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Parabolic Muckenhoupt Weights Characterized by Parabolic Fractional Maximal and Integral Operators with Time Lag

Weiyi Kong, Dachun Yang*, Wen Yuan and Chenfeng Zhu

Abstract In this article, motivated by the regularity theory of the solutions of doubly nonlinear parabolic partial differential equations the authors introduce the off-diagonal two-weight version of the parabolic Muckenhoupt class with time lag. Then the authors introduce the uncentered parabolic fractional maximal operator with time lag and characterize its two-weighted boundedness (including the endpoint case) in terms of these weights under an additional mild assumption (which is not necessary for one-weight case). The most novelty of this article exists in that the authors further introduce a new parabolic shaped domain and its corresponding parabolic fractional integral with time lag and, moreover, applying the aforementioned (two-)weighted boundedness of the parabolic fractional maximal operator with time lag, the authors characterize the (two-)weighted boundedness (including the endpoint case) of these parabolic fractional integrals in terms of the off-diagonal (two-weight) parabolic Muckenhoupt class with time lag; as applications, the authors further establish a parabolic weighted Sobolev embedding and a priori estimate for the solution of the heat equation. The key tools to achieve these include the parabolic Calderón–Zygmund-type decomposition, the chaining argument, and the parabolic Welland inequality which is obtained by making the utmost of the geometrical relation between the parabolic shaped domain and the parabolic rectangle.

Contents

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*Corresponding author, E-mail: dcyang@bnu.edu.cn/March 6, 2025/Final version.

1 Introduction

The Muckenhoupt class, introduced by Muckenhoupt [75], is of fundamental importance in harmonic analysis and partial differential equations. Let $q \in (1, \infty)$ and ω be a *weight* on $\mathbb{R}^n$, that is, a nonnegative locally integrable function. It is well known that some classical operators (for instance, the Hardy–Littlewood maximal operator and the Calderón–Zygmund operators) are bounded on the *weighted Lebesgue space*

$$L^q(\mathbb{R}^n, \omega) := \left\{ f : \|f\|_{L^q(\mathbb{R}^n,\omega)} := \left[\int_{\mathbb{R}^n} |f(x)|^q \omega(x)\, dx \right]^{\frac{1}{q}} < \infty \right\}$$

if and only if ω belongs to the *Muckenhoupt class* $A_q(\mathbb{R}^n)$, that is,

$$(1.1) \qquad [\omega]_{A_q(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \omega(y)\, dy \left\{ \frac{1}{|Q|} \int_Q [\omega(y)]^{\frac{1}{1-q}}\, dy \right\}^{q-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ whose edges are all parallel to the coordinate axis. Furthermore, the Muckenhoupt weights have deep connections with the elliptic partial differential equation

$$(1.2) \qquad \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = 0,$$

where $p \in (1, \infty)$. So far, the Muckenhoupt classes and the theory of weighted function spaces have been developed in a comprehensive manner; see, for example, [19, 20, 21, 22, 38, 39, 40, 41]. Moreover, there exists a well-established theory related to the Muckenhoupt weights with applications in partial differential equations; see, for example, [28, 36, 37, 54, 57, 71].

From the perspective of partial differential equations, in addition to the Muckenhoupt classes related to (1.2), there also exist parabolic Muckenhoupt classes with time lag, introduced by Kinnunen and Saari [50], tailored to the doubly nonlinear parabolic partial differential equation

$$(1.3) \qquad \frac{\partial}{\partial t} \left(|u|^{p-2} u \right) - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = 0.$$

Here and thereafter, we *always fix* $p \in (1, \infty)$. The definition of parabolic Muckenhoupt weights with time lag is based on the following definition of parabolic rectangles. For any $x \in \mathbb{R}^n$ and $L \in (0, \infty)$, let $Q(x, L)$ be the *cube* in $\mathbb{R}^n$ centered at x with edge length $2L$.

Definition 1.1. Let $(x, t) \in \mathbb{R}^{n+1}$ and $L \in (0, \infty)$. A *parabolic rectangle* R centered at (x, t) with edge length L is defined by setting

$$R := R(x, t, L) := Q(x, L) \times \left(t - L^p, t + L^p \right).$$

Let $\gamma \in [0, 1)$. The γ*-upper part* $R^+(\gamma)$ and the γ*-lower part* $R^-(\gamma)$ of R are defined, respectively, by setting $R^+(\gamma) := Q(x, L) \times (t + \gamma L^p, t + L^p)$ and

$$R^-(\gamma) := Q(x, L) \times (t - L^p, t - \gamma L^p),$$

where γ is called the *time lag*.

Denote by $\mathcal{R}_p^{n+1}$ the set of all parabolic rectangles in $\mathbb{R}^{n+1}$. For any locally integrable function f on $\mathbb{R}^{n+1}$ and for any measurable set $A \subset \mathbb{R}^{n+1}$ with $|A| \in (0, \infty)$, let

$$⨍_A f := \frac{1}{|A|} \int_A f.$$

Here and thereafter, we *always omit* the differential $dx\, dt$ in all integral representations to simplify the presentation if there exists no confusion. The following is the definition of parabolic Muckenhoupt classes with time lag; see also [50, Definition 3.2].

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Definition 1.2. Let $\gamma \in [0,1)$ and $q \in (1,\infty)$. The *parabolic Muckenhoupt class* $A_q^+(\gamma)$ is defined to be the set of all nonnegative locally integrable functions ω on $\mathbb{R}^{n+1}$ such that

$$[\omega]_{A_q^+(\gamma)} := \sup_{R\in\mathcal{R}_p^{n+1}} \fint_{R^-(\gamma)} \omega \left[\fint_{R^+(\gamma)} \omega^{\frac{1}{1-q}}\right]^{q-1} < \infty. \tag{1.4}$$

If the above condition is satisfied with the direction of the time axis reversed, then $\omega \in A_q^-(\gamma)$ which is also called the *parabolic Muckenhoupt class* and consists of all such ω.

Different from the classical case, in (1.4), Euclidean cubes in (1.1) are substituted by parabolic rectangles, which respects the natural geometry of (1.3). Indeed, if $u(x,t)$ is a solution of (1.3), then so does $u(\lambda x, \lambda^p t)$ for any $\lambda \in (0,\infty)$. It turns out in Moser [72, 73] and Trudinger [89] that any nonnegative weak solution u of (1.3) satisfies a scale and location invariant Harnack inequality, that is, for any given $\gamma \in (0,1)$, there is a positive constant C such that, for any $R \in \mathcal{R}_p^{n+1}$,

$$\operatorname*{ess\,sup}_{(x,t)\in R^-(\gamma)} u(x,t) \le C \operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} u(x,t),$$

where the time lag γ appears naturally. The Harnack inequality further implies that any nonnegative weak solution of (1.3) is a parabolic Muckenhoupt weight with time lag. Kinnunen and Saari [50] also introduced the centered parabolic Hardy–Littlewood maximal operators with time lag and showed that these operators are bounded on the weighted Lebesgue space if and only if the weight belongs to the corresponding parabolic Muckenhoupt class with time lag. Their results in [50] were streamlined and complemented by Kinnunen and Myyryläinen [45] in which they replaced the centered maximal operator by the uncentered version to include the endpoint case. On the other hand, as proved in [50, Lemma 7.4], the parabolic Muckenhoupt classes with time lag give a Coifman–Rochberg type characterization of the function space with parabolic bounded mean oscillation which was explicitly defined by Fabes and Garofalo [27] and is essential in the regularity theory for (1.3). We refer to [11, 12, 13, 14, 15, 32, 44, 53, 74, 91] for more studies about (1.3), to [2, 47, 52, 77, 84, 85] for more studies of parabolic function spaces, and to [45, 46, 48, 49, 50, 61] for recent studies of the parabolic Muckenhoupt classes with time lag.

The other motivation to study the parabolic Muckenhoupt classes with time lag is due to the theory of the one-sided Muckenhoupt classes introduced by Sawyer [86] in connection with ergodic theory. Recall that, for any $q \in (1,\infty)$, the *one-sided Muckenhoupt class* $A_q^+(\mathbb{R})$ is defined to be the set of all nonnegative locally integrable functions ω on $\mathbb{R}$ such that

$$[\omega]_{A_q^+(\mathbb{R})} := \sup_{x\in\mathbb{R},\, h\in(0,\infty)} \frac{1}{h}\int_{x-h}^{x} \omega(y)\,dy \left\{\frac{1}{h}\int_{x}^{x+h} [\omega(y)]^{\frac{1}{1-q}}\,dy\right\}^{q-1}$$

is finite. Actually, the parabolic Muckenhoupt classes with time lag are higher dimensional generalizations of the one-sided Muckenhoupt classes in some sense. The one-sided weighted theory has been extensively investigated; see, for example, [3, 17, 25, 63, 64, 65, 67, 68, 69, 81]. There also exists several inspirational studies about higher-dimensional extensions of the one-sided weights and related topics; see, for example, [8, 9, 29, 30, 31, 56, 59, 78].

On the other hand, both the fractional maximal operators and the fractional integral operators occupy an important position in potential theory, harmonic analysis, and partial differential equations; see, for example, [1, 10, 16, 35, 90]. Recall that, for any given $\beta \in (0,n)$, the *fractional maximal operator* M_β and the *fractional integral operator* I_β are defined, respectively, by setting, for any locally integrable function f on $\mathbb{R}^n$ and any $x \in \mathbb{R}^n$,

$$M_\beta(f)(x) := \sup_{L\in(0,\infty)} \frac{1}{|Q(x,L)|^{1-\frac{\beta}{n}}} \int_{Q(x,L)} |f(y)|\,dy$$

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and

$$I_\beta(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}\,dy.$$

Let $\beta \in (0,n)$, $1 < r \le q < \infty$ with $\frac{1}{r} - \frac{1}{q} = \frac{\beta}{n}$, and ω be a weight on $\mathbb{R}^n$. It is well known that the fractional integral operator I_β (or the fractional maximal operator M_β) is bounded from $L^r(\mathbb{R}^n, \omega^r)$ to $L^q(\mathbb{R}^n, \omega^q)$ if and only if ω belongs to the *off-diagonal Muckenhoupt class* $A_{r,q}(\mathbb{R}^n)$, that is,

$$(1.5)\qquad [\omega]_{A_{r,q}(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q [\omega(x)]^q\,dx \left\{\frac{1}{|Q|}\int_Q [\omega(x)]^{-r'}\,dx\right\}^{\frac{q}{r'}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ whose edges are all parallel to the coordinate axis. We refer to [23, 60, 76, 92] for more studies on the one-weight case and to [23, 26, 70, 51, 55, 80, 87, 88] for more investigations on the two-weight case of the weighted boundedness of the fractional integral operators and the fractional maximal operators. As for the one-sided situation, Andersen and Sawyer [6] obtained the characterizations of the weighted boundedness of one-sided fractional maximal operators and the Weyl (and the Riemann–Liouville) fractional integral operators in terms of the one-sided off-diagonal Muckenhoupt classes. For more studies of the one-sided fractional maximal operators and the one-sided fractional integral operators, see, for example, [58, 62, 66, 79, 82]. In the parabolic setting, Ma et al. [61] introduced the centered parabolic fractional maximal operator with time lag and showed that it is bounded on the weighted Lebesgue spaces if and only if the weight belongs to the corresponding off-diagonal parabolic Muckenhoupt class with time lag. Inspired by these, it is natural to ask what is the most appropriate definition of parabolic fractional integral operators with time lag and whether or not the weighted boundedness of such operators can characterize the parabolic off-diagonal Muckenhoupt class with time lag. We give positive answers to these two questions in this article (see Definition 5.1 and Theorems 5.8 and 5.10).

The main goal of this article are twofold. One is to generalize the parabolic Muckenhoupt class with time lag in [45, 50] and the off-diagonal parabolic Muckenhoupt class with time lag in [61] to the two-weight case. The other is to characterize such two-weight parabolic Muckenhoupt class with time lag in terms of the weighted boundedness of some fractional operators, namely the centered and the uncentered parabolic fractional maximal operators with time lag and the parabolic fractional integral operators with time lag. More precisely, inspired by the regularity theory of (1.3), we introduce the off-diagonal two-weight version of the parabolic Muckenhoupt class with time lag. Then we introduce the uncentered parabolic fractional maximal operator with time lag and characterize its two-weighted boundedness (including the endpoint case) in terms of these weights under an additional mild assumption (which is not necessary for one-weight case). The most novelty of this article exists in that we further introduce a new parabolic shaped domain and its corresponding parabolic fractional integral with time lag and, moreover, applying the aforementioned two-weighted boundedness of the uncentered parabolic fractional maximal operator with time lag, we characterize the (two-)weighted boundedness (including the endpoint case) of these parabolic fractional integrals in terms of the off-diagonal (two-weight) parabolic Muckenhoupt class with time lag; as applications, we further establish a parabolic weighted Sobolev embedding and a priori estimate for the solution of the heat equation. The key tools to achieve these include the parabolic Calderón–Zygmund-type decomposition, the chaining argument, and the parabolic Welland inequality which is obtained by making the utmost of the geometrical relation between the parabolic shaped domain and the parabolic rectangle.

The organization of the remainder of this article is as follows.

In Section 2, we introduce the concept of parabolic Muckenhoupt two-weight classes with time lag. Several elementary properties of parabolic Muckenhoupt two weights, such as the nested property, the duality property, and the forward in time doubling property, are presented. Moreover, we give a characterization of the parabolic Muckenhoupt two-weight class with time lag in the endpoint case in terms of the uncentered parabolic maximal operator with time lag; see Proposition 2.5.

In Section 3, under an additional mild assumption (which is not necessary for one-weight case), applying the chaining argument we show that the parabolic Muckenhoupt two-weight class is independent of the choice of the time lag; see Theorem 3.1. As an application, we obtain the self-improving property of the parabolic Muckenhoupt two-weight; see Corollary 3.5.

Section 4 is devoted to characterizing the parabolic Muckenhoupt two-weight class with time lag in terms of the uncentered parabolic fractional maximal operator with time lag; see Theorem 4.1. To achieve this, we utilize a covering argument in [50] and Theorem 3.1 to change the time lag. As a corollary, we prove the strong-type parabolic weighted norm inequality for the uncentered parabolic fractional maximal operator with time lag; see Corollary 4.4. All these results are both the fractional variants of the counterparts in [45, 50] and the generalization of the counterpart of [61] from the centered one to the uncentered one. We also obtain the weak-type parabolic two-weighted norm inequality for the centered parabolic fractional maximal operator with time lag; see Theorem 4.7. Notice that Theorems 4.1 and 4.7 are respectively the generalizations of the counterparts of [45, 61] from the one weight to two weights.

In Section 5, based on a new parabolic shaped domain, we introduce the parabolic forward in time and back in time fractional integral operators with time lag; see Definition 5.1. Then we establish the pointwise relation between the centered parabolic fractional maximal operator with time lag and the parabolic fractional integral operator with time lag by showing a parabolic Welland type inequality; see Lemmas 5.3 and 5.4. Using this, we prove the weak-type parabolic two-weighted inequality and the strong-type parabolic weighted inequality for the parabolic fractional integrals with time lag; see Theorems 5.5 and 5.7 and Corollary 5.6.

The aims of Section 6 are two aspects. One is to establish the parabolic weighted boundedness of the parabolic Riesz potentials introduced in [42] for a special class of parabolic Muckenhoupt weights with time lag; see Theorem 6.4. Applying this, we establish a parabolic weighted Sobolev embedding theorem and obtain a priori estimate for the solution of the heat equation; see, respectively, Corollaries 6.6 and 6.7.

