be satisfied. This equation is equivalent to

$$\frac{1}{8r^3(1-\kappa r^2)^{3/2}} = \frac{1}{2\sqrt{2}r^3(1-\frac{\kappa r^2}{2})^{3/2}},$$

which leads to $3\kappa r^2 = 2$. For $\mathbb{S}^2_\kappa$, it leads to $r = \sqrt{2/3}\kappa^{-1/2}$. Since $r < \kappa^{-1/2}$, such a solution always exists in $\mathbb{S}^2_\kappa$. ∎

4.4 The Case of One Negligible Mass

Let $m_1, m_2, m_3 = m > 0$ and assume that $m_4 = 0$. Then the equations of motion take the form

$$
\text{(4.8)} \quad \begin{cases}
\ddot{x}_1 = \dfrac{m_2\left[x_2 - \left(1-\frac{\kappa\rho_{12}^2}{2}\right)x_1\right]}{\left(1-\frac{\kappa\rho_{12}^2}{4}\right)^{3/2}\rho_{12}^3} + \dfrac{m_3\left[x_3 - \left(1-\frac{\kappa\rho_{31}^2}{2}\right)x_1\right]}{\left(1-\frac{\kappa\rho_{31}^2}{4}\right)^{3/2}\rho_{31}^3} - \kappa(\dot{x}_1{}^2 + \dot{y}_1{}^2 + \kappa B_1)x_1,\\
\ddot{y}_1 = \dfrac{m_2\left[y_2 - \left(1-\frac{\kappa\rho_{12}^2}{2}\right)y_1\right]}{\left(1-\frac{\kappa\rho_{12}^2}{4}\right)^{3/2}\rho_{12}^3} + \dfrac{m_3\left[y_3 - \left(1-\frac{\kappa\rho_{31}^2}{2}\right)y_1\right]}{\left(1-\frac{\kappa\rho_{31}^2}{4}\right)^{3/2}\rho_{31}^3} - \kappa(\dot{x}_1{}^2 + \dot{y}_1{}^2 + \kappa B_1)y_1,
\end{cases}
$$

$$
\text{(4.9)} \quad \begin{cases}
\ddot{x}_2 = \dfrac{m_1\left[x_1 - \left(1-\frac{\kappa\rho_{12}^2}{2}\right)x_2\right]}{\left(1-\frac{\kappa\rho_{12}^2}{4}\right)^{3/2}\rho_{12}^3} + \dfrac{m_3\left[x_3 - \left(1-\frac{\kappa\rho_{32}^2}{2}\right)x_2\right]}{\left(1-\frac{\kappa\rho_{32}^2}{4}\right)^{3/2}\rho_{32}^3} - \kappa(\dot{x}_2{}^2 + \dot{y}_2{}^2 + \kappa B_2)x_2,\\
\ddot{y}_2 = \dfrac{m_1\left[y_1 - \left(1-\frac{\kappa\rho_{12}^2}{2}\right)y_2\right]}{\left(1-\frac{\kappa\rho_{12}^2}{4}\right)^{3/2}\rho_{12}^3} + \dfrac{m_3\left[y_3 - \left(1-\frac{\kappa\rho_{32}^2}{2}\right)y_2\right]}{\left(1-\frac{\kappa\rho_{32}^2}{4}\right)^{3/2}\rho_{32}^3} - \kappa(\dot{x}_2{}^2 + \dot{y}_2{}^2 + \kappa B_2)y_2,
\end{cases}
$$

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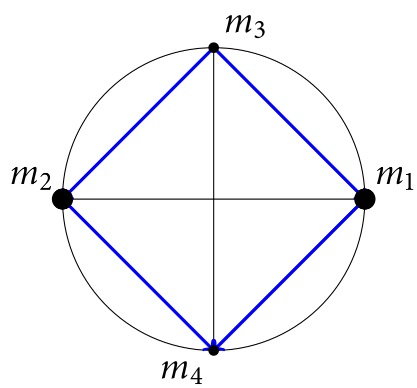

