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ON INFINITE GROUPS OF FIBONACCI TYPE

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1. Introduction

Let $F_n=\langle a_1,a_2,\ldots,a_n\rangle$ denote the free group of rank n, and let θ denote the automorphism of F_n which permutes the generators cyclically, in other words:

$$a_1^\theta=a_2,\ a_2^\theta=a_3,\ldots,\ a_n^\theta=a_1.$$

If w is a word in F_n, let $N_n(w)$ denote the normal closure of

$$\langle w, w^\theta, w^{\theta^2},\ldots,w^{\theta^{n-1}}\rangle$$

in F_n, and let $G_n(w)$ denote the factor group $F_n/N_n(w)$.

If w is the word $a_1a_2\ldots a_ra_{r+1}^{-1}$ (where the subscripts are always reduced modulo n), then $G_n(w)$ is the *Fibonacci group* $F(r,n)$. In [**13**, Theorem C], it is shown that:

If $d=(r+1,n)$, then $F(r,n)$ is infinite whenever either:

(i) $d>3$, or:

(ii) $d=3$ and n is even.

Since $F(3k-1,3)$ is a homomorphic image of $F(3k-1,3u)$ for $u\geqq 1$ (see [**11**, Theorem 6]), and since $F(3k-1,3)$ is infinite for $k>1$ ([**12**, Theorem 6]), we also have:

If $d=(r+1,n)$, then $F(r,n)$ is infinite whenever $d=3$ and $r>2$.

The purpose of this paper is to extend these results to some generalizations of the Fibonacci groups, namely the groups

$$H(r,n,s)=G_n(a_1a_2\ldots a_r(a_{r+1}a_{r+2}\ldots a_{r+s})^{-1}),$$

where $r>s\geqq 1$ (see [**2, 6**]), and the groups

$$F(r,n,k)=G_n(a_1a_2\ldots a_ra_{r+k}^{-1}),$$

where $r\geqq 2$, $k\geqq 0$ (see [**3, 4, 5, 9**]). Note that the groups $H(r,n,1)$ and $F(r,n,1)$ are each isomorphic to $F(r,n)$.

It is clear that, for any word w in F_n, θ induces an automorphism of $G_n(w)$, and hence we may take a semi-direct product of $G_n(w)$ with a cyclic group of order n with action induced by θ. The results in [13] were proved by considering the group:

$$E(r,n)=\langle x,t \mid xt^r=tx^r, t^n=1\rangle,$$

which is the semi-direct product of $F(r,n)$ with C_n, the cyclic group of order n. Here we consider the group:

$$I(r,n,s)=\langle x,t \mid x^s t^r=t^s x^r, t^n=1\rangle,$$

which is a semi-direct product of $H(r,n,s)$ with C_n, and the group:

$$E(r,n,k)=\langle x,t \mid xt^{r+k-1}=t^k x^r, t^n=1\rangle,$$

which is a semi-direct product of $F(r,n,k)$ with C_n, and use these to determine sufficient conditions for $H(r,n,s)$ and $F(r,n,k)$ to be infinite. As a necessary part of this investigation, we also study the groups:

$$G(a,b,c,k,l)=\langle x,y \mid x^a=y^b=(x^k y^l)^c=1\rangle,$$

and prove the following result (see Theorem 2.5):

Theorem. *The group $G(a,b,c,k,l)$ is finite if and only if one of the following three conditions holds:*

(i) $(a,k)=(b,l)=1$ and $1/a+1/b+1/c>1$.

(ii) $(a,kc)=1$ *and b divides l.*

(iii) $(b,lc)=1$ *and a divides k.*

The notation used in this paper is reasonably standard; we use C_n to denote the cyclic group of order n, $A*B$ to denote the free product of the groups A and B, and $(u_1,u_2,\ldots,u_n)$ to denote the highest common factor of the integers $u_1,u_2,\ldots,u_n$.

2. The groups $G(a,b,c,k,l)$

In this section, we consider the groups:

$$G(a,b,c,k,l)=\langle x,y \mid x^a=y^b=(x^k y^l)^c=1\rangle,$$

where $a,b,c>1$. The group $G(a,b,c,1,1)$ is the polyhedral group:

$$\langle x,y \mid x^a=y^b=(xy)^c=1\rangle,$$

which is finite if and only if $1/a+1/b+1/c>1$ (see [7]).