At the end of this introduction, we make some conventions on notation. Throughout this article, let $\mathbb{N} := \{1,2,\dots\}$, $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$, and $\mathbb{R}^{n+1}_+ := \mathbb{R}^n\times(0,\infty)$. For any $s\in\mathbb{R}$, the symbol $\lceil s\rceil$ denotes the smallest integer not less than s. For any $r\in[1,\infty]$, let r' be the conjugate number of r, that is, $\frac{1}{r}+\frac{1}{r'}=1$. For any $x:=(x_1,\dots,x_n)\in\mathbb{R}^n$, let $\|x\|_\infty := \max\{|x_1|,\dots,|x_n|\}$ and $|x| := \sqrt{|x_1|^2+\cdots+|x_n|^2}$. For any $A\subset\mathbb{R}^{n+1}$, let

$$\mathrm{pr}_x(A) := \{x\in\mathbb{R}^n:\ \text{there exists } t\in\mathbb{R} \text{ such that } (x,t)\in A\}$$

and

$$\mathrm{pr}_t(A) := \{t\in\mathbb{R}:\ \text{there exists } x\in\mathbb{R}^n \text{ such that } (x,t)\in A\}$$

be the projections of A, respectively, onto the space $\mathbb{R}^n$ and the time axis $\mathbb{R}$. Let $\mathbf{0}$ denote the origin of $\mathbb{R}^n$. For any $A\subset\mathbb{R}^{n+1}$ and $a\in\mathbb{R}$, let $A-(\mathbf{0},a) := \{(x,t-a):\ (x,t)\in A\}$. For any measurable set $A\subset\mathbb{R}^{n+1}$, we denote by $|A|$ its $(n+1)$-dimensional Lebesgue measure. For any $R\in\mathcal{R}^{n+1}_p$, denote by (x_R,t_R) its center and by $l(R)$ its edge length. The *top* of $R\in\mathcal{R}^{n+1}_p$ means $Q(x_R,l(R))\times\{t_R+[l(R)]^p\}$ and the *bottom* of R means $Q(x_R,l(R))\times\{t_R-[l(R)]^p\}$. Let $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ (resp. $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$) be the set of all locally integrable functions on $\mathbb{R}^{n+1}$ (resp. $\mathbb{R}^{n+1}_+$). The symbol $f\lesssim g$ means that there exists a positive constant C such that $f\le Cg$ and, if $f\lesssim g\lesssim f$, we then write $f\sim g$. Finally, when we show a theorem (and the like), in its proof we always use the same symbols as those appearing in the statement itself of that theorem (and the like).

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2 Parabolic Muckenhoupt Two-Weight Classes with Time Lag

In this section, we introduce the concept of parabolic Muckenhoupt two-weight classes with time lag and present their several basic properties. We begin with the following definition.

Definition 2.1. Let $\gamma \in [0, 1)$.

(i) Let $1 < r \le q < \infty$. The *parabolic Muckenhoupt two-weight class* $TA^+_{r,q}(\gamma)$ *with time lag* is defined to be the set of all pairs (u, v) of nonnegative functions on $\mathbb{R}^{n+1}$ such that

$$[u,v]_{TA^+_{r,q}(\gamma)} := \sup_{R\in\mathcal{R}^{n+1}_p} \fint_{R^-(\gamma)} u^q \left[\fint_{R^+(\gamma)} v^{-r'}\right]^{\frac{q}{r'}} < \infty.$$

If the above condition holds with the time axis reversed, that is,

$$[u,v]_{TA^-_{r,q}(\gamma)} := \sup_{R\in\mathcal{R}^{n+1}_p} \fint_{R^+(\gamma)} u^q \left[\fint_{R^-(\gamma)} v^{-r'}\right]^{\frac{q}{r'}} < \infty,$$

then $(u, v) \in TA^-_{r,q}(\gamma)$ which is also called the *parabolic Muckenhoupt two-weight class with time lag* and consists of all such (u, v).

(ii) Let $q \in [1, \infty)$. The *parabolic Muckenhoupt two-weight class* $TA^+_{1,q}(\gamma)$ *with time lag* is defined to be the set of all pairs (u, v) of nonnegative functions on $\mathbb{R}^{n+1}$ such that

$$[u,v]_{TA^+_{1,q}(\gamma)} := \sup_{R\in\mathcal{R}^{n+1}_p} \fint_{R^-(\gamma)} u^q \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} v(x,t)\right]^{-q} < \infty.$$

If the above condition holds with the time axis reversed, that is,

$$[u,v]_{TA^-_{1,q}(\gamma)} := \sup_{R\in\mathcal{R}^{n+1}_p} \fint_{R^+(\gamma)} u^q \left[\operatorname*{ess\,inf}_{(x,t)\in R^-(\gamma)} v(x,t)\right]^{-q} < \infty,$$

then $(u, v) \in TA^-_{1,q}(\gamma)$ which is also called the *parabolic Muckenhoupt two-weight class with time lag* and consists of all such (u, v).

Remark 2.2. (i) Let $\gamma \in [0, 1)$ and $1 < r \le q < \infty$. Recall that the *parabolic Muckenhoupt class* $A^+_{r,q}(\gamma)$ *with time lag*, introduced in [61], is defined to be the set of all nonnegative functions ω on $\mathbb{R}^{n+1}$ such that

$$\sup_{R\in\mathcal{R}^{n+1}_p} \fint_{R^-(\gamma)} \omega^q \left[\fint_{R^+(\gamma)} \omega^{-r'}\right]^{\frac{q}{r'}} < \infty$$

and the *parabolic Muckenhoupt class* $A^+_1(\gamma)$ *with time lag*, introduced in [45, 50], is defined to be the set of all nonnegative functions ω on $\mathbb{R}^{n+1}$ such that

$$\sup_{R\in\mathcal{R}^{n+1}_p} \fint_{R^-(\gamma)} \omega \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} \omega(x,t)\right]^{-1} < \infty.$$

Then we are easy to prove that, if $\omega \in A^+_{r,q}(\gamma)$, then $(\omega, \omega) \in TA^+_{r,q}(\gamma)$ and, if $w \in A^+_1(\gamma)$, then, for any $s \in [1, \infty)$, $(w^{\frac{1}{s}}, w^{\frac{1}{s}}) \in TA^+_{1,s}(\gamma)$. Thus, the parabolic Muckenhoupt two-weight class with time lag in Definition 2.1 is indeed a generalization of both $A^+_{r,q}(\gamma)$ and $A^+_1(\gamma)$.

(ii) Let $0 \le \gamma_1 \le \gamma_2 < 1$ and $1 \le r \le q < \infty$. Then $TA^+_{r,q}(\gamma_1) \subset TA^+_{r,q}(\gamma_2)$. Moreover, for any $(u, v) \in TA^+_{r,q}(\gamma_1)$, $[u,v]_{TA^+_{r,q}(\gamma_2)} \le (\frac{1-\gamma_1}{1-\gamma_2})^{1+\frac{q}{r'}}[u,v]_{TA^+_{r,q}(\gamma_1)}$.

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For the parabolic Muckenhoupt two-weight class with time lag, we have the following nested property which can be directly deduced from Definition 2.1 and the Hölder inequality.

Proposition 2.3. *Let $\gamma \in [0, 1)$.*

(i) *Let $1 \le \widetilde{r} < r \le q < \infty$. Then $TA^+_{\widetilde{r},q}(\gamma) \subset TA^+_{r,q}(\gamma)$. Moreover, for any $(u, v) \in TA^+_{\widetilde{r},q}(\gamma)$, $[u, v]_{TA^+_{r,q}(\gamma)} \le [u, v]_{TA^+_{\widetilde{r},q}(\gamma)}$.*

(ii) *Let $1 \le r \le \widetilde{q} < q < \infty$. Then $TA^+_{r,q}(\gamma) \subset TA^+_{r,\widetilde{q}}(\gamma)$. Moreover, for any $(u, v) \in TA^+_{r,q}(\gamma)$, $[u, v]_{TA^+_{r,\widetilde{q}}(\gamma)} \le [u, v]^{\frac{\widetilde{q}}{q}}_{TA^+_{r,q}(\gamma)}$.*

Next, we introduce the uncentered parabolic fractional maximal function with time lag.

Definition 2.4. Let $\gamma, \beta \in [0, 1)$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, the *uncentered forward in time parabolic fractional maximal function $M^{\gamma+}_{\beta}(f)$ with time lag* and the *uncentered back in time parabolic fractional maximal function $M^{\gamma-}_{\beta}(f)$ with time lag* of f are defined, respectively, by setting, for any $(x, t) \in \mathbb{R}^{n+1}$,

$$M^{\gamma+}_{\beta}(f)(x, t) := \sup_{\substack{R\in\mathcal{R}^{n+1}_p \\ (x,t)\in R^-(\gamma)}} \left|R^+(\gamma)\right|^{\beta} \fint_{R^+(\gamma)} |f|$$

and

$$M^{\gamma-}_{\beta}(f)(x, t) := \sup_{\substack{R\in\mathcal{R}^{n+1}_p \\ (x,t)\in R^+(\gamma)}} \left|R^-(\gamma)\right|^{\beta} \fint_{R^-(\gamma)} |f|.$$

Notice that, for any $\gamma \in [0, 1)$, $M^{\gamma+}_0$ coincides with the uncentered parabolic parabolic forward in time maximal operator $M^{\gamma+}$ with time lag introduced in [45, Definition 2.2]. We present the following characterization of the parabolic Muckenhoupt two-weight class $TA^+_{1,q}(\gamma)$ in terms of the uncentered back in time parabolic maximal operator $M^{\gamma-}_0$.

Proposition 2.5. *Let $\gamma \in [0, 1)$, $q \in [1, \infty)$, and (u, v) be a pair of nonnegative functions on $\mathbb{R}^{n+1}$. Then $(u, v) \in TA^+_{1,q}(\gamma)$ if and only if there exists $C \in (0, \infty)$ such that, for almost every $(x, t) \in \mathbb{R}^{n+1}$,*

$$M^{\gamma-}_0(u^q)(x, t) \le C[v(x, t)]^q. \tag{2.1}$$

Proof. We first show the sufficiency. Assume that (u, v) satisfies (2.1). By this and Definition 2.4, we find that, for any $R \in \mathcal{R}^{n+1}_p$ and for almost every $(x, t) \in R^+(\gamma)$,

$$\fint_{R^-(\gamma)} u^q \le M^{\gamma-}_0(u^q)(x, t) \le C[v(x, t)]^q.$$

Taking the essential infimum over all $(x, t) \in R^+(\gamma)$, we then obtain

$$\fint_{R^-(\gamma)} u^q \le C\left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} v(x, t)\right]^q$$

and hence $(u, v) \in TA^+_{1,q}(\gamma)$ with $[u, v]_{TA^+_{1,q}(\gamma)} \le C$. This finishes the proof of the sufficiency.

Now, we prove the necessity. Assume that $(u, v) \in TA^+_{1,q}(\gamma)$ and let

$$\mathcal{N} := \left\{(x, t) \in \mathbb{R}^{n+1} : \ M^{\gamma-}_0\,(u^q)\,(x, t) > [u, v]_{TA^+_{1,q}(\gamma)}[v(x, t)]^q\right\}.$$

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To show that (2.1) holds almost everywhere for some $C \in (0,\infty)$, it suffices to prove that $|\mathcal{N}| = 0$. Indeed, from Definition 2.4, we infer that, for any given $(x,t) \in \mathcal{N}$, there exist $R_{(x,t)} \in \mathcal{R}_p^{n+1}$ and $\epsilon \in (0,\infty)$ such that $(x,t) \in R^+_{(x,t)}(\gamma)$ and

$$\fint_{R^-_{(x,t)}(\gamma)} u^q > \frac{1}{|R^-_{(x,t)}(\gamma)| + \epsilon} \int_{R^-_{(x,t)}(\gamma)} u^q > [u,v]_{TA^+_{1,q}(\gamma)}[v(x,t)]^q. \tag{2.2}$$

Since $\mathrm{pr}_t(R^+_{(x,t)}(\gamma))$ is an open interval and $(x,t) \in R^+_{(x,t)}(\gamma)$, it follows that there exists $\widetilde{R}_{(x,t)} := Q(x_0,L_0) \times (t_0 - L_0^p, t_0 + L_0^p) \in \mathcal{R}_p^{n+1}$ with $(x_0,t_0) \in \mathbb{R}^{n+1}$ and $L_0 \in (0,\infty)$ such that the following statements hold:

(i) $(x,t) \in \widetilde{R}^+_{(x,t)}(\gamma)$;

(ii) $R^-_{(x,t)}(\gamma) \subset \widetilde{R}^-_{(x,t)}(\gamma)$ and $|\widetilde{R}^-_{(x,t)}(\gamma) \setminus R^-_{(x,t)}(\gamma)| < \epsilon$;

(iii) all the vertices of $Q(x_0,L_0)$ belong to $\mathbb{Q}^n$ and $t_0 - L_0^p \in \mathbb{Q}$.

Combining (2.2), (ii), and $(u,v) \in TA^+_{1,q}(\gamma)$, we conclude that

$$\begin{aligned}[u,v]_{TA^+_{1,q}(\gamma)}[v(x,t)]^q &< \frac{1}{|R^-_{(x,t)}(\gamma)| + \epsilon} \int_{R^-_{(x,t)}(\gamma)} u^q \\ &< \fint_{\widetilde{R}^-_{(x,t)}(\gamma)} u^q \le [u,v]_{TA^+_{1,q}(\gamma)} \left[\operatorname*{ess\,inf}_{(y,s)\in \widetilde{R}^+_{(x,t)}(\gamma)} v(y,s) \right]^q,\end{aligned}$$

which further implies that

$$v(x,t) < \operatorname*{ess\,inf}_{(y,s)\in \widetilde{R}^+_{(x,t)}(\gamma)} v(y,s). \tag{2.3}$$

Let $\{R_k\}_{k\in\mathbb{N}}$ be the sequence of all $R := Q(z,L) \times (r - L^p, r + L^p) \in \mathcal{R}_p^{n+1}$ with $(z,r) \in \mathbb{R}^{n+1}$ and $L \in (0,\infty)$ such that all the vertices of $Q(z,L)$ belong to $\mathbb{Q}^n$ and $r - L^p \in \mathbb{Q}$ and, for any $k \in \mathbb{N}$, let

$$\mathcal{N}_k := \left\{ (x,t) \in R_k^+(\gamma) : \ v(x,t) < \operatorname*{ess\,inf}_{(y,s)\in R_k^+(\gamma)} v(y,s) \right\}.$$

Then $|\mathcal{N}_k| = 0$ for any $k \in \mathbb{N}$. Moreover, from (i), (iii), and (2.3), it follows that $\mathcal{N} \subset \bigcup_{k\in\mathbb{N}} \mathcal{N}_k$, which further implies that $|\mathcal{N}| \le \sum_{k\in\mathbb{N}} |\mathcal{N}_k| = 0$. This finishes the proof of the necessity and hence Proposition 2.5. □

Throughout this article, we *always omit* the variables (x,t) in the notation if there is no ambiguity and, for instance, for any $A \subset \mathbb{R}^{n+1}$, any function f on $\mathbb{R}^{n+1}$, and any $\lambda \in \mathbb{R}$, we simply write

$$A \cap \{f > \lambda\} := \{(x,t) \in A : \ f(x,t) > \lambda\}.$$

The following proposition indicates that the parabolic Muckenhoupt two-weight class is closed under taking the maximum and the minimum.