Figure 2: The case of two equal masses and two negligible masses.

$$
(4.10)\quad \begin{cases}
\ddot{x}_3 = \dfrac{m_1\left[x_1-\left(1-\frac{\kappa\rho_{13}^2}{2}\right)x_3\right]}{\left(1-\frac{\kappa\rho_{13}^2}{4}\right)^{3/2}\rho_{13}^3} + \dfrac{m_2\left[x_2-\left(1-\frac{\kappa\rho_{32}^2}{2}\right)x_3\right]}{\left(1-\frac{\kappa\rho_{32}^2}{4}\right)^{3/2}\rho_{32}^3} - \kappa(\dot{x}_3{}^2+\dot{y}_3{}^2+\kappa B_3)x_3,\\
\ddot{y}_3 = \dfrac{m_1\left[y_1-\left(1-\frac{\kappa\rho_{13}^2}{2}\right)y_3\right]}{\left(1-\frac{\kappa\rho_{13}^2}{4}\right)^{3/2}\rho_{13}^3} + \dfrac{m_2\left[y_2-\left(1-\frac{\kappa\rho_{32}^2}{2}\right)y_3\right]}{\left(1-\frac{\kappa\rho_{32}^2}{4}\right)^{3/2}\rho_{32}^3} - \kappa(\dot{x}_3{}^2+\dot{y}_3{}^2+\kappa B_3)y_3,
\end{cases}
$$

$$
(4.11)\quad \begin{cases}
\ddot{x}_4 = \dfrac{m_1\left[x_1-\left(1-\frac{\kappa\rho_{14}^2}{2}\right)x_4\right]}{\left(1-\frac{\kappa\rho_{14}^2}{4}\right)^{3/2}\rho_{14}^3} + \dfrac{m_2\left[x_2-\left(1-\frac{\kappa\rho_{42}^2}{2}\right)x_4\right]}{\left(1-\frac{\kappa\rho_{42}^2}{4}\right)^{3/2}\rho_{42}^3} + \dfrac{m_3\left[x_3-\left(1-\frac{\kappa\rho_{43}^2}{2}\right)x_4\right]}{\left(1-\frac{\kappa\rho_{43}^2}{4}\right)^{3/2}\rho_{43}^3}\\
\qquad\qquad -\kappa(\dot{x}_4{}^2+\dot{y}_4{}^2+\kappa B_4)x_4,\\
\ddot{y}_4 = \dfrac{m_1\left[y_1-\left(1-\frac{\kappa\rho_{14}^2}{2}\right)y_4\right]}{\left(1-\frac{\kappa\rho_{14}^2}{4}\right)^{3/2}\rho_{14}^3} + \dfrac{m_2\left[y_2-\left(1-\frac{\kappa\rho_{42}^2}{2}\right)y_4\right]}{\left(1-\frac{\kappa\rho_{42}^2}{4}\right)^{3/2}\rho_{42}^3} + \dfrac{m_3\left[y_3-\left(1-\frac{\kappa\rho_{43}^2}{2}\right)y_4\right]}{\left(1-\frac{\kappa\rho_{43}^2}{4}\right)^{3/2}\rho_{43}^3}\\
\qquad\qquad -\kappa(\dot{x}_4{}^2+\dot{y}_4{}^2+\kappa B_4)y_4.
\end{cases}
$$

We will next show that if the non-negligible masses are equal, then there exist some kite-shaped relative equilibria.