Proposition 2.1. *$G(a,b,c,k,l)$ is isomorphic to $G(b,a,c,l,k)$.*

Proof. This follows by a simple sequence of Tietze transformations.

Proposition 2.2.

(i) *$G(a,b,c,k,l)$ is isomorphic to $G(a,b,c,k,1)$ if $(b,l)=1$.*

(ii) *$G(a,b,c,k,l)$ is isomorphic to $G(a,b,c,1,l)$ if $(a,k)=1$.*

Proof. We shall prove (i), (ii) then following from Proposition 2.1.

Assume that $(b,l)=1$, and let $y_1=y^l$. Then $G=G(a,b,c,k,l)$ has presentation:

$$\langle x,y,y_1 \mid x^a=y^b=(x^k y_1)^c=1, y_1=y^l\rangle.$$

Since $y_1=y^l$ and $y^b=1$, we have:

$$\langle x,y,y_1 \mid x^a=y^b=y_1^b=(x^k y_1)^c=1, y_1=y^l\rangle.$$

Now let m be such that $lm\equiv 1$ (mod. b). Then we have:

$$\langle x,y,y_1 \mid x^a=y^b=y_1^b=(x^k y_1)^c=1, y=y_1^m\rangle.$$

The relation $y^b=1$ is now redundant, so that we have:

$$\langle x,y,y_1 \mid x^a=y_1^b=(x^k y_1)^c=1, y=y_1^m\rangle.$$

We delete the superfluous generator y to get:

$$\langle x,y_1 \mid x^a=y_1^b=(x^k y_1)^c=1\rangle,$$

which is $G(a,b,c,k,1)$ as required.

Proposition 2.3.

(i) *If b divides l, then $G(a,b,c,k,l)$ is isomorphic to $C_d * C_b$, where $d=(a,kc)$.*

(ii) *If a divides k, then $G(a,b,c,k,l)$ is isomorphic to $C_a * C_e$, where $e=(b,lc)$.*

Proof. Again, (ii) follows from (i) and Proposition 2.1.

If b divides l, then we have:

$$\langle x,y \mid x^a=y^b=(x^k)^c=1\rangle,$$

in other words, the group:

$$\langle x,y \mid x^d=y^b=1\rangle,$$

where $d=(a,kc)$.

Proposition 2.4.

(i) *If* $b=ef$, *where* $1<e=(b,l)<b$, *then* $G(a,b,c,k,l)$ *has a normal subgroup of index* e *isomorphic to the free product of the groups*

$$\langle x_i, z \mid x_i^a = z^f = (x_i^k z)^c = 1 \rangle$$

$(i=0,1,\dots,e-1)$ *with* $\langle z \rangle$ *amalgamated.*

(ii) *If* $a=ef$, *where* $1<e=(a,k)<a$, *then* $G(a,b,c,k,l)$ *has a normal subgroup of index* e *isomorphic to the free product of the groups*

$$\langle z, y_i \mid z^f = y_i^b = (z y_i^l)^c = 1 \rangle$$

$(i=0,1,\dots,e-1)$ *with* $\langle z \rangle$ *amalgamated.*

Proof. Yet again, we need only prove (i), and (ii) will follow from Proposition 2.1.

Let $x_1 = yxy^{-1}$, $x_2 = y^2xy^{-2}, \dots, x_{e-1} = y^{e-1}xy^{1-e}$, $z = y^e$ in the group:

$$\langle x, y \mid x^a = y^{ef} = (x^k y^{eg})^c = 1 \rangle,$$

where $eg = l$, and let:

$$N = \langle x, x_1, x_2, \dots, x_{e-1}, z \rangle.$$

It is clear that N has index e in $G(a,b,c,k,l)$ with coset representatives $1, y, y^2, \dots, y^{e-1}$. From the relation $x^a = 1$, we get as relations for the subgroup N:

$$x^a = x_1^a = x_2^a = \cdots = x_{e-1}^a = 1.$$

From the relation $y^{ef} = 1$, we get the single relation $z^f = 1$ for N, and, from the relation $(x^k y^{eg})^c = 1$, we get the relations:

$$(x^k z^g)^c = (x_1^k z^g)^c = \cdots = (x_{e-1}^k z^g)^c = 1.$$

So N has presentation:

$$\begin{aligned}\langle x, x_1, x_2, \dots, x_{e-1}, z \mid x^a = x_1^a = x_2^a = \cdots &= x_{e-1}^a \\ &= z^f = (x^k z^g)^c = (x_1^k z^g)^c = (x_2^k z^g)^c \\ &= \cdots = (x_{e-1}^k z^g)^c = 1 \rangle.\end{aligned}$$