Proposition 2.6. *Let $\gamma \in [0,1)$ and $1 \le r \le q < \infty$. Then, for any $(u,v), (\widetilde{u},\widetilde{v}) \in TA^+_{r,q}(\gamma)$,*

(i) $(\max\{u,\widetilde{u}\}, \max\{v,\widetilde{v}\}) \in TA^+_{r,q}(\gamma)$ *and*

$$[\max\{u,\widetilde{u}\}, \max\{v,\widetilde{v}\}]_{TA^+_{r,q}(\gamma)} \le [u,v]_{TA^+_{r,q}(\gamma)} + [\widetilde{u},\widetilde{v}]_{TA^+_{r,q}(\gamma)}.$$

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(ii) $(\min\{u, \widetilde{u}\}, \min\{v, \widetilde{v}\}) \in TA^+_{r,q}(\gamma)$. *Moreover, if* $r = 1$, *then*

$$[\min\{u, \widetilde{u}\}, \min\{v, \widetilde{v}\}]_{TA^+_{1,q}(\gamma)} \le \max\left\{[u, v]_{TA^+_{1,q}(\gamma)}, [\widetilde{u}, \widetilde{v}]_{TA^+_{1,q}(\gamma)}\right\}$$

and, if $r \in (1, \infty)$, *then* $[\min\{u, \widetilde{u}\}, \min\{v, \widetilde{v}\}]_{TA^+_{r,q}(\gamma)} \le [u, v]_{TA^+_{r,q}(\gamma)} + [\widetilde{u}, \widetilde{v}]_{TA^+_{r,q}(\gamma)}$.

Proof. Let $(u, v), (\widetilde{u}, \widetilde{v}) \in TA^+_{r,q}(\gamma)$. We first show (i). Let $W_1 := \max\{u, \widetilde{u}\}$ and $W_2 := \max\{v, \widetilde{v}\}$. We consider the following two cases for r.

Case 1) $r = 1$. In this case, by Definition 2.1(ii), we find that, for any given $R \in \mathcal{R}^{n+1}_p$,

$$\begin{aligned}
\fint_{R^-(\gamma)} W_1^q &\le \fint_{R^-(\gamma)} u^q + \fint_{R^-(\gamma)} \widetilde{u}^q \\
&\le [u, v]_{TA^+_{1,q}(\gamma)} \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} v(x, t)\right]^q \\
&\quad + [\widetilde{u}, \widetilde{v}]_{TA^+_{1,q}(\gamma)} \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} \widetilde{v}(x, t)\right]^q \\
&\le \left\{[u, v]_{TA^+_{1,q}(\gamma)} + [\widetilde{u}, \widetilde{v}]_{TA_{1,q}(\gamma)}\right\} \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} W_2(x, t)\right]^q.
\end{aligned}$$

Taking the supremum over all $R \in \mathcal{R}^{n+1}_p$, we conclude that $(W_1, W_2) \in TA^+_{1,q}(\gamma)$ and

$$[W_1, W_2]_{TA^+_{1,q}(\gamma)} \le [u, v]_{TA^+_{1,q}(\gamma)} + [\widetilde{u}, \widetilde{v}]_{TA^+_{1,q}(\gamma)}.$$

Case 2) $r \in (1, \infty)$. In this case, from Definition 2.1(i), we deduce that, for any given $R \in \mathcal{R}^{n+1}_p$,

$$\begin{aligned}
\fint_{R^-(\gamma)} W_1^q \left[\fint_{R^+(\gamma)} W_2^{-r'}\right]^{\frac{q}{r'}} &\le \left[\fint_{R^-(\gamma)} u^q + \fint_{R^-(\gamma)} \widetilde{u}^q\right]\left[\fint_{R^+(\gamma)} W_2^{-r'}\right]^{\frac{q}{r'}} \\
&\le \fint_{R^-(\gamma)} u^q \left[\fint_{R^+(\gamma)} v^{-r'}\right]^{\frac{q}{r'}} \\
&\quad + \fint_{R^-(\gamma)} \widetilde{u}^q \left[\fint_{R^+(\gamma)} \widetilde{v}^{-r'}\right]^{\frac{q}{r'}} \\
&\le [u, v]_{TA^+_{r,q}(\gamma)} + [\widetilde{u}, \widetilde{v}]_{TA^+_{r,q}(\gamma)}.
\end{aligned}$$

Taking the supremum over all $R \in \mathcal{R}^{n+1}_p$, we obtain $(W_1, W_2) \in TA^+_{r,q}(\gamma)$ and

$$[W_1, W_2]_{TA^+_{r,q}(\gamma)} \le [u, v]_{TA^+_{r,q}(\gamma)} + [\widetilde{u}, \widetilde{v}]_{TA^+_{r,q}(\gamma)}.$$

Combining the above two cases, we then finish the proof of (i).

Next, we prove (ii). Let $w_1 := \min\{u, \widetilde{u}\}$ and $w_2 := \min\{v, \widetilde{v}\}$. Similarly to the proof of (i), we consider the following two cases for r.

Case 1) $r = 1$. In this case, using Definition 2.1(ii), we conclude that, for any $R \in \mathcal{R}^{n+1}_p$,

$$\begin{aligned}
\fint_{R^-(\gamma)} w_1^q &\le \min\left\{\fint_{R^-(\gamma)} u^q, \fint_{R^-(\gamma)} \widetilde{u}^q\right\} \\
&\le \min\left\{[u, v]_{TA^+_{1,q}(\gamma)} \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} v(x, t)\right]^q,\right. \\
&\qquad \left.[\widetilde{u}, \widetilde{v}]_{TA^+_{1,q}(\gamma)} \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} \widetilde{v}(x, t)\right]^q\right\} \\
&\le \max\left\{[u, v]_{TA^+_{1,q}(\gamma)}, [\widetilde{u}, \widetilde{v}]_{TA^+_{1,q}(\gamma)}\right\} \left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\gamma)} w_2(x, t)\right]^q.
\end{aligned}$$

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Taking the supremum over all $R \in \mathcal{R}_p^{n+1}$, we find that $(w_1, w_2) \in TA_{1,q}^+(\gamma)$ and

$$[w_1, w_2]_{TA_{1,q}^+(\gamma)} \le \max\left\{[u, v]_{TA_{1,q}^+(\gamma)},\ [\widetilde{u}, \widetilde{v}]_{TA_{1,q}^+(\gamma)}\right\}.$$

Case 2) $r \in (1, \infty)$. In this case, from Definition 2.1(i), we infer that, for any $R \in \mathcal{R}_p^{n+1}$,

$$\begin{aligned}
&\fint_{R^-(\gamma)} w_1^q \left[\fint_{R^+(\gamma)} w_2^{-r'}\right]^{\frac{q}{r'}} \\
&\quad \le \fint_{R^-(\gamma)} w_1^q \left[\frac{1}{|R^+(\gamma)|}\int_{R^+(\gamma)\cap\{v>\widetilde{v}\}} \widetilde{v}^{-r'}\right]^{\frac{q}{r'}} \\
&\qquad + \fint_{R^-(\gamma)} w_1^q \left[\frac{1}{|R^+(\gamma)|}\int_{R^+(\gamma)\cap\{v\le\widetilde{v}\}} v^{-r'}\right]^{\frac{q}{r'}} \\
&\quad \le \fint_{R^-(\gamma)} \widetilde{u}^q \left[\fint_{R^+(\gamma)} \widetilde{v}^{-r'}\right]^{\frac{q}{r'}} + \fint_{R^-(\gamma)} u^q \left[\fint_{R^+(\gamma)} v^{-r'}\right]^{\frac{q}{r'}} \\
&\quad \le [\widetilde{u}, \widetilde{v}]_{TA_{r,q}^+(\gamma)} + [u, v]_{TA_{r,q}^+(\gamma)}.
\end{aligned}$$

Taking the supremum over all $R \in \mathcal{R}_p^{n+1}$, we obtain $(w_1, w_2) \in TA_{r,q}^+(\gamma)$ and

$$[w_1, w_2]_{TA_{r,q}^+(\gamma)} \le [u, v]_{TA_{r,q}^+(\gamma)} + [\widetilde{u}, \widetilde{v}]_{TA_{r,q}^+(\gamma)}.$$

Combining the above two cases then completes the proof of (ii) and hence Proposition 2.6. □

The following duality property follows directly from Definition 2.1(i); we omit the details.

Proposition 2.7. *Let $\gamma \in [0, 1)$ and $1 < r \le q < \infty$. Assume that (u, v) is a pair of positive functions on $\mathbb{R}^{n+1}$. Then $(u, v) \in TA_{r,q}^+(\gamma)$ if and only if $(v^{-1}, u^{-1}) \in TA_{q',r'}^-(\gamma)$.*

At the end of this section, we give a characterization of the diagonal parabolic Muckenhoupt two-weight class in terms of a forward in time doubling condition. In what follows, for any nonnegative function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and any measurable set $E \subset \mathbb{R}^{n+1}$, we denote $\int_E f$ by $f(E)$.

Proposition 2.8. *Let $\gamma \in [0, 1)$ and (u, v) be a pair of positive functions on $\mathbb{R}^{n+1}$. Then the following statements are equivalent.*

(i) *There exists $q \in (1, \infty)$ such that $(u, v) \in TA_{q,q}^+(\gamma)$.*

(ii) *There exist $C \in (0, \infty)$, $\delta \in (0, 1)$, and $r \in (\frac{1}{\delta}, \infty)$ such that, for any $R \in \mathcal{R}_p^{n+1}$ and any measurable set $E \subset R^+(\gamma)$,*

$$\frac{|E|}{|R^+(\gamma)|} \le C\left[\frac{(v^r)(E)}{(u^r)(R^-(\gamma))}\right]^{\delta}. \tag{2.4}$$

Proof. We first show (i) $\Longrightarrow$ (ii). Assume that $(u, v) \in TA_{q,q}^+(\gamma)$ for some $q \in (1, \infty)$. Then, from the Hölder inequality, we deduce that, for any $R \in \mathcal{R}_p^{n+1}$ and any measurable set $E \subset R^+(\gamma)$,

$$\begin{aligned}
\frac{|E|}{|R^+(\gamma)|} &= \fint_{R^+(\gamma)} \mathbf{1}_E = \fint_{R^+(\gamma)} v^{-1} v \mathbf{1}_E \le \left[\fint_{R^+(\gamma)} v^q \mathbf{1}_E\right]^{\frac{1}{q}} \left[\fint_{R^+(\gamma)} v^{-q'}\right]^{\frac{1}{q'}} \\
&\le \left[\fint_{R^+(\gamma)} v^q \mathbf{1}_E\right]^{\frac{1}{q}} [u, v]_{TA_{q,q}^+(\gamma)}^{\frac{1}{q}} \left[\fint_{R^-(\gamma)} u^q\right]^{-\frac{1}{q}}
\end{aligned}$$

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$$= [u,v]_{TA^+_{q,q}(\gamma)}^{\frac{1}{q}} \left[\frac{(v^q)(E)}{(u^q)(R^-(\gamma))}\right]^{\frac{1}{q}},$$

which further implies that (2.4) with $r := q$, $\delta \in (0, \frac{1}{q}]$, and $C := [u,v]^{\delta}_{TA^+_q(\gamma)}$ holds. This finishes the proof of (i) $\Longrightarrow$ (ii).

Now, we prove (ii) $\Longrightarrow$ (i). Assume that (ii) holds. According to (2.4), we conclude that, for any $R \in \mathcal{R}^{n+1}_p$ and any measurable set $E \subset R^+(\gamma)$,

$$C^{-\frac{1}{\delta}} (u^r)(R^-(\gamma)) \left[\frac{|E|}{|R^+(\gamma)|}\right]^{\frac{1}{\delta}} \le (v^r)(E). \tag{2.5}$$

Fix $R \in \mathcal{R}^{n+1}_p$ and, for any given $\lambda \in (0,\infty)$, let $E_\lambda := R^+(\gamma) \cap \{v^{-r} > \lambda\}$. Then $(v^r)(E_\lambda) \le |E_\lambda|/\lambda$. Combining this and (2.5), we find that, for any $\lambda \in (0,\infty)$,

$$C^{-\frac{1}{\delta}} (u^r)(R^-(\gamma)) \left[\frac{|E_\lambda|}{|R^+(\gamma)|}\right]^{\frac{1}{\delta}} \le (v^r)(E_\lambda) \le \frac{|E_\lambda|}{\lambda},$$

which further implies that

$$|E_\lambda| \le \frac{C^{\frac{1}{1-\delta}} |R^-(\gamma)|^{\frac{1}{1-\delta}}}{\lambda^{\frac{\delta}{1-\delta}} [(u^r)(R^-(\gamma))]^{\frac{\delta}{1-\delta}}}. \tag{2.6}$$

Since $r \in (\frac{1}{\delta}, \infty)$, it follows that $\frac{r'}{r} \in (0, \frac{\delta}{1-\delta})$. From the Cavalieri principle (see, for example, [34, Proposition 1.1.4]), the obvious fact that $E_\lambda \subset R^+(\gamma)$ for any $\lambda \in (0,\infty)$, and (2.6), we infer that

$$\begin{aligned}
\int_{R^+(\gamma)} v^{-r'} &= \frac{r'}{r} \int_0^\infty \lambda^{\frac{r'}{r}-1} \left|R^+(\gamma) \cap \{v^{-r} > \lambda\}\right| d\lambda \\
&= \frac{r'}{r} \left[\int_0^{\frac{1}{(u^r)_{R^-(\gamma)}}} + \int_{\frac{1}{(u^r)_{R^-(\gamma)}}}^\infty\right] \lambda^{\frac{r'}{r}-1} |E_\lambda|\, d\lambda \\
&\le \frac{r'}{r} \left|R^+(\gamma)\right| \int_0^{\frac{1}{(u^r)_{R^-(\gamma)}}} \lambda^{\frac{r'}{r}-1}\, d\lambda \\
&\quad + \frac{r'}{r} \frac{C^{\frac{1}{1-\delta}} |R^-(\gamma)|^{\frac{1}{1-\delta}}}{[(u^r)(R^-(\gamma))]^{\frac{\delta}{1-\delta}}} \int_{\frac{1}{(u^r)_{R^-(\gamma)}}}^\infty \lambda^{\frac{r'}{r}-\frac{1}{1-\delta}}\, d\lambda \\
&\lesssim \left|R^+(\gamma)\right| \left[\fint_{R^-(\gamma)} u^r \right]^{-\frac{r'}{r}},
\end{aligned}$$

where the implicit positive constant depends only on C, r, and δ. Taking the supremum over all $R \in \mathcal{R}^{n+1}_p$, we conclude that $(u,v) \in TA^+_{r,r}(\gamma)$, which completes the proof of the sufficiency and hence Proposition 2.8. □

3 Independence of Choices of Time Lag and Self-improving Property of $TA^+_{r,q}(\gamma)$

In this section, under an additional mild assumption (which is not necessary for one-weight case), we show that the parabolic Muckenhoupt two-weight class is independent of the time lag and the distance between the upper and the lower parts of parabolic rectangles. As an application, we prove that the parabolic Muckenhoupt two-weight class has the self-improving property. Recall that, for any $\gamma \in [0,1)$, $A^+_\infty(\gamma) := \bigcup_{q\in(1,\infty)} A^+_q(\gamma)$ and, for any $A \subset \mathbb{R}^{n+1}$ and $a \in \mathbb{R}$, $A - (\mathbf{0}, a) := \{(x, t-a) : (x,t) \in A\}$.