Theorem 4.3 *Consider the curved 4-body problem with masses $m_1 = m_2 = m_3 := m > 0$ and $m_4 = 0$. Then, in $\mathbb{S}_\kappa^2$, there exists at least one kite-shaped relative equilibrium for which the equal masses lie at the vertices of an equilateral triangle, whereas the negligible mass is at the intersection of the extension of one height of the triangle with the circle on which all the bodies move.*

Proof We will check a solution of the form

$$
\begin{aligned}
x_1 &= r\cos\alpha t, & y_1 &= r\sin\alpha t,\\
x_2 &= r\cos\left(\alpha t + \frac{2\pi}{3}\right), & y_2 &= r\sin\left(\alpha t + \frac{2\pi}{3}\right),\\
x_3 &= r\cos\left(\alpha t + \frac{4\pi}{3}\right), & y_3 &= r\sin\left(\alpha t + \frac{4\pi}{3}\right),
\end{aligned}
$$

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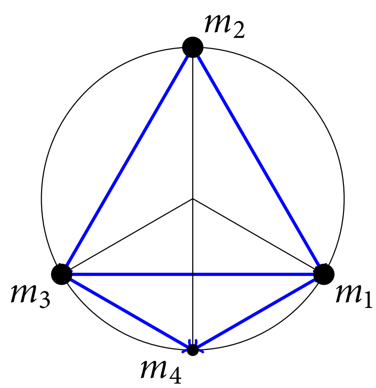

Figure 3: A kite configuration of 3 equal masses and one negligible mass.

$$x_4 = r\cos\left(\alpha t - \frac{\pi}{3}\right), \qquad y_4 = r\sin\left(\alpha t - \frac{\pi}{3}\right),$$

where

$$\rho_{12}^2 = \rho_{13}^2 = \rho_{23}^2 = 3r^2, \quad \rho_{43}^2 = \rho_{41}^2 = r^2, \quad \rho_{24}^2 = 4r^2.$$

Substituting these expressions into the above system, we are led to the conclusion that the following two equations must be satisfied,

$$\alpha^2 = \frac{m}{\sqrt{3}r^3(1-\frac{3\kappa r^2}{4})^{3/2}} \quad \text{and} \quad \alpha^2 = \frac{m}{4r^3(1-\kappa r^2)^{3/2}} + \frac{m}{r^3(1-\frac{\kappa r^2}{4})^{3/2}}.$$

Comparing these equations we obtain the condition for the existence of the kite-shaped relative equilibria,

$$\frac{1}{\sqrt{3}(1-\frac{3\kappa r^2}{4})^{3/2}} = \frac{1}{4(1-\kappa r^2)^{3/2}} + \frac{1}{(1-\frac{\kappa r^2}{4})^{3/2}}.$$

Straightforward computations show that r is a solution of this equation if it is a root of the polynomial

$$P(r) = a_{24}r^{24} + a_{22}r^{22} + a_{20}r^{20} + a_{18}r^{18} + a_{16}r^{16} + a_{14}r^{14} + a_{12}r^{12} + a_{10}r^{10} + a_8r^8 + a_6r^6 + a_4r^4 + a_2r^2 + a_0,$$

$$\begin{aligned}
a_{24} &= \tfrac{6697290145}{16777216}\kappa^{12}, & a_{22} &= -\tfrac{2884257825}{524288}\kappa^{11}, & a_{20} &= \tfrac{18063189465}{524288}\kappa^{10},\\
a_{18} &= -\tfrac{4241985935}{32768}\kappa^{9}, & a_{16} &= \tfrac{21267471735}{65536}\kappa^{8}, & &\\
a_{14} &= -\tfrac{584429805}{1024}\kappa^{7}, & a_{12} &= \tfrac{737853351}{1024}\kappa^{6}, & a_{10} &= -\tfrac{41995431}{64}\kappa^{5},\\
a_{8} &= \tfrac{109080063}{256}\kappa^{4}, & a_{6} &= -\tfrac{1530101}{8}\kappa^{3}, & &\\
a_{4} &= \tfrac{446217}{8}\kappa^{2}, & a_{2} &= -9318\kappa, & a_0 &= 649
\end{aligned}$$

belonging to the interval $r \in (0, \kappa^{-1/2})$ for $\mathbb{S}_\kappa^2$. To find out if we have such a root, we make the substitution $x = r^2$, and obtain the polynomial

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$$Q(x) = a_{24}x^{12} + a_{22}x^{11} + a_{20}x^{10} + a_{18}x^9 + a_{16}x^8 + a_{14}x^7 + a_{12}x^6 + a_{10}x^5 + a_8x^4 + a_6x^3 + a_4x^2 + a_2x + a_0.$$

By Descartes' rule of signs, the number of positive roots depends on the number of changes of sign of the coefficients, which in turn depends on the sign of κ. So let us discuss the two cases separately.