We see that N is a free product of the groups:

$$\langle x_i, z \mid x_i^a = z^f = (x_i^k z^g)^c = 1 \rangle$$

$(i=0,1,\ldots,e-1)$ with $\langle z\rangle$ amalgamated. Since $(f,g)=1$, we may replace z by z^g as in Proposition 2.2 to get that N is the free product of the groups:

$$\langle x_i, z \mid x_i^a = z^f = (x_i^k z)^c = 1\rangle$$

$(i=0,1,\ldots,e-1)$ with $\langle z\rangle$ amalgamated as required.

As a consequence of Propositions 2.2, 2.3 and 2.4, we have:

Theorem 2.5. *The group $G(a,b,c,k,l)$ is finite if and only if one of the following three conditions holds:*

(i) $(a,k)=(b,l)=1$ *and* $1/a+1/b+1/c>1$.

(ii) $(a,kc)=1$ *and* b *divides* l.

(iii) $(b,lc)=1$ *and* a *divides* k.

3. The groups $H(r,n,s)$

In this section, we consider the groups $H(r,n,s)=G_n(w)$, where $w=a_1a_2\ldots a_r(a_{r+1}a_{r+2}\ldots a_{r+s})^{-1}$ and $r>s\geqq 1$. In Lemmas 3 and 4 of **[6]**, it is shown that $H(r,n,s)$ is infinite if any of the following three conditions holds:

(3.1)

(i) $(r,n,s)>1$.

(ii) $r+s\equiv 0\,(\text{mod}.\, n), n\geqq 5$.

(iii) $r+s\equiv 0\,(\text{mod}.\, 8), n=4$.

We generalize these results by proving:

Theorem 3.2. *$H(r,n,s)$ is infinite if any of the following three conditions holds:*

(i) $(r+s,n)>3$.

(ii) $(r+s,n)=3$ *with* $r+s>3$.

(iii) $(r+s,n)>1$ *with* $(r,s)>1$.

We let d denote $(r+s,n)$. Note that, if $r+s=3$, then we have $r=2$, $s=1$, and so we have the group $H(2,n,1)$, which is the Fibonacci group $F(2,n)$. In the case where 3 divides n, so that $d=3$, $F(2,3)$ is isomorphic to the quaternion group Q_8, $F(2,6)$ is infinite, $F(2,9)$ is unknown, and $F(2,3u)$ is infinite for $u\geqq 4$ (see **[8, 10, 11]** for example).

Putting $s=1$, we get:

Corollary 3.3. *$F(r,n)$ is infinite if $(r+1,n)>3$, or if $(r+1,n)=3$ with $r>2$.*

Note that this is a slightly stronger result than in **[13]**.

If we consider θ acting on $a_1, a_2, \ldots, a_n$, we may rewrite the relation:

$$a_1 a_2 \ldots a_r = a_{r+1} a_{r+2} \ldots a_{r+s}$$

as:

$$(a_1\theta^{-1})^r\theta^r = \theta^{-r}(a_1\theta^{-1})^s\theta^{r+s}.$$

If we write $a_1\theta^{-1}$ as x^{-1} and θ as t, this becomes:

$$x^s t^r = t^s x^r.$$

So, if we take the semi-direct product of $H(r,n,s)$ by a cyclic group of order n with action induced by θ, we get the group:

$$I(r,n,s) = \langle x, t \mid x^s t^r = t^s x^r,\ t^n = 1 \rangle.$$

If $(d,r) > 1$, then $H(r,n,s)$ is infinite by (3.1)(i); so, in proving Theorem 3.2, we may assume that $(d,r) = 1$. We add the relation $t^d = 1$ to the presentation for $I(r,n,s)$ to get:

$$\langle x, t \mid x^s t^r = t^s x^r, t^d = 1 \rangle,$$

in other words:

$$\langle x, t \mid x^s t^r = t^{-r} x^r, t^d = 1 \rangle,$$

and hence:

$$\langle x, t \mid (x^r t^{-r})^2 = x^{r+s}, t^d = 1 \rangle.$$

Replacing t by t^{-1}, we have:

$$\langle x, t \mid (x^r t^r)^2 = x^{r+s}, t^d = 1 \rangle,$$

which has as a homomorphic image:

$$\langle x, t \mid x^{r+s} = t^d = (x^r t^r)^2 = 1 \rangle,$$

which is a presentation for the group $G(r+s, d, 2, r, r)$. So, if the group $G(r+s, d, 2, r, r)$ is infinite, then $H(r,n,s)$ is infinite.

Assume that $d > 1$. Then, since $r+s$ and d do not divide r, Theorem 2.5 gives that $G = G(r+s, d, 2, r, r)$ is finite if and only if:

$$(r+s, r) = (d, r) = 1 \text{ and } \frac{1}{r+s} + \frac{1}{d} + \frac{1}{2} > 1.$$

So $H(r,n,s)$ is infinite if $(r,s)>1$. On the other hand, if $r+s=dv$, then G is infinite unless:

$$\frac{1}{dv}+\frac{1}{d}>\frac{1}{2},$$

in other words:

$$d<2+\frac{2}{v}.$$

If $v>1$, then we must have $d<3$ for $H(r,n,s)$ to be finite. If $v=1$, we have that $r+s=d$. In this case, for $H(r,n,s)$ to be finite, we must have that $d<4$, and hence that $r+s=3$.
This completes the proof of Theorem 3.2.

4. The groups $F(r,n,k)$

In this section, we consider the groups $F(r,n,k)=G_n(w)$, where $w=a_1a_2\ldots a_ra_{r+k}^{-1}$, and $r\geqq 2$, $k\geqq 0$. In [9], it is shown that $F(r,n,k)$ is infinite if either of the following two conditions holds:

(4.1)

(i) $(r-1,n,k)>1$,

(ii) $v_2(r+1)>v_2(k-1)<v_2(n)$,

where $v_2(m)=\alpha$ if $m=2^\alpha q$ with q odd, $v_2(0)=\infty$.
We shall prove:

Theorem 4.2. *$F(r,n,k)$ is infinite if $(r+1,n,k-1)>3$, or if $(r+1,n,k-1)=3$ with $r>2$.*

Theorem 4.2 follows immediately from Corollary 3.3 and the following result:

Theorem 4.3. *If $d=(n,k-1)$, and if $F(r,d)$ is infinite, then $F(r,n,k)$ is infinite.*

If we consider θ acting on $a_1,a_2,\ldots,a_n$, we may rewrite the relation:

$$a_1a_2\ldots a_r=a_{r+k}$$

as:

$$(a_1\theta^{-1})^r\theta^r=\theta^{1-r-k}(a_1\theta^{-1})\theta^{r+k}.$$

If we write $a_1\theta^{-1}$ as x^{-1} and θ as t, this becomes:

$$xt^{r+k-1}=t^kx^r.$$

So, if we take the semi-direct product of $F(r,n,k)$ by a cyclic group of order n with action induced by θ, we get the group:

$$E(r,n,k)=\langle x,t \mid xt^{r+k-1}=t^kx^r, t^n=1\rangle.$$

Let $d=(n,k-1)$, and add the relation $t^d=1$ to get the group:

$$\langle x,t\,|\,xt^r=tx^r, t^d=1\rangle,$$

i.e. the group $E(r,d)$. So $E(r,n,k)$, and hence $F(r,n,k)$, is infinite if $E(r,d)$ is infinite, i.e. if $F(r,d)$ is infinite.

Note. If $r=2$, then we know that $F(2,d)$ is infinite unless $d<6, d=7$, or (possibly) $d=9$ (see [**1, 8, 10, 11**]). Theorem 4.3 now gives that $F(2,n,k)$ is infinite if $(k-1,n)=6$ or 8, or if $(k-1,n)\geqq 10$.

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