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Theorem 3.1. *Let* $0<\gamma<\alpha<1$, $\tau\in[1,\infty)$, *and* (u,v) *be a pair of nonnegative functions on* $\mathbb{R}^{n+1}$. *Assume that* $u\in A_\infty^+(\gamma)$. *Then the following statements hold.*

(i) *If* $1<r\le q<\infty$, *then* $(u,v)\in TA_{r,q}^+(\gamma)$ *if and only if there exists a positive constant* C *such that, for any* $R\in\mathcal{R}_p^{n+1}$,

$$\fint_{R^+(\alpha)-(\mathbf{0},\tau(1+\alpha)[l(R)]^p)} u^q\left[\fint_{R^+(\alpha)} v^{-r'}\right]^{\frac{q}{r'}}\le C. \tag{3.1}$$

(ii) *If* $q\in[1,\infty)$, *then* $(u,v)\in TA_{1,q}^+(\gamma)$ *if and only if there exists a positive constant* C *such that, for any* $R\in\mathcal{R}_p^{n+1}$,

$$\fint_{R^+(\alpha)-(\mathbf{0},\tau(1+\alpha)[l(R)]^p)} u^q\left[\operatorname*{ess\,inf}_{(x,t)\in R^+(\alpha)} v(x,t)\right]^{-q}\le C. \tag{3.2}$$

Proof. We first show the necessity of (i) and (ii). Assume that $(u,v)\in TA_{r,q}^+(\gamma)$. Fix $R:=R(x,t,L)\in\mathcal{R}_p^{n+1}$ with $(x,t)\in\mathbb{R}^{n+1}$ and $L\in(0,\infty)$. Let

$$P:=R\left(x,t-\frac{(\tau-1)(1+\alpha)L^p}{2},\left[1+\frac{(\tau-1)(1+\alpha)}{2}\right]^{\frac1p}L\right).$$

Then

$$R^+(\alpha)\subset P^+(\widetilde{\alpha})\ \text{ and }\ R^+(\alpha)-(\mathbf{0},\tau(1+\alpha)L^p)\subset P^-(\widetilde{\alpha}), \tag{3.3}$$

where $\widetilde{\alpha}:=\frac{2\alpha+(\tau-1)(1+\alpha)}{2+(\tau-1)(1+\alpha)}\in(\gamma,1)$. Moreover, we have

$$\left|P^\pm(\widetilde{\alpha})\right|=2^n\left[1+\frac{(\tau-1)(1+\alpha)}{2}\right]^{\frac np}\left|R^\pm(\alpha)\right|. \tag{3.4}$$

From Remark 2.2(ii), we deduce that $(u,v)\in TA_{r,q}^+(\widetilde{\alpha})$, which, combined with (3.3) and (3.4), further implies that

$$\begin{aligned}
&\fint_{R^+(\alpha)-(\mathbf{0},\tau(1+\alpha)L^p)} u^q\left[\fint_{R^+(\alpha)} v^{-r'}\right]^{\frac{q}{r'}}\\
&\quad\le\left\{2^n\left[1+\frac{(\tau-1)(1+\alpha)}{2}\right]^{\frac np}\right\}^{1+\frac{q}{r'}}\fint_{P^-(\widetilde{\alpha})} u^q\left[\fint_{P^+(\widetilde{\alpha})} v^{-r'}\right]^{\frac{q}{r'}}\\
&\quad\le\left\{2^n\left[1+\frac{(\tau-1)(1+\alpha)}{2}\right]^{\frac np}\right\}^{1+\frac{q}{r'}}[u,v]_{TA_{r,q}^+(\widetilde{\alpha})}.
\end{aligned}$$

This finishes the proof of the necessity of (i). Letting $r\to1^+$ in the above argument, we find that the necessity of (ii) holds. Here and thereafter, $r\to1^+$ means that $r\in(1,\infty)$ and $r\to1$.

Then we prove the sufficiency of (i) by the chaining argument. Suppose that (3.1) holds. Fix $R\in\mathcal{R}_p^{n+1}$. Without loss of generality, we may suppose that R is centered at the origin $(\mathbf{0},0)$. Let $m\in\mathbb{N}$ be such that there exists $\epsilon\in[0,1)$ satisfying

$$m=\log_2\left[\frac{\tau(1+\alpha)}{1-\alpha}\right]+\frac{1}{p-1}\left\{1+\log_2\left[\frac{\tau(1+\alpha)}{\gamma}\right]\right\}+2+\epsilon. \tag{3.5}$$

Partition each spatial edge of $R^+(\gamma)$ into 2^m equally long intervals and divide the temporal edge of $R^+(\gamma)$ into $J:=\lceil(1-\gamma)2^{pm}/(1-\alpha)\rceil$ equally long intervals. Then we obtain $2^{nm}J$ subrectangles of

$R^+(\gamma)$ and denote them by $\{V^+_{i,j}\}_{i\in\mathbb{N}\cap[1,2^{nm}],j\in\mathbb{N}\cap[1,J]}$. Fix $i \in \mathbb{N}\cap[1,2^{nm}]$ and $j \in \mathbb{N}\cap[1,J]$. Notice that there exists $R_{i,j} \in \mathcal{R}^{n+1}_p$ such that the tops of both $R_{i,j}$ and $V^+_{i,j}$ coincide, $V^+_{i,j} \subset R^+_{i,j}(\alpha)$, and

$$\frac{|V^+_{i,j}|}{|R^+_{i,j}(\alpha)|} = \frac{(1-\gamma)2^{pm}}{(1-\alpha)J}. \tag{3.6}$$

Divide $R^-(\gamma)$ in the same manner as we partition $R^+(\gamma)$. Then we obtain $2^{nm}J$ subrectangles of $R^-(\gamma)$ and denote them by $\{U^-_{k,\iota}\}_{k\in\mathbb{N}\cap[1,2^{nm}],\iota\in\mathbb{N}\cap[1,J]}$. Fix $k \in \mathbb{N}\cap[1,2^{nm}]$ and $\iota \in \mathbb{N}\cap[1,J]$. Observe that there exists $\widetilde{R}_{k,\iota} \in \mathcal{R}^{n+1}_p$ such that the bottoms of both $\widetilde{R}^+_{k,\iota}(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(\widetilde{R}_{k,\iota})]^p)$ and $U^-_{k,\iota}$ coincide, $U^-_{k,\iota} \subset \widetilde{R}^+_{k,\iota}(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(\widetilde{R}_{k,\iota})]^p)$, and

$$\frac{|U^-_{k,\iota}|}{|\widetilde{R}^+_{k,\iota}(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(\widetilde{R}_{k,\iota})]^p)|} = \frac{(1-\gamma)2^{pm}}{(1-\alpha)J}. \tag{3.7}$$

We claim that there exists $\mathfrak{R} \in \mathcal{R}^{n+1}_p$ satisfying that, for any $i,k \in \mathbb{N}\cap[1,2^{nm}]$ and $j,\iota \in \mathbb{N}\cap[1,J]$, there exist $N_j, \widetilde{N}_\iota \in \mathbb{N}$ and chains $\{P^{i,j}_d\}_{d=0}^{N_j}$ and $\{\widetilde{P}^{k,\iota}_d\}_{d=0}^{\widetilde{N}_\iota}$ consisting of congruent parabolic rectangles such that the following statements hold.

(i) $\mathfrak{R} \subset R^+(0)$ and $\mathfrak{R}^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(\mathfrak{R})]^p) \subset R^+(0)$.

(ii) For any $i \in \mathbb{N}\cap[1,2^{nm}]$ and $j \in \mathbb{N}\cap[1,J]$, $P^{i,j}_0 = R_{i,j}$ and $P^{i,j}_{N_j} = \mathfrak{R}$. For any $k \in \mathbb{N}\cap[1,2^{nm}]$ and $\iota \in \mathbb{N}\cap[1,J]$, $\widetilde{P}^{k,\iota}_0 = \widetilde{R}_{k,\iota}$ and $\widetilde{P}^{k,\iota}_{\widetilde{N}_\iota} = \mathfrak{R}$.

(iii) For any $j \in \mathbb{N}\cap[1,J]$,

$$N_j \le 2^{\frac{p}{p-1}+3p+1}\left[\frac{\tau(1+\alpha)}{1-\alpha}\right]^p\left[\frac{\tau(1+\alpha)}{\gamma}\right]^{\frac{p}{p-1}} =: C_1.$$

For any $\iota \in \mathbb{N}\cap[1,J]$, $\widetilde{N}_\iota \le 2C_1$.

(iv) For any $i \in \mathbb{N}\cap[1,2^{nm}]$, $j \in \mathbb{N}\cap[1,J]$, and $d \in \mathbb{N}\cap[1,N_j]$,

$$\frac{|(P^{i,j}_d)^+(\alpha)\cap(S^{i,j}_{d-1})^-(\alpha)|}{|(P^{i,j}_d)^+(\alpha)|} \in \left[\frac{1}{2^{n+1}}, 1\right],$$

where $(S^{i,j}_h)^-(\alpha) := (P^{i,j}_h)^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(P^{i,j}_h)]^p)$ for any $h \in \mathbb{Z}_+\cap[0,N_j]$. Moreover, for any $k \in \mathbb{N}\cap[1,2^{nm}]$, $\iota \in \mathbb{N}\cap[1,J]$, and $d \in \mathbb{N}\cap[1,\widetilde{N}_\iota]$,

$$\frac{|(\widetilde{S}^{k,\iota}_d)^-(\alpha)\cap(\widetilde{P}^{k,\iota}_{d-1})^+(\alpha)|}{|(\widetilde{S}^{k,\iota}_d)^-(\alpha)|} \in \left[\frac{1}{2^{n+1}}, 1\right],$$

where $(\widetilde{S}^{k,\iota}_h)^-(\alpha) := (\widetilde{P}^{k,\iota}_h)^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(\widetilde{P}^{k,\iota}_h)]^p)$ for any $h \in \mathbb{Z}_+\cap[0,\widetilde{N}_\iota]$.

Indeed, fix $i \in \mathbb{N}\cap[1,2^{nm}]$ and $j \in \mathbb{N}\cap[1,J]$. We will specify $\mathfrak{R}$ and N_j and construct the chain $\{P^{i,j}_d\}_{d=0}^{N_j}$ in the following two steps.

Step 1. In this step, we construct the chain corresponding to spatial variables. Assume that $R_{i,j} = Q(x_i, l) \times (t_j - 2l^p, t_j)$ with $(x_i, t_j) \in \mathbb{R}^{n+1}$ and $l := l(R)/2^m$. Let $Q^i_0 := Q(x_i, l)$. For any $d \in \mathbb{N}\cap[1,M_i]$ with $M_i \in \mathbb{N}$ determined later, let

$$Q^i_d := Q^i_{d-1} - \frac{x_i}{|x_i|}\frac{\theta_i l}{2}, \tag{3.8}$$

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where $\theta_i \in [1, \sqrt{n}]$, depending only on the angle between x_i and the spatial axes, is such that the center of Q^i_d belongs to the boundary of Q^i_{d-1}. Notice that there exists $b_i \in \mathbb{N} \cap [1, 2^m - 1]$, depending only on $|x_i|$, such that

$$|x_i| = \frac{\theta_i}{2}[l(R) - b_i l]. \tag{3.9}$$

From this and (3.8), it follows that, to ensure that $Q^i_{M_i} = Q(\mathbf{0}, l)$, we need to choose

$$M_i := \frac{2|x_i|}{\theta_i l} = \frac{l(R)}{l} - b_i = 2^m - b_i. \tag{3.10}$$

Observe that, for any $d \in \mathbb{N} \cap [1, M_i]$,

$$\frac{1}{2^n} \le \frac{|Q^i_d \cap Q^i_{d-1}|}{|Q^i_d|} \le \frac{1}{2}. \tag{3.11}$$

Then $\{Q^i_d\}_{d=0}^{M_i}$ forms a chain in $\mathbb{R}^n$ starting from $Q(x_i, l)$ and ending with $Q(\mathbf{0}, l)$.

Step 2. In this step, we specify $\mathfrak{R}$ and N_j and construct the chain $\{P^{i,j}_d\}_{d=0}^{N_j}$ based on the chains in Step 1. For any $d \in \mathbb{Z}_+ \cap [0, M_i]$, let

$$P^{i,j}_d := Q^i_d \times \left(t_j - d\tau(1+\alpha)l^p - 2l^p, t_j - d\tau(1+\alpha)l^p\right).$$

Then $P^{i,j}_0 = R_{i,j}$ and $\mathrm{pr}_x(P^{i,j}_{M_i}) = Q(\mathbf{0}, l)$. To determine $\mathfrak{R}$, we first assume that $j = 1$ and $i \in \mathbb{N} \cap [1, 2^{nm}]$ such that $Q(x_i, l)$ intersects with the boundary of $\mathrm{pr}_x(R) = Q(\mathbf{0}, l(R))$. From (3.9) and (3.10), we infer that $b_i = 1$ and hence $M_i = 2^m - 1$ in this case. Let $N := 2^m - 1$ and $\mathfrak{R} := P^{i,1}_N$. We show that (i) holds. Indeed, on the one hand, notice that

$$Q_N = Q(\mathbf{0}, l) \subset Q(\mathbf{0}, l(R)) \text{ and } t_1 - N\tau(1+\alpha)l^p < t_1 < [l(R)]^p. \tag{3.12}$$

On the other hand, by (3.5), we obtain

$$m \ge \frac{1}{p-1}\left\{1 + \log_2\left\lceil\frac{\tau(1+\alpha)}{\gamma}\right\rceil\right\},$$

which further implies that $2^{(p-1)m} \ge \frac{2\tau(1+\alpha)}{\gamma}$ and hence $\gamma - \frac{2\tau(1+\alpha)}{2^{(p-1)m}} \ge 0$. From this, the definitions of both J and l, and the fact that $\lceil s \rceil \le 2s$ for any $s \in [1, \infty)$, we deduce that

$$\begin{aligned}
&t_1 - (N+1)\tau(1+\alpha)l^p - (1-\alpha)l^p \\
&= \left[\gamma + \frac{1-\gamma}{\lceil(1-\gamma)2^{pm}/(1-\alpha)\rceil} - \frac{(N+1)\tau(1+\alpha)}{2^{pm}} - \frac{(1-\alpha)}{2^{pm}}\right][l(R)]^p \\
&\ge \left[\gamma + \frac{1-\gamma}{(1-\gamma)2^{pm+1}/(1-\alpha)} - \frac{(N+1)\tau(1+\alpha)}{2^{pm}} - \frac{(1-\alpha)}{2^{pm}}\right][l(R)]^p \\
&= \left[\gamma - \frac{1-\alpha}{2^{pm+1}} - \frac{\tau(1+\alpha)}{2^{(p-1)m}}\right][l(R)]^p \ge \left[\gamma - \frac{2\tau(1+\alpha)}{2^{(p-1)m}}\right][l(R)]^p \ge 0.
\end{aligned}$$

This, together with both the fact that the bottom of $\mathfrak{R}^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(\mathfrak{R})]^p)$ is

$$Q(\mathbf{0}, l) \times \{t_1 - (N+1)\tau(1+\alpha)l^p - (1-\alpha)l^p\}$$

and (3.12), further implies that (i) holds.

Then we suppose that $j = 1$ and $i \in \mathbb{N} \cap [1, 2^{nm}]$ such that $Q(x_i, l)$ does not intersect with the boundary of $\mathrm{pr}_x(R) = Q(\mathbf{0}, l(R))$. In this case, $b_i \ne 1$ and hence $M_i \in \mathbb{N} \cap [1, N)$. For any $d \in \mathbb{N} \cap [M_i, N-1]$, let $P^{i,j}_{d+1} := P^{i,j}_d - (\mathbf{0}, \tau(1+\alpha)l^p)$. Then we are easy to see that $P^{i,j}_N = \mathfrak{R}$.