In $\mathbb{S}^2_\kappa$, *i.e.,* for $\kappa > 0$, there are twelve changes of sign, so Q can have twelve, ten, eight, six, four, two, or zero positive roots, so this does not guarantee the existence of a positive root. However, we can notice that $Q(\kappa^{-1}/2) = -2.4959 < 0$ and $Q(0) = 649 > 0$, so a root must exist for $x \in (0, \kappa^{-1/2})$, *i.e.,* for $r \in (0, \kappa^{-1})$, a remark that proves the existence of at least one kite-shaped relative equilibrium. ■

5 The Case of Negative Curvature

As we mentioned in Section 3, the relative equilibria on $\mathbb{H}^2_\kappa$ can be of three different kinds, depending of the special group of isometry that is acting on that surface. In this way, we can have elliptic, parabolic, or hyperbolic relative equilibria. In [15], the authors prove the no existence of parabolic relative equilibria. More recently, Perez-Chavela et al. proved the non existence of polygonal hyperbolic relative equilibria. Therefore, in this paper, we have restricted our analysis to the case of elliptic relative equilibria on $\mathbb{H}^2_\kappa$, also known as hyperbolic elliptic relative equilibria.

5.1 The Case of Two Negligible Masses on $\mathbb{H}^2_\kappa$

It is known (see, for instance, [5]), that for any $N \in \mathbb{N}$, $m > 0$ and $z > 1$, there are two values of ω, one positive and one negative such that, the isometry matrix $A(\omega t)$ defined by equation (3.1) generates relative equilibria where the masses are located at the vertices of a regular N-gon. For these reason and the results proved in the previous section, we believe that it should be possible to extend those results to the case of negative curvature. Unexpectedly, this is the case.

In order to facilitate the notation, from here on we will assume without loss of generality that the negative curvature is equal to -1. In this subsection we consider the case when $m_1 = m_2 = m > 0$ and $m_3 = m_4 = 0$. We must check the existence or non-existence of a solution of the form

$$\begin{aligned}
&\mathbf{q} = (q_1, q_2, q_3, q_4) \in \mathbb{H}^2_\kappa, && \mathbf{q_i} = (x_i, y_i), \quad i = 1,2,3,4.\\
&x_1 = r\cos\alpha t, && y_1 = r\sin\alpha t,\\
&x_2 = -r\cos\alpha t, && y_2 = -r\sin\alpha t,\\
&x_3 = r\cos(\alpha t + \pi/2) = -r\sin\alpha t, && y_3 = r\sin(\alpha t + \pi/2) = r\cos\alpha t,\\
&x_4 = -r\cos(\alpha t + \pi/2) = r\sin\alpha t, && y_4 = -r\sin(\alpha t + \pi/2) = -r\cos\alpha t,
\end{aligned}$$

where

$$x_i^2 + y_i^2 = r^2, \quad \rho^2 = \rho_{13}^2 = \rho_{14}^2 = \rho_{23}^2 = \rho_{24}^2 = 2r^2, \quad \rho_{12}^2 = \rho_{34}^2 = 4r^2.$$

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Substituting these expressions into system (4.4)–(4.7) for $\kappa = -1 < 0$, the first four equations lead us to

$$\alpha^2 = \frac{m}{4r^3(1+r^2)^{3/2}},$$

whereas the last four equations yield

$$\alpha^2 = \frac{m}{\sqrt{2}r^3(1+\frac{r^2}{2})^{3/2}}.$$

So, to have a solution, the equation

$$\frac{m}{4r^3(1+r^2)^{3/2}} = \frac{m}{\sqrt{2}r^3(1+\frac{r^2}{2})^{3/2}},$$

must be satisfied. This equation is equivalent to

$$4(1+r^2)^{3/2} = \sqrt{2}\Big(1+\frac{r^2}{2}\Big)^{3/2},$$

which leads to $3r^2 = -2$, which is a contradiction. Hence, these orbits do not exist on $\mathbb{H}^2_{-1}$.