Now, we consider the residual situation $j \in \mathbb{N} \cap [2, J]$ and $i \in \mathbb{N} \cap [1, 2^{nm}]$. In this case, we first extend the chain $\{P_d^{i,j}\}_{d=0}^{M_i}$ to the chain $\{P_d^{i,j}\}_{d=0}^{N}$ in the same way as we did in the above case. Then we can easily verify that $\mathrm{pr}_x(P_N^{i,j}) = \mathrm{pr}_x(P_N^{i,1})$. However, in the temporal variable, the distance between the top of $P_N^{i,j}$ and that of $P_N^{i,1}$ is $(j-1)(1-\gamma)[l(R)]^p/J$. We can shift every $P_d^{i,j}$ for $d \in \mathbb{N} \cap [1, 2^{m-1}]$ and add $\widetilde{M}_j$ parabolic rectangles into the chain to guarantee that the eventual parabolic rectangle is $\mathfrak{R}$. To be specific, we can choose $\beta_j \in [0,1)$ and $\widetilde{M}_j \in \mathbb{Z}_+$ such that

$$\frac{2^{m-1}\beta_j(1-\alpha)[l(R)]^p}{2^{pm}} + \frac{\widetilde{M}_j\tau(1+\alpha)[l(R)]^p}{2^{pm}} = \frac{(j-1)(1-\gamma)[l(R)]^p}{J}. \tag{3.13}$$

Indeed, notice that there exists $\eta \in [1,2)$ such that

$$J = \left\lceil \frac{(1-\gamma)2^{pm}}{1-\alpha} \right\rceil = \frac{\eta(1-\gamma)2^{pm}}{1-\alpha}$$

and hence we can rewrite (3.13) as

$$2^{m-1}\beta_j(1-\alpha) + \widetilde{M}_j\tau(1+\alpha) = \frac{(j-1)(1-\alpha)}{\eta}. \tag{3.14}$$

Choose $\widetilde{M}_j \in \mathbb{Z}_+$ such that

$$\widetilde{M}_j\tau(1+\alpha) \le \frac{(j-1)(1-\alpha)}{\eta} < (\widetilde{M}_j+1)\tau(1+\alpha).$$

That is, let $\widetilde{M}_j \in \mathbb{Z}_+$ be such that

$$\begin{aligned} \frac{(j-1)(1-\alpha)}{2\tau(1+\alpha)} - 1 \le \frac{(j-1)(1-\alpha)}{\eta\tau(1+\alpha)} - 1 &< \widetilde{M}_j \\ &\le \frac{(j-1)(1-\alpha)}{\tau(1+\alpha)} \le \frac{(j-1)(1-\alpha)}{\tau(1+\alpha)}. \end{aligned} \tag{3.15}$$

Let $\xi_j := \frac{(j-1)(1-\alpha)}{\eta} - \widetilde{M}_j(1+\alpha)$. Using this and the choice of $\widetilde{M}_j$, we find that $\xi_j \in [0, \tau(1+\alpha))$. Select $\beta_j \in [0,\infty)$ such that $\xi_j = 2^{m-1}\beta_j(1-\alpha)$. From this and (3.5), we infer that

$$\beta_j = 2^{\frac{-1}{p-1}-1-\epsilon}\left(\frac{\gamma}{1+\alpha}\right)^{\frac{1}{p-1}} \frac{\xi_j}{\tau(1+\alpha)}.$$

By this and $\xi_j \in [0, \tau(1+\alpha))$, we conclude that

$$0 \le \beta_j < 2^{-1}\left(\frac{\gamma}{1+\alpha}\right)^{\frac{1}{p-1}} < \frac{1}{2},$$

which, together with the choices of both $\widetilde{M}_j$ and β_j, further implies that (3.14) holds. Finally, for any $d \in \mathbb{N} \cap [1, 2^{m-1}]$, we modify the definition of $P_d^{i,j}$ by setting

$$\begin{aligned} P_d^{i,j} := Q_d^i \times \Big(&t_j - d\left[\tau(1+\alpha) + \beta_j(1-\alpha)\right] l^p - 2l^p, \\ &t_j - d\left[\tau(1+\alpha) + \beta_j(1-\alpha)\right] l^p\Big). \end{aligned}$$

If $\widetilde{M}_j \in \mathbb{N}$, then, for any $d \in \mathbb{N} \cap [N, N+\widetilde{M}_j - 1]$, let $P_{d+1}^{i,j} := P_d^{i,j} - (\mathbf{0}, \tau(1+\alpha)l^p)$. Then $P_{N+\widetilde{M}_j}^{i,j} = \mathfrak{R}$. For convenience, let $\widetilde{M}_1 := 0$.

In conclusion, for any $i \in \mathbb{N} \cap [1, 2^{nm}]$ and $j \in \mathbb{N} \cap [1, J]$, we have constructed a chain $\{P_d^{i,j}\}_{d=0}^{N+\widetilde{M}_j}$ starting from $R_{i,j}$ and ending with $\mathfrak{R}$. Let $N_j := N + \widetilde{M}_j$. Applying the definitions of both N and J, (3.15), and (3.5), we obtain

$$\begin{aligned} N_j &= 2^m - 1 + \widetilde{M}_j \le 2^m + \frac{(j-1)(1-\alpha)}{\tau(1+\alpha)} \\ &\le 2^m + \frac{(J-1)(1-\alpha)}{\tau(1+\alpha)} \le 2^m + \frac{(1-\gamma)2^{pm}}{1-\alpha} \frac{1-\alpha}{\tau(1+\alpha)} \\ &\le 2^{pm+1} \le 2^{\frac{p}{p-1}+3p+1} \left[\frac{\tau(1+\alpha)}{1-\alpha}\right]^p \left[\frac{\tau(1+\alpha)}{\gamma}\right]^{\frac{p}{p-1}}. \end{aligned}$$

From (3.11), the fact that $\beta_j \in [0, \frac{1}{2})$, and the construction of $\{P_d^{i,j}\}_{d=0}^{N_j}$, we deduce that, for any $i \in \mathbb{N} \cap [1, 2^{nm}]$, $j \in \mathbb{N} \cap [1, J]$, and $d \in \mathbb{N} \cap [1, N_j]$,

$$\frac{|(P_d^{i,j})^+(\alpha) \cap (S_{d-1}^{i,j})^-(\alpha)|}{|(P_d^{i,j})^+(\alpha)|} \in \left[\frac{1}{2^{n+1}}, 1\right],$$

Similarly, for any $k \in \mathbb{N} \cap [1, 2^{nm}]$ and $\iota \in \mathbb{N} \cap [1, J]$, we can construct a chain $\{\widetilde{P}_d^{k,\iota}\}_{d=0}^{\widetilde{N}_\iota}$ such that $\widetilde{N}_\iota \in \mathbb{N} \cap [1, 2C_1]$, $\widetilde{P}_0^{k,\iota} = \widetilde{R}_{k,\iota}$, $\widetilde{P}_{\widetilde{N}_\iota}^{k,\iota} = \mathfrak{R}$, and

$$\frac{|(\widetilde{S}_d^{k,\iota})^-(\alpha) \cap (\widetilde{P}_{d-1}^{k,\iota})^+(\alpha)|}{|(\widetilde{S}_d^{k,\iota})^-(\alpha)|} \in \left[\frac{1}{2^{n+1}}, 1\right]$$

for any $d \in \mathbb{N} \cap [1, \widetilde{N}_\iota]$. This finishes the proof of (ii)-(iv) and hence the above claim.

Next, based on (i)-(iv), we can build a chain connecting $R_{i,j}$ and $\widetilde{R}_{k,\iota}$ for any $i, k \in \mathbb{N} \cap [1, 2^{nm}]$ and $j, \iota \in \mathbb{N} \cap [1, J]$. More precisely, for any $i, k \in \mathbb{N} \cap [1, 2^{nm}]$, $j, \iota \in \mathbb{N} \cap [1, J]$, and $d \in \mathbb{Z}_+ \cap [0, \widetilde{N}_\iota + N_j]$, let

$$P_d^{i,j,k,\iota} := \begin{cases} \widetilde{P}_d^{k,\iota} & \text{if } d \in \mathbb{Z}_+ \cap \left[0, \widetilde{N}_\iota\right], \\ P_{\widetilde{N}_\iota + N_j - d}^{i,j} & \text{if } d \in \mathbb{Z}_+ \cap \left[\widetilde{N}_\iota + 1, \widetilde{N}_\iota + N_j\right] \end{cases}$$

and $(S_d^{i,j,k,\iota})^-(\alpha) := (P_d^{i,j,k,\iota})^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(P_d^{i,j,k,\iota})]^p)$. Then, for any $i, k \in \mathbb{N} \cap [1, 2^{nm}]$ and $j, \iota \in \mathbb{N} \cap [1, J]$, we have constructed a chain $\{P_d^{i,j,k,\iota}\}_{d=0}^{\widetilde{N}_\iota + N_j}$ consisting of congruent parabolic rectangles. From (i) through (iv), it follows that the following statements hold.

(v) For any $i, k \in \mathbb{N} \cap [1, 2^{nm}]$, $j, \iota \in \mathbb{N} \cap [1, J]$, and $d \in \mathbb{Z}_+ \cap [0, \widetilde{N}_\iota + N_j]$, $P_d^{i,j,k,\iota} \subset R$.

(vi) For any $i, k \in \mathbb{N} \cap [1, 2^{nm}]$ and $j, \iota \in \mathbb{N} \cap [1, J]$, $P_0^{i,j,k,\iota} = \widetilde{R}_{k,\iota}$ and $P_{\widetilde{N}_\iota + N_j}^{i,j,k,\iota} = R_{i,j}$.

(vii) For any $i, k \in \mathbb{N} \cap [1, 2^{nm}]$, $j, \iota \in \mathbb{N} \cap [1, J]$, and $d \in \mathbb{N} \cap [1, \widetilde{N}_\iota + N_j]$,

$$\frac{|(S_d^{i,j,k,\iota})^-(\alpha) \cap (P_{d-1}^{i,j,k,\iota})^+(\alpha)|}{|(S_d^{i,j,k,\iota})^-(\alpha)|} \in \left[\frac{1}{2^{n+1}}, 1\right].$$

Now, we prove that there exists a positive constant C_2 such that, for any given $i, k \in \mathbb{N} \cap [1, 2^{nm}]$ and $j, \iota \in \mathbb{N} \cap [1, J]$,

$$\fint_{\widetilde{R}_{k,\iota}^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(\widetilde{R}_{k,\iota})]^p)} u^q \le C_2 \fint_{R_{i,j}^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)[l(R_{i,j})]^p)} u^q. \tag{3.16}$$

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Indeed, by [50, Lemma 7.4], [45, Theorems 3.1 and 4.1], and the assumption that $u \in A_\infty^+(\gamma)$, we find that there exist $K, \delta \in (0, \infty)$ such that, for any $R \in \mathcal{R}_p^{n+1}$ and any measurable set $E \subset R^+(\alpha)$,

$$(u^q)\,(R^+(\alpha) - (0, \tau(1+\alpha)l(R)^p)) \le K\left[\frac{|R^+(\alpha)|}{|E|}\right]^\delta (u^q)\,(E).$$

From this and (vii), we infer that, for any $d \in \mathbb{N} \cap [1, \widetilde{N}_\iota + N_j]$,

$$\begin{aligned}\fint_{(S_{d-1}^{i,j,k,\iota})^-(\alpha)} u^q &= \frac{1}{|(S_{d-1}^{i,j,k,\iota})^-(\alpha)|}\,(u^q)\left(\left(S_{d-1}^{i,j,k,\iota}\right)^-(\alpha)\right)\\ &\le \frac{K}{|(S_{d-1}^{i,j,k,\iota})^-(\alpha)|}\left[\frac{|(P_{d-1}^{i,j,k,\iota})^+(\alpha)|}{|(S_d^{i,j,k,\iota})^-(\alpha) \cap (P_{d-1}^{i,j,k,\iota})^+(\alpha)|}\right]^\delta\\ &\quad\times (u^q)\left(\left(S_d^{i,j,k,\iota}\right)^-(\alpha) \cap \left(P_{d-1}^{i,j,k,\iota}\right)^+(\alpha)\right)\\ &\le 2^{(n+1)\delta}K \fint_{(S_d^{i,j,k,\iota})^-(\alpha)} u^q.\end{aligned}$$

Iterating this inequality and using (vi) and (iii), we obtain

$$\begin{aligned}\fint_{\widetilde{R}_{k,\iota}^+(\alpha)-(\mathbf{0},\tau(1+\alpha)[l(\widetilde{R}_{k,\iota})]^p)} u^q &= \fint_{(S_0^{i,j,k,\iota})^-(\alpha)} u^q\\ &\le \left[2^{(n+1)\delta}K\right]^{\widetilde{N}_\iota + N_j} \fint_{(S_{\widetilde{N}_\iota+N_j}^{i,j,k,\iota})^-(\alpha)} u^q\\ &\le \left[2^{(n+1)\delta}K\right]^{3C_1} \fint_{R_{i,j}^+(\alpha)-(\mathbf{0},\tau(1+\alpha)[l(R_{i,j})]^p)} u^q.\end{aligned}$$

Hence (3.16) holds with $C_2 := [2^{(n+1)\delta}K]^{3C_1}$.

To show $(u, v) \in TA_{r,q}^+(\gamma)$, we still need the following estimate. From the fact that $\lceil s \rceil \le 2s$ for any $s \in [1, \infty)$, (3.5), and the definition of J, we deduce that

$$\begin{aligned}2^{nm}J = 2^{nm}\left\lceil\frac{(1-\gamma)2^{pm}}{1-\alpha}\right\rceil &\le \frac{1-\gamma}{1-\alpha}2^{(n+p)m+1}\\ &\le \frac{1-\gamma}{1-\alpha}\left(\frac{1+\alpha}{1-\alpha}\right)^{n+p}\left(\frac{1+\alpha}{\gamma}\right)^{n+p} 2^{\frac{n+p}{p-1}+3(n+p)+1} =: C_3.\end{aligned}\tag{3.17}$$

Applying the obvious facts that

$$R^+(\gamma) = \bigcup_{i=1}^{2^{nm}}\bigcup_{j=1}^{J} V_{i,j}^+ \text{ and } R^-(\gamma) = \bigcup_{k=1}^{2^{nm}}\bigcup_{\iota=1}^{J} U_{k,\iota}^-,$$

$V_{i,j}^+ \subset R_{i,j}^+(\alpha)$ for any $i \in \mathbb{N} \cap [1, 2^{nm}]$ and $j \in \mathbb{N} \cap [1, J]$, (3.6), $U_{k,\iota}^- \subset \widetilde{R}_{k,\iota}^+(\alpha) - (\mathbf{0}, \tau(1+\alpha)l(\widetilde{R}_{k,\iota})^p)$ for any $k \in \mathbb{N} \cap [1, 2^{nm}]$ and $\iota \in \mathbb{N} \cap [1, J]$, (3.7), (3.16), (3.17), (3.1), the definition of J, and the fact that $\lceil s \rceil \le 2s$ for any $s \in [1, \infty)$, we conclude that

$$\begin{aligned}&\fint_{R^-(\gamma)} u^q \left[\fint_{R^+(\gamma)} v^{-r'}\right]^{\frac{q}{r'}}\\ &\quad\le \left[\sum_{k=1}^{2^{nm}}\sum_{\iota=1}^{J}\frac{|U_{k,\iota}^-|}{|R^-(\gamma)|}\fint_{U_{k,\iota}^-} u^q\right]\left[\sum_{i=1}^{2^{nm}}\sum_{j=1}^{J}\frac{|V_{i,j}^+|}{|R^+(\gamma)|}\fint_{V_{i,j}^+} v^{-r'}\right]^{\frac{q}{r'}}\end{aligned}\tag{3.18}$$

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$$
\begin{aligned}
&\le \left(\frac{1}{2^{nm}J}\right)^{1+\frac{q}{r'}}\left[\frac{(1-\alpha)J}{2^{pm}(1-\gamma)}\right]^{1+\frac{q}{r'}}\\
&\quad\times\left[\sum_{k=1}^{2^{nm}}\sum_{\iota=1}^{J}\fint_{\widetilde{R}^+_{k,\iota}(\alpha)-(\mathbf{0},\tau(1+\alpha)[l(\widetilde{R}_{k,\iota})]^p)}u^q\right]\left[\sum_{i=1}^{2^{nm}}\sum_{j=1}^{J}\fint_{R^+_{i,j}(\alpha)}v^{-r'}\right]^{\frac{q}{r'}}\\
&\le \left(\frac{1}{2^{nm}J}\right)^{1+\frac{q}{r'}}\left[\frac{(1-\alpha)J}{2^{pm}(1-\gamma)}\right]^{1+\frac{q}{r'}}\max\left\{1,\ (2^{nm}J)^{\frac{q}{r'}-1}\right\}\\
&\quad\times\sum_{i,k=1}^{2^{nm}}\sum_{j,\iota=1}^{J}\fint_{\widetilde{R}^+_{k,\iota}(\alpha)-(\mathbf{0},\tau(1+\alpha)[l(\widetilde{R}_{k,\iota})]^p)}u^q\left[\fint_{R^+_{i,j}(\alpha)}v^{-r'}\right]^{\frac{q}{r'}}\\
&\le \left(\frac{1}{2^{nm}J}\right)^{1+\frac{q}{r'}}\left[\frac{(1-\alpha)J}{2^{pm}(1-\gamma)}\right]^{1+\frac{q}{r'}}\max\left\{1,\ (2^{nm}J)^{\frac{q}{r'}-1}\right\}C_2\\
&\quad\times\sum_{i,k=1}^{2^{nm}}\sum_{j,\iota=1}^{J}\fint_{R^+_{i,j}(\alpha)-(\mathbf{0},\tau(1+\alpha)[l(R_{i,j})]^p)}u^q\left[\fint_{R^+_{i,j}(\alpha)}v^{-r'}\right]^{\frac{q}{r'}}\\
&\le 2^{1+\frac{q}{r'}}\max\left\{1,\ C_3^{1-\frac{q}{r'}}\right\}C_2C.
\end{aligned}
$$

Taking the supremum over all $R\in\mathcal{R}_p^{n+1}$, we obtain $(u,v)\in TA^+_{r,q}(\gamma)$ and

$$[u,v]_{TA^+_{r,q}(\gamma)}\le 2^{1+\frac{q}{r'}}\max\left\{1,\ C_3^{1-\frac{q}{r'}}\right\}C_2C,$$

which completes the proof of the sufficiency of (i).

Finally, to prove the present theorem, it remains to show the sufficiency of (ii). Indeed, using [34, Exercises 1.1.3(a)] and letting $r\to 1^+$ in (3.18) with the assumption therein replaced by (3.2), we find that the sufficiency of (ii) holds, which then completes the proof of Theorem 3.1. □

Remark 3.2. (i) The assumption that $u\in A^+_\infty(\gamma)$ is only used to prove the sufficiency of Theorem 3.1.

(ii) Theorem 3.1 when both $r=q$ and $u=v$ coincides with [45, Theorem 3.1].

The following is a direct consequence of Theorem 3.1.

Corollary 3.3. *Let $\gamma\in(0,1)$, $1\le r\le q<\infty$, and (u,v) be a pair of nonnegative functions on $\mathbb{R}^{n+1}$. Assume that $u\in A^+_\infty(\gamma)$. Then the following statements hold.*

(i) *For any $\alpha\in(0,1)$, $(u,v)\in TA^+_{r,q}(\gamma)$ if and only if $(u,v)\in TA^+_{r,q}(\alpha)$.*

(ii) *$(u,v)\in TA^+_{r,q}(\gamma)$ if and only if there exists a positive constant C such that, for any $R\in\mathcal{R}_p^{n+1}$,*

$$\fint_{R^{--}(\gamma)}u^q\left[\fint_{R^+(\gamma)}v^{-r'}\right]^{\frac{q}{r'}}\le C,$$

where $R^{--}(\gamma):=R^-(\gamma)-(\mathbf{0},(1+\gamma)[l(R)]^p)$.

Remark 3.4. Corollary 3.3 when $u=v$ coincides with [61, Lemma 2.1(3)].

Using Corollary 3.3, we obtain the following self-improving property of the parabolic Muckenhoupt two-weight class with time lag.

Corollary 3.5. *Let $\gamma\in(0,1)$, $1<r\le q<\infty$, and $(u,v)\in TA^+_{r,q}(\gamma)$. If $u\in A^+_\infty(\gamma)$, then there exists $\delta_0\in(0,\infty)$, depending only on n, p, γ, r, q, and $[u,v]_{TA^+_{r,q}(\gamma)}$, such that, for any $\delta\in(0,\delta_0)$, $(u,v)\in TA^+_{r,q+\delta}(\gamma)$.*

Proof. From the assumption that $u \in A_\infty^+(\gamma)$ and [50, Lemma 7.4], it follows that there exists $q_u \in (1,\infty)$ such that $u^q \in A_{q_u}^+(\gamma)$. By this and [45, Corollary 5.3], we conclude that there exist positive constants C and δ_0, depending only on n, p, γ, q, and the weight constant of u^q, such that, for any $R \in \mathcal{R}_p^{n+1}$,

$$\left[\fint_{R^-(\gamma)} u^{q(1+\delta_0)}\right]^{\frac{1}{1+\delta_0}} \le C \fint_{R^+(\gamma)} u^q,$$

which further implies that

$$\begin{aligned}&\left[\fint_{R^{--}(\gamma)} u^{q(1+\delta_0)}\right]^{\frac{1}{q(1+\delta_0)}} \left[\fint_{R^+(\gamma)} v^{-r'}\right]^{\frac{1}{r'}}\\ &\quad\le C\left[\fint_{R^-(\gamma)} u^q\right]^{\frac{1}{q}} \left[\fint_{R^+(\gamma)} v^{-r'}\right]^{\frac{1}{r'}} \le C[u,v]_{TA_{r,q}^+(\gamma)}^{\frac{1}{q}}.\end{aligned}$$

From this and Corollary 3.3(ii), we infer that $(u,v) \in TA_{r,q+\delta_0}^+(\gamma)$. This, together with Proposition 2.3(ii), further implies that, for any $\delta \in (0,\delta_0)$, $(u,v) \in TA_{r,q+\delta}^+(\gamma)$. This finishes the proof of Corollary 3.5. □

4 Characterizations of Weighted Boundedness of Parabolic Fractional Maximal Operators with Time Lag

In this section, we characterize the parabolic Muckenhoupt (two-weight) class with time lag in terms of the (weak-type) weighted boundedness of the parabolic fractional maximal operator. Recall that, for any given $q \in [1,\infty)$ and any nonnegative locally integrable function ω on $\mathbb{R}^{n+1}$, the *weighted weak Lebesgue space* $L^{q,\infty}(\mathbb{R}^{n+1},\omega)$ is defined to be the set of all measurable functions f on $\mathbb{R}^{n+1}$ such that

$$\|f\|_{L^{q,\infty}(\mathbb{R}^{n+1},\omega)} := \sup_{\lambda\in(0,\infty)} \lambda\left[\omega(\{|f|>\lambda\})\right]^{\frac{1}{q}} < \infty.$$

Specifically, if $\omega \equiv 1$, then $L^{q,\infty}(\mathbb{R}^{n+1},1)$ is exactly the *weak Lebesgue space* and we simply write $L^{q,\infty}(\mathbb{R}^{n+1}) := L^{q,\infty}(\mathbb{R}^{n+1},1)$. The following theorem is the main result of this section. We borrow some ideas and techniques from [29, 45].

Theorem 4.1. *Let $\gamma \in (0,1)$, $\beta \in [0,1)$, $1 \le r \le q < \infty$, $\beta = \frac{1}{r} - \frac{1}{q}$, and (u,v) be a pair of nonnegative functions on $\mathbb{R}^{n+1}$. If $u \in A_\infty^+(\gamma)$, then $(u,v) \in TA_{r,q}^+(\gamma)$ if and only if there exists a positive constant C such that, for any $f \in L^r(\mathbb{R}^{n+1},v^r)$,*

$$\left\|M_\beta^{\gamma+}(f)\right\|_{L^{q,\infty}(\mathbb{R}^{n+1},u^q)} \le C\|f\|_{L^r(\mathbb{R}^{n+1},v^r)}. \tag{4.1}$$

Proof. We first show the sufficiency. Assume that (4.1) holds for any $f \in L^r(\mathbb{R}^{n+1},v^r)$. Fix $R \in \mathcal{R}_p^{n+1}$. From Definition 2.4, we deduce that, for any $\lambda \in (0, |R^+(\gamma)|^\beta (v^{-r'})_{R^+(\gamma)})$ and for almost every $(x,t) \in R^-(\gamma)$,

$$\lambda < \left|R^+(\gamma)\right|^\beta \left(v^{-r'}\right)_{R^+(\gamma)} = \left|R^+(\gamma)\right|^\beta \fint_{R^+(\gamma)} v^{-r'} \le M_\beta^{\gamma+}\left(v^{-r'}\mathbf{1}_{R^+(\gamma)}\right)(x,t),$$

which further implies that $R^-(\gamma) \subset \{M_\beta^{\gamma+}(v^{-r'}\mathbf{1}_{R^+(\gamma)}) > \lambda\}$ up to a set of measure zero. Combining this and (4.1), we obtain

$$\int_{R^-(\gamma)} u^q \le (u^q)\left(\left\{M_\beta^{\gamma+}\left(v^{-r'}\mathbf{1}_{R^+(\gamma)}\right) > \lambda\right\}\right) \le \frac{C}{\lambda^q}\left[\int_{R^+(\gamma)} v^{-r'}\right]^{\frac{q}{r}}.$$

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Letting $\lambda \to |R^+(\gamma)|^\beta \langle v^{-r'}\rangle_{R^+(\gamma)}$, dividing both sides by $|R^+(\gamma)|$, and using $\frac{1}{r} - \frac{1}{q} = \beta$, we find that

$$\fint_{R^-(\gamma)} u^q \left[\fint_{R^+(\gamma)} v^{-r'}\right]^{\frac{q}{r'}} \le C.$$

Taking the supremum over all $R \in \mathcal{R}_p^{n+1}$, we conclude that $(u,v) \in TA_{r,q}^+(\gamma)$ and $[u,v]_{TA_{r,q}^+(\gamma)} \le C$, which completes the proof of the sufficiency.

Next, we prove the necessity. Let $(u,v) \in TA_{r,q}^+(\gamma)$, $f \in L^r(\mathbb{R}^{n+1}, v^r)$, and $\lambda \in (0,\infty)$. We need to show that (4.1) holds. To this end, we divide the proof into the following five steps.

Step 1. In this step, we make the following simplification.

(i) We may assume that f is bounded and has compact support. Indeed, for any $k \in \mathbb{N}$, let

$$f_k := \max\{|f|,\, k\}\, \mathbf{1}_{R(\mathbf{0},0,k)}.$$

Then, for any $k \in \mathbb{N}$, f_k is bounded and has compact support and $f_k \to f$ almost everywhere on $\mathbb{R}^{n+1}$ as $k \to \infty$. From this and the monotone convergence theorem, it follows that

$$\int_{\mathbb{R}^{n+1}} |f_k|^r v^r \to \int_{\mathbb{R}^{n+1}} |f|^r v^r \tag{4.2}$$

as $k \to \infty$. In addition, by an argument similar to that used in the proof of [18, Lemma 3.30] and the monotone convergence theorem again, we find that

$$(u^q)\left(\left\{M_\beta^{\gamma+}(f_k) > \lambda\right\}\right) \to (u^q)\left(\left\{M_\beta^{\gamma+}(f) > \lambda\right\}\right)$$

as $k \to \infty$. From this and (4.2), we deduce that, to prove (4.1) for f, it suffices to show that (4.1) holds for any f_k with $k \in \mathbb{N}$ and the positive constant C independent of k, λ, and f.

(ii) We may assume that both u^q and v^r have a lower bound A for some $A \in (0,\infty)$. Indeed, applying Proposition 2.6, we conclude that $(\max\{u, A^{\frac{1}{q}}\}, \max\{v, A^{\frac{1}{r}}\}) \in TA_{r,q}^+(\gamma)$. If we have

$$(\max\{u^q, A\})\left(\left\{M_\beta^{\gamma+}(f) > \lambda\right\}\right) \lesssim \frac{1}{\lambda^q}\left[\int_{\mathbb{R}^{n+1}} |f|^r \max\{v^r, A\}\right]^{\frac{q}{r}},$$

where the implicit positive constant is independent of A, λ, and f, then we obtain (4.1) by letting $A \to 0$ and taking the supremum over $\lambda \in (0,\infty)$.

(iii) Fix $a \in (0,1)$. Let $M_{\beta,a}^{\gamma+}$ denote the *truncated uncentered parabolic forward in time maximal operator*, that is, for any $g \in L^1_{\rm loc}(\mathbb{R}^{n+1})$ and $(x,t) \in \mathbb{R}^{n+1}$,

$$M_{\beta,a}^{\gamma+}(g)(x,t) := \sup_{\substack{R\in\mathcal{R}_p^{n+1} \\ (x,t)\in R^-(\gamma),\, l(R)\in[a,\infty)}} \left|R^+(\gamma)\right|^\beta \fint_{R^+(\gamma)} |g|.$$

To prove (4.1), we only need to show that

$$(u^q)\left(\left\{M_{\beta,a}^{\gamma+}(f) > \lambda\right\}\right) \lesssim \frac{1}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v^r\right)^{\frac{q}{r}}, \tag{4.3}$$

where the implicit positive constant is independent of a, λ, and f. Indeed, in (4.3), letting $a \to 0$ and taking the supremum over $\lambda \in (0,\infty)$, we then obtain (4.1).

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(iv) To prove (4.3), it suffices to show that

$$(4.4)\qquad (u^q)\left(\left\{\lambda < M^{\gamma+}_{\beta,a}(f) \le 2\lambda\right\}\right) \le \frac{1}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v^r\right)^{\frac{q}{r}},$$

where the implicit positive constant is independent of a, λ, and f. Indeed, from (4.4), we infer that

$$\begin{aligned}(u^q)\left(\left\{M^{\gamma+}_{\beta,a}(f) > \lambda\right\}\right) &\le \sum_{k\in\mathbb{Z}_+} (u^q)\left(\left\{2^k\lambda < M^{\gamma+}_{\beta,a}(f) \le 2^{k+1}\lambda\right\}\right)\\ &\lesssim \sum_{k\in\mathbb{Z}_+} \frac{1}{2^{kq}\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v^r\right)^{\frac{q}{r}}\\ &\sim \frac{1}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v^r\right)^{\frac{q}{r}}\end{aligned}$$

and hence (4.3) holds.

(v) Fix any compact set $K \subset \{\lambda < M^{\gamma+}_{\beta,a}(f) \le 2\lambda\}$. Then, to prove (4.4), it suffices to show

$$(4.5)\qquad (u^q)(K) \lesssim \frac{1}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v\right)^{\frac{q}{r}},$$

where the implicit positive constant is independent of K, a, λ, and f. Indeed, using the inner regularity of the Lebesgue measure (see, for example, [83, Theorem 2.14(d)]) and taking the supremum over all compact subsets of $\{\lambda < M^{\gamma+}_{\beta,a}(f) \le 2\lambda\}$, we then obtain (4.4).