5.2 The Case of One Negligible Mass on $\mathbb{H}^2_{-1}$

Let $m_1, m_2, m_3 = m > 0$ and assume that $m_4 = 0$. Without loss of generality, we can restrict our study to the unit hyperbolic sphere for negative curvature. Then we will check a solution of the form

$$\begin{aligned}
x_1 &= r\cos\alpha t, & y_1 &= r\sin\alpha t,\\
x_2 &= r\cos\Big(\alpha t + \frac{2\pi}{3}\Big), & y_2 &= r\sin\Big(\alpha t + \frac{2\pi}{3}\Big)\\
x_3 &= r\cos\Big(\alpha t + \frac{4\pi}{3}\Big), & y_3 &= r\sin\Big(\alpha t + \frac{4\pi}{3}\Big),\\
x_4 &= r\cos\Big(\alpha t - \frac{\pi}{3}\Big), & y_4 &= r\sin\Big(\alpha t - \frac{\pi}{3}\Big),
\end{aligned}$$

where

$$\rho^2_{12} = \rho^2_{13} = \rho^2_{23} = 3r^2, \quad \rho^2_{43} = \rho^2_{41} = r^2, \quad \rho^2_{24} = 4r^2.$$

Substituting these expressions into system (4.8)–(4.11) for $\kappa = -1 < 0$, we are led to the conclusion that the following two equations must be satisfied,

$$\alpha^2 = \frac{m}{\sqrt{3}r^3(1+\frac{3r^2}{4})^{3/2}} \quad \text{and} \quad \alpha^2 = \frac{m}{4r^3(1+r^2)^{3/2}} + \frac{m}{r^3(1+\frac{r^2}{4})^{3/2}}.$$

Comparing these equations we obtain the condition for the existence of the kite-shaped relative equilibria,

$$\frac{1}{\sqrt{3}(1+\frac{3r^2}{4})^{3/2}} = \frac{1}{4(1+r^2)^{3/2}} + \frac{1}{(1+\frac{r^2}{4})^{3/2}}.$$

Straightforward computations show that r is a solution of this equation if it is a root of the polynomial

$$P(r) = a_{24}r^{24} + a_{22}r^{22} + a_{20}r^{20} + a_{18}r^{18} + a_{16}r^{16} + a_{14}r^{14} + a_{12}r^{12}$$

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$$+ a_{10}r^{10} + a_8r^8 + a_6r^6 + a_4r^4 + a_2r^2 + a_0,$$

$$a_{24} = \frac{6697290145}{16777216}, \quad a_{22} = \frac{2884257825}{524288}, \quad a_{20} = \frac{18063189465}{524288},$$
$$a_{18} = \frac{4241985935}{32768}, \quad a_{16} = \frac{21267471735}{65536},$$
$$a_{14} = \frac{584429805}{1024}, \quad a_{12} = \frac{737853351}{1024}, \quad a_{10} = \frac{41995431}{64},$$
$$a_8 = \frac{109080063}{256}, \quad a_6 = \frac{1530101}{8},$$
$$a_4 = \frac{446217}{8}, \quad a_2 = 9318, \quad a_0 = 649.$$

Since all coefficients of the polynomial $P(r)$ are positives, by Descartes' rule of signs, this polynomial does not have positive roots. Therefore, there are no kite solutions in $\mathbb{H}^2_{-1}$.

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