Step 2. In this step, we aim to prove that there exist $N \in \mathbb{N}$ and a sequence $\{P_i\}_{i=1}^N$ of parabolic rectangles such that

$$(4.6)\qquad (u^q)(K) \le 2\sum_{i=1}^N (u^q)\left(P_i^-(\alpha)\right),$$

where $\alpha := \frac{\gamma}{5^p}$. To begin with, from the definitions of both K and $M^{\gamma+}_{\beta,a}$, we deduce that, for any given $(x,t) \in K$, there exists $R_{(x,t)} \in \mathcal{R}_p^{n+1}$ such that $(x,t) \in R^-_{(x,t)}(\gamma)$, $l(R_{(x,t)}) \in [a,\infty)$, and

$$(4.7)\qquad \lambda < \left|R^+_{(x,t)}(\gamma)\right|^\beta \fint_{R^+_{(x,t)}(\gamma)} |f| \le 2\lambda.$$

Using the assumption on f in (i) of Step 1, we find that $f \in L^1(\mathbb{R}^{n+1})$. This, together with (4.7), further implies that

$$\begin{aligned}2^n(1-\gamma)\left[l(R_{(x,t)})\right]^{n+p} = \left|R^+_{(x,t)}(\gamma)\right| &< \left[\frac{1}{\lambda}\int_{R^+_{(x,t)}(\gamma)} |f|\right]^{\frac{1}{1-\beta}}\\ &\le \left[\frac{\|f\|_{L^1(\mathbb{R}^{n+1})}}{\lambda}\right]^{\frac{1}{1-\beta}} < \infty.\end{aligned}$$

Combining this and the fact that $l(R_{(x,t)}) \in [a,\infty)$, we conclude that

$$(4.8)\qquad a \le l\left(R_{(x,t)}\right) \le \left[\frac{\|f\|_{L^1(\mathbb{R}^{n+1})}}{\lambda}\right]^{\frac{1}{(n+p)(1-\beta)}}\left[\frac{1}{2^n(1-\gamma)}\right]^{\frac{1}{n+p}}.$$

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Let $P_{(x,t)} := 5R_{(x,t)}$. Here and thereafter, for any $R := \prod_{i=1}^{n}[y_i - L_i, y_i + L_i] \times (s - L^p, s + L^p) \subset \mathbb{R}^{n+1}$ with $\{y_i\}_{i=1}^n \subset \mathbb{R}$, $\{L_i\}_{i=1}^n \subset (0,\infty)$, $s \in \mathbb{R}$, and $L \in (0,\infty)$ and for any $\Lambda \in (0,\infty)$, define $\Lambda R := \prod_{i=1}^{n}[y_i - \Lambda L_i, y_i + \Lambda L_i] \times (s - (\Lambda L)^p, s + (\Lambda L)^p)$. Then it is easy to verify that $R^+_{(x,t)}(\gamma) \subset P^+_{(x,t)}(\alpha)$ and $R^-_{(x,t)}(\gamma) \subset P^-_{(x,t)}(\alpha)$. In addition, from the fact that K is compact and (4.8), it follows that $\bigcup_{(x,t)\in K} P_{(x,t)}$ is bounded and hence u^q is integrable on $\bigcup_{(x,t)\in K} P_{(x,t)}$. Applying this, the absolute continuity of the Lebesgue integral, (4.8), and (ii) in Step 1, we find that there exists $\epsilon \in (0,1)$ such that, for any $(x,t) \in K$,

$$\begin{aligned}(u^q)\left((1+\epsilon)P^-_{(x,t)}(\alpha) \setminus P^-_{(x,t)}(\alpha)\right)\\ \le A2^n 5^{n+p} a^{n+p}(1-\alpha) \le A\left|P^-_{(x,t)}(\alpha)\right| \le (u^q)\left(P^-_{(x,t)}(\alpha)\right),\end{aligned}$$

which further implies that

$$(u^q)\left((1+\epsilon)P^-_{(x,t)}(\alpha)\right) \le 2\,(u^q)\left(P^-_{(x,t)}(\alpha)\right). \tag{4.9}$$

On the other hand, let

$$\widetilde{\epsilon} := \min\left\{\left(\frac{1-\alpha}{2}\right)^{\frac{1}{p}} [(1+\epsilon)^p - 1]^{\frac{1}{p}},\ \epsilon\right\}. \tag{4.10}$$

From the finite covering theorem and the fact that

$$K \subset \bigcup_{(x,t)\in K} \operatorname{int}(R(x,t,5a\widetilde{\epsilon})),$$

we deduce that there exist $N_1 \in \mathbb{N}$ and $\{(x_k,t_k)\}_{k=1}^{N_1} \subset K$ such that

$$K \subset \bigcup_{k=1}^{N_1} \operatorname{int}(R(x_k,t_k,5a\widetilde{\epsilon})) \subset \bigcup_{k=1}^{N_1} R(x_k,t_k,5a\widetilde{\epsilon}), \tag{4.11}$$

where, for any $E \subset \mathbb{R}^{n+1}$, $\operatorname{int}(E)$ denotes the interior of E.

Now, we select a subsequence of $\{R_{(x_k,t_k)}\}_{k=1}^{N_1}$ by two steps. Assume that, for any $k \in \mathbb{N}\cap[1,N_1]$, $R_{(x_k,t_k)} := R(y_k, s_k, L_k)$ with $(y_k, s_k) \in \mathbb{R}^{n+1}$ and $L_k \in (0,\infty)$. Without loss of generality, we may assume that the tops of $\{R_{(x_k,t_k)}\}_{k=1}^{N_1}$ are monotonically descending, that is, for any $k, j \in \mathbb{N}\cap[1,N_1]$ with $k \le j$, $s_k + L_k^p \ge s_j + L_j^p$; otherwise, we can rearrange $\{R_{(x_k,t_k)}\}_{k=1}^{N_1}$ in terms of the t-coordinates of their tops.

Then we make the first selection inductively. Select $R_{(x_1,t_1)}$ and denote it by $R_{(x_{k_1},t_{k_1})}$. Suppose that we have selected the subsequence $\{R_{(x_{k_i},t_{k_i})}\}_{i=1}^m$ of $\{R_{(x_k,t_k)}\}_{k=1}^{N_1}$, where $m \in \mathbb{N}\cap[1,N_1)$ and $k_m < N_1$. Let

$$j_m := \min\left\{j \in \mathbb{N}:\ k_m + j \le N_1 \text{ and } \left(x_{k_m+j}, t_{k_m+j}\right) \notin \bigcup_{i=1}^m P^-_{(x_{k_i},t_{k_i})}(\alpha)\right\}$$

with the convention $\inf\emptyset = \infty$. If $j_m \in \mathbb{N}$, then select $R_{(x_{k_m+j_m}, t_{k_m+j_m})}$ and denote it by $R_{(x_{k_{m+1}}, t_{k_{m+1}})}$; otherwise, we terminate the selection process. In this manner, we have obtained a subsequence $\{R_{(x_{k_i},t_{k_i})}\}_{i=1}^{N_2}$ of $\{R_{(x_k,t_k)}\}_{k=1}^{N_1}$, where $N_2 \in \mathbb{N}\cap[1,N_1]$. This finishes the first step of selection process.

Without loss of generality, we may assume that the edge lengths of $\{R_{(x_{k_i},t_{k_i})}\}_{i=1}^{N_2}$ are monotonically decreasing; otherwise, we can rearrange $\{R_{(x_{k_i},t_{k_i})}\}_{i=1}^{N_2}$ in terms of their edge lengths. Then we make the second selection inductively. Select $R_{(x_{k_1},t_{k_1})}$ and denote it by $R_{(x_{k_{r_1}},t_{k_{r_1}})}$. Assume that we have selected the subsequence $\{R_{(x_{k_{r_i}},t_{k_{r_i}})}\}_{i=1}^m$ of $\{R_{(x_{k_i},t_{k_i})}\}_{i=1}^{N_2}$, where $m \in \mathbb{N}\cap[1,N_2)$ and $k_{r_m} < N_2$. Let

$$j_{r_m} := \min\Bigg\{j \in \mathbb{N}:\ k_{r_m} + j \le N_2 \text{ and}$$

$$P^-_{(x_{k_{r_m+j}},t_{k_{r_m+j}})}(\alpha)\not\subset\bigcup_{i=1}^m P^-_{(x_{k_{r_i}},t_{k_{r_i}})}(\alpha)\Bigg\}$$

with the convention $\inf\emptyset=\infty$. If $j_{r_m}\in\mathbb{N}$, then select $R_{(x_{k_{r_m+j_{r_m}}},t_{k_{r_m+j_{r_m}}})}$ and denote it by $R_{(x_{k_{r_{m+1}}},t_{k_{r_{m+1}}})}$; otherwise, we stop the selection process. By this way, we have obtained a subsequence $\{R_{(x_{k_{r_i}},t_{k_{r_i}})}\}_{i=1}^N$ of $\{R_{(x_{k_i},t_{k_i})}\}_{i=1}^{N_2}$, where $N\in\mathbb{N}\cap[1,N_2]$, which completes the second step of selection process. For convenience, for any $i\in\mathbb{N}\cap[1,N]$, we simply write $R_{(x_{k_{r_i}},t_{k_{r_i}})}$ as R_i. In conclusion, we have selected a subsequence $\{R_i\}_{i=1}^N$ of $\{R_{(x_k,t_k)}\}_{k=1}^{N_1}$.

According to the above selection, we conclude that the following statements hold.

(i) For any $i,j\in\mathbb{N}\cap[1,N]$ with $i\neq j$, $R_i^-(\gamma)\not\subset R_j^-(\gamma)$, which is a direct consequence of the first selection.

(ii) For any $k\in\mathbb{N}\cap[1,N_1]$, there exists $i\in\mathbb{N}\cap[1,N]$ such that $(x_k,t_k)\in P_i^-(\alpha)$. Indeed, if $R_{(x_k,t_k)}\notin\{R_{(x_{k_i},t_{k_i})}\}_{i=1}^{N_2}$, then, based on the first selection, there exists $i\in\mathbb{N}\cap[1,N_2]$ such that $(x_k,t_k)\in P^-_{(x_{k_i},t_{k_i})}(\alpha)$. If $R_{(x_{k_i},t_{k_i})}\notin\{R_i\}_{i=1}^N$, then, in light of the second selection, we have $P^-_{(x_{k_i},t_{k_i})}(\alpha)\subset\bigcup_{i=1}^N P_i^-(\alpha)$.

(iii) For any $k\in\mathbb{N}\cap[1,N_2]$, there exists $i\in\mathbb{N}\cap[1,N]$ such that

$$R(x_k,t_k,5a\widetilde{\epsilon})\subset(1+\epsilon)P_i^-(\alpha). \tag{4.12}$$

Indeed, let $i\in\mathbb{N}\cap[1,N]$ be such that

$$\begin{aligned}(x_k,t_k)\in P_i^-(\alpha)&=P^-_{(x_{k_{r_i}},t_{k_{r_i}})}(\alpha)\\&=Q\left(y_{k_{r_i}},5L_{k_{r_i}}\right)\times\left(s_{k_{r_i}}-\left(5L_{k_{r_i}}\right)^p,s_{k_{r_i}}-\alpha\left(5L_{k_{r_i}}\right)^p\right).\end{aligned}$$

Then $\|x_k-y_{k_{r_i}}\|_\infty\in[0,5L_{k_{r_i}}]$ and $s_{k_{r_i}}-t_k\in(\alpha(5L_{k_{r_i}})^p,(5L_{k_{r_i}})^p)$. From this, (4.8), and (4.10), we infer that, for any $(y,s)\in R(x_k,t_k,5a\widetilde{\epsilon})$,

$$\begin{aligned}s<t_k+(5a)^p\widetilde{\epsilon}^p&<s_{k_{r_i}}-\alpha\left(5L_{k_{r_i}}\right)^p+(5a)^p\widetilde{\epsilon}^p\\&\le s_{k_{r_i}}+(\widetilde{\epsilon}^p-\alpha)\left(5L_{k_{r_i}}\right)^p\\&\le s_{k_{r_i}}-\frac{1+\alpha}{2}\left(5L_{k_{r_i}}\right)^p+\frac{1-\alpha}{2}(1+\epsilon)^p\left(5L_{k_{r_i}}\right)^p,\end{aligned}$$

$$\begin{aligned}s>t_k-(5a)^p\widetilde{\epsilon}^p&>s_{k_{r_i}}-\left(5L_{k_{r_i}}\right)^p-(5a)^p\widetilde{\epsilon}^p\\&\ge s_{k_{r_i}}-\left(5L_{k_{r_i}}\right)^p(1+\widetilde{\epsilon}^p)\\&\ge s_{k_{r_i}}-\frac{1+\alpha}{2}\left(5L_{k_{r_i}}\right)^p-\frac{1-\alpha}{2}(1+\epsilon)^p\left(5L_{k_{r_i}}\right)^p,\end{aligned}$$

and

$$\begin{aligned}\left\|y-y_{k_{r_i}}\right\|_\infty&\le\|y-x_k\|_\infty+\left\|x_k-y_{k_{r_i}}\right\|_\infty\\&\le5a\widetilde{\epsilon}+5L_{k_{r_i}}\le5L_{k_{r_i}}(1+\epsilon),\end{aligned}$$

which further implies that $(y,s)\in(1+\epsilon)P_i^-(\alpha)$. By the arbitrariness of (y,s), we obtain $R(x_k,t_k,5a\widetilde{\epsilon})\subset(1+\epsilon)P_i^-(\alpha)$.

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(iv) For any given $k \in \mathbb{Z}$ and $R_i, R_j \in \{R_i\}_{i=1}^N$ with $l(R_i), l(R_j) \in (\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, we have

$$R_i^-(\gamma) \cap R_j^-(\gamma) = \emptyset. \tag{4.13}$$

We show this by contradiction. Indeed, from the first selection, we deduce that $R_i^-(\gamma) \not\subset P_j^-(\alpha)$ or $R_j^-(\gamma) \not\subset P_i^-(\alpha)$. Without loss of generality, we may assume that $R_i^-(\gamma) \not\subset P_j^-(\alpha)$. Suppose that there exist $k_0 \in \mathbb{Z}$ and $R_i, R_j \in \{R_i\}_{i=1}^N$ satisfying $l(R_i), l(R_j) \in (\frac{1}{2^{k_0+1}}, \frac{1}{2^{k_0}}]$ and $R_i^-(\gamma) \cap R_j^-(\gamma) \neq \emptyset$. Fix $(y_0, s_0) \in R_i^-(\gamma) \cap R_j^-(\gamma)$. Then, for any $(y, s) \in R_i^-(\gamma) = R^-_{(x_{k_{r_i}}, t_{k_{r_i}})}(\gamma)$,

$$s_{k_{r_j}} - s \geq \gamma \left(L_{k_{r_j}}\right)^p = \alpha \left(5L_{k_{r_j}}\right)^p,$$

$$\begin{aligned}
s - s_{k_{r_j}} &= s - s_0 + s_0 - s_{k_{r_j}} > s - s_0 - \left(L_{k_{r_j}}\right)^p \\
&> -(1-\gamma)\left(L_{k_{r_i}}\right)^p - \left(L_{k_{r_j}}\right)^p \\
&\geq -\left(2^p(1-\gamma)+1\right)\left(L_{k_{r_j}}\right)^p > -\left(5L_{k_{r_j}}\right)^p,
\end{aligned}$$

and

$$\left\|y - y_{k_{r_j}}\right\|_\infty \leq \|y - y_0\|_\infty + \left\|y_0 - y_{k_{r_j}}\right\|_\infty \leq 2L_{k_{r_i}} + L_{k_{r_j}} \leq 5L_{k_{r_j}},$$

which implies that $R_i^-(\gamma) \subset P_j^-(\alpha)$. This contradicts $R_i^-(\gamma) \not\subset P_j^-(\alpha)$. Thus, (4.13) holds.

Using (4.11) and (4.12), we conclude that

$$K \subset \bigcup_{k=1}^{N_1} R(x_k, t_k, 5a\widetilde{\epsilon}) \subset \bigcup_{i=1}^{N} (1+\epsilon) P_i^-(\alpha),$$

which, combined with (4.9), completes the proof of (4.6) and hence Step 2.

Step 3. In this step, we prove that there exists a positive constant C such that, for any $i \in \mathbb{N} \cap [1, N]$,

$$\sum_{j\in\Gamma_i} \left[\int_{R_j^+(\gamma)} |f|\right]^{\frac{1}{1-\beta}} \leq C_1 \left[\int_{R_i^+(\gamma)} |f|\right]^{\frac{1}{1-\beta}}, \tag{4.14}$$

where $\Gamma_i := \{j \in \mathbb{N} \cap [1, N] : \ R_i^+(\gamma) \cap R_j^+(\gamma) \neq \emptyset, \ l(R_j) < l(R_i)\}$. Fix $i \in \mathbb{N} \cap [1, N]$ and let

$$\Gamma_{i,1} := \left\{j \in \Gamma_i : \ R_j^+(\gamma) \not\subset R_i^+(\gamma)\right\} \ \text{ and } \ \Gamma_{i,2} := \left\{j \in \Gamma_i : \ R_j^+(\gamma) \subset R_i^+(\gamma)\right\}.$$

Then $\Gamma_{i,1} \cup \Gamma_{i,2} = \Gamma_i$ and $\Gamma_{i,1} \cap \Gamma_{i,2} = \emptyset$. To show (4.14), it suffices to prove that there exist positive constants C_2 and C_3, depending only on n, p, and γ, such that, for any $h \in \{1, 2\}$,

$$\sum_{j\in\Gamma_{i,h}} \left|R_j^+(\gamma)\right| \leq C_{h+1} \left|R_i^+(\gamma)\right|. \tag{4.15}$$

Indeed, from this and (4.7), we infer that

$$\begin{aligned}
\sum_{j\in\Gamma_i} \left[\int_{R_j^+(\gamma)} |f|\right]^{\frac{1}{1-\beta}} &= \left(\sum_{j\in\Gamma_{i,1}} + \sum_{j\in\Gamma_{i,2}}\right)\left[\int_{R_j^+(\gamma)} |f|\right]^{\frac{1}{1-\beta}} \\
&\leq (2\lambda)^{\frac{1}{1-\beta}} \left(\sum_{j\in\Gamma_{i,1}} + \sum_{j\in\Gamma_{i,2}}\right) \left|R_j^+(\gamma)\right|
\end{aligned}$$

$$\leq (C_2+C_3)(2\lambda)^{\frac{1}{1-\beta}}\left|R_i^+(\gamma)\right|$$
$$\leq (C_2+C_3)2^{\frac{1}{1-\beta}}\left[\int_{R_i^+(\gamma)}|f|\right]^{\frac{1}{1-\beta}}$$

and hence (4.14) holds with $C_1 := (C_2+C_3)2^{\frac{1}{1-\beta}}$.

Next, we turn to show (4.15). Let $k_0 \in \mathbb{Z}$ be such that $l(R_i) \in (\frac{1}{2^{k_0+1}}, \frac{1}{2^{k_0}}]$. We first consider the case $h = 1$. We claim that, for any $k \in \mathbb{Z} \cap [k_0, \infty)$, there exists a measurable set $E_k \subset \mathbb{R}^{n+1}$ such that

$$\bigcup_{j\in\Gamma_{i,1},\, l(R_j)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]} R_j \subset E_k$$

and

$$|E_k| \leq \frac{2\times 3^n(3-\gamma)}{2^{kp}}[l(R_i)]^n + \frac{2^3 3^{n-1}(2-\gamma)n}{2^k}[l(R_i)]^{n-1+p}.$$

Indeed, fix $k \in \mathbb{Z}\cap[k_0,\infty)$. Denote the $2n+2$ faces of $R_i^+(\gamma)$ by $\{S_m\}_{m=1}^{2n+2}$, where S_1 and S_2 denote, respectively, the top and the bottom of $R_i^+(\gamma)$. For any given $j \in \Gamma_{i,1}$ such that $l(R_j) \in (\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, since $R_j^+(\gamma) \not\subset R_i^+(\gamma)$ and $R_j^+(\gamma) \cap R_i^+(\gamma) \neq \emptyset$, it follows that $R_j^+(\gamma)$ intersects with the boundary of $R_i^+(\gamma)$, that is,

$$R_j^+(\gamma) \cap \left(\bigcup_{m=1}^{2n+2} S_m\right) \neq \emptyset. \tag{4.16}$$

For any S_m with $m \in \mathbb{N}\cap[1, 2n+2]$, there exists a rectangle $E_{k,m} \subset \mathbb{R}^{n+1}$ such that

$$\bigcup_{j\in\Gamma_{i,1},\, l(R_j)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}],\, R_j^+(\gamma)\cap S_m\neq\emptyset} R_j \subset E_{k,m}$$

and

$$\left|E_{k,m}\right| = \begin{cases} (3-\gamma)\left[l(R_i)+2l(R_j)\right]^n [l(R_j)]^p & \text{if } m\in\{1,2\},\\ 2l(R_j)\left[l(R_i)+2l(R_j)\right]^{n-1}\Big\{(1-\gamma) & \\ \quad\times[l(R_i)]^p + (3-\gamma)[l(R_j)]^p\Big\} & \text{if } m \in \mathbb{N}\cap[3,2n+2]. \end{cases}$$

Applying this, (4.16), $l(R_j) < l(R_i)$, and $l(R_j) \in (\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, we find that

$$\bigcup_{\substack{j\in\Gamma_{i,1}\\ l(R_j)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]}} R_j = \bigcup_{m=1}^{2n+2}\ \bigcup_{\substack{j\in\Gamma_{i,1}\\ l(R_j)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}],\, R_j^+(\gamma)\cap S_m\neq\emptyset}} R_j \subset \bigcup_{m=1}^{2n+2} E_{k,m} =: E_k$$

and

$$|E_k| \leq \sum_{m=1}^{2n+2}\left|E_{k,m}\right|$$
$$\leq 2[3l(R_i)]^n\frac{3-\gamma}{2^{kp}} + 2n\frac{2}{2^k}[3l(R_i)]^{n-1}(4-2\gamma)[l(R_i)]^p$$
$$= \frac{2\times 3^n(3-\gamma)}{2^{kp}}[l(R_i)]^n + \frac{2^3 3^{n-1}(2-\gamma)n}{2^k}[l(R_i)]^{n-1+p}.$$

This finishes the proof of the above claim.

Observe that, for any $j\in\Gamma_{i,1}$, $l(R_j)<l(R_i)$ and hence

$$\Gamma_{i,1}=\bigcup_{k=k_0}^{\infty}\left\{j\in\Gamma_{i,1}:\ l\left(R_j\right)\in\left(\frac{1}{2^{k+1}},\frac{1}{2^k}\right]\right\}.$$

From this, (iv) in Step 2, the above claim, and $l(R_i)\in(\frac{1}{2^{k_0+1}},\frac{1}{2^{k_0}}]$, we deduce that

$$\begin{aligned}\sum_{j\in\Gamma_{i,1}}\left|R_j^+(\gamma)\right|&=\sum_{k=k_0}^{\infty}\ \sum_{\{j\in\Gamma_{i,1}:\ l(R_j)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]\}}\left|R_j^+(\gamma)\right|\\&=\sum_{k=k_0}^{\infty}\ \sum_{\{j\in\Gamma_{i,1}:\ l(R_j)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]\}}\left|R_j^-(\gamma)\right|\le\sum_{k=k_0}^{\infty}|E_k|\\&\le\sum_{k=k_0}^{\infty}\left\{\frac{3^n(3-\gamma)}{2^{kp-1}}+\frac{3^{n-1}(2-\gamma)n}{2^{k-3}}[l(R_i)]^{p-1}\right\}[l(R_i)]^n\\&=\left\{\frac{2^{2p+1}3^n(3-\gamma)}{2^p-1}\frac{1}{2^{(k_0+1)p}}\right.\\&\qquad\left.+2^5 3^{n-1}(2-\gamma)n[l(R_i)]^{p-1}\frac{1}{2^{k_0+1}}\right\}[l(R_i)]^n\\&\le\left[\frac{2^{2p+1}3^n(3-\gamma)}{2^p-1}+2^5 3^{n-1}(2-\gamma)n\right][l(R_i)]^{n+p}\\&=\frac{2^{2p+1}3^n(3-\gamma)+2^5 3^{n-1}(2-\gamma)(2^p-1)n}{2^n(1-\gamma)(2^p-1)}\left|R_i^+(\gamma)\right|\\&=:C_2\left|R_i^+(\gamma)\right|,\end{aligned}\tag{4.17}$$

which completes the proof of (4.15) when $h=1$.

Now, we prove (4.15) in the case $h=2$. Let

$$\Omega_{k_0}:=\left\{j\in\Gamma_{i,2}:\ l(R_j)\in\left(\frac{1}{2^{k_0+1}},\frac{1}{2^{k_0}}\right]\right\}.$$

For any $k\in\mathbb{N}\cap[k_0+1,\infty)$, we define Ω_k inductively by setting

$$\Omega_k:=\left\{j\in\Gamma_{i,2}:\ l(R_j)\in\left(\frac{1}{2^{k+1}},\frac{1}{2^k}\right]\text{ and }\right.$$
$$\left.R_j^-(\gamma)\cap\bigcup_{d=k_0}^{k-1}\bigcup_{m\in\Omega_d}R_m^-(\gamma)=\emptyset\right\}.$$

Let $\Omega:=\bigcup_{k=k_0}^{\infty}\Omega_k$. Notice that $\{\Omega_k\}_{k=k_0}^{\infty}$ are pairwise disjoint. In addition, for any $j\in\Omega$, we have $R_j^+(\gamma)\subset R_i^+(\gamma)$ and hence $R_j^-(\gamma)\subset R_i$. From these two facts and (iv) of Step 2, it follows that

$$\begin{aligned}\sum_{j\in\Omega}\left|R_j^+(\gamma)\right|&=\sum_{k=k_0}^{\infty}\sum_{j\in\Omega_k}\left|R_j^+(\gamma)\right|=\sum_{k=k_0}^{\infty}\sum_{j\in\Omega_k}\left|R_j^-(\gamma)\right|\\&=\sum_{k=k_0}^{\infty}\left|\bigcup_{j\in\Omega_k}R_j^-(\gamma)\right|=\left|\bigcup_{k=k_0}^{\infty}\bigcup_{j\in\Omega_k}R_j^-(\gamma)\right|\\&\le\left|\bigcup_{j\in\Gamma_{i,2}}R_j^-(\gamma)\right|\le\left|R_i^-(\gamma)\right|=\left|R_i^+(\gamma)\right|.\end{aligned}\tag{4.18}$$

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Next, we show that, for any $m \in \Gamma_{i,2} \setminus \Omega$, there exists $j \in \Omega$ such that $R_j^-(\gamma) \cap R_m^-(\gamma) \neq \emptyset$, $R_m^-(\gamma) \not\subset R_j^-(\gamma)$, and $l(R_j) \in (l(R_m), \infty)$. Indeed, let $k_{\min} \in [k_0, \infty)$ be the smallest integer such that there exists $j \in \Omega_{k_{\min}}$ satisfying $R_m^-(\gamma) \cap R_j^-(\gamma) \neq \emptyset$. From (i) and (iv) in Step 2, we deduce that $k_{\min} > k_0$ and $R_m^-(\gamma) \not\subset R_j^-(\gamma)$. Moreover, $l(R_j) \in (l(R_m), \infty)$. Otherwise, by the definition of $k_{\min}$ and (iv) in Step 2, we conclude that there exists $k' \in \mathbb{N} \cap [k_0, k_{\min})$ such that $m \in \Omega_{k'}$, which contradicts the assumption that $m \in \Gamma_{i,2} \setminus \Omega$. Thus, $l(R_j) \in (l(R_m), \infty)$.

For any $j \in \Omega$, define

$$\widetilde{\Omega}_j := \left\{m \in \Gamma_{i,2} \setminus \Omega : \ R_m^-(\gamma) \cap R_j^-(\gamma) \neq \emptyset, \ R_m^-(\gamma) \not\subset R_j^-(\gamma), \text{ and } l(R_m) < l(R_j)\right\}.$$

Let $j \in \Omega$. Then there exists $\widetilde{k} \in \mathbb{Z} \cap [k_0, \infty)$ such that $l(R_j) \in (\frac{1}{2^{\widetilde{k}+1}}, \frac{1}{2^{\widetilde{k}}}]$. We claim that, for any $k \in \mathbb{Z} \cap [\widetilde{k}, \infty)$, there exists a measurable set $\widetilde{E}_k \subset \mathbb{R}^{n+1}$ such that

$$\bigcup_{m\in\widetilde{\Omega}_j,\, l(R_m)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]} R_m^-(\gamma) \subset \widetilde{E}_k \tag{4.19}$$

and

$$\left|\widetilde{E}_k\right| \le \frac{2^2 3^n(1-\gamma)}{2^{kp}}[l(R_j)]^n + \frac{2^2 3^n(1-\gamma)n}{2^k}[l(R_j)]^{n-1+p}. \tag{4.20}$$

Indeed, fix $k \in \mathbb{Z} \cap [\widetilde{k}, \infty)$. Denote the $2n+2$ faces of $R_j^-(\gamma)$ by $\{\widetilde{S}_d\}_{d=1}^{2n+2}$, where $\widetilde{S}_1$ and $\widetilde{S}_2$ denote, respectively, the top and the bottom of $R_j^-(\gamma)$. For any given $m \in \widetilde{\Omega}_j$ such that $l(R_m) \in (\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, since $R_m^-(\gamma) \not\subset R_j^-(\gamma)$ and $R_m^-(\gamma) \cap R_j^-(\gamma) \neq \emptyset$, it follows that R_m^- intersects with the boundary of $R_j^-(\gamma)$, that is,

$$R_m^-(\gamma) \cap \left(\bigcup_{d=1}^{2n+2} \widetilde{S}_d\right) \neq \emptyset. \tag{4.21}$$

For any $\widetilde{S}_d$ with $d \in \mathbb{N} \cap [1, 2n+2]$, there exists a rectangle $\widetilde{E}_{k,d} \subset \mathbb{R}^{n+1}$ such that

$$\bigcup_{\substack{m\in\widetilde{\Omega}_j \\ l(R_m)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}],\, R_m^-(\gamma)\cap \widetilde{S}_d\neq\emptyset}} R_m^-(\gamma) \subset \widetilde{E}_{k,d}$$

and

$$\left|\widetilde{E}_{k,d}\right| = \begin{cases} \left[l(R_j) + 2l(R_m)\right]^n 2(1-\gamma)[l(R_m)]^p & \text{if } d \in \{1,2\}, \\ 2l(R_m)\left[l(R_j) + 2l(R_m)\right]^{n-1}\Big\{(1-\gamma) & \\ \quad \times [l(R_j)]^p + 2(1-\gamma)[l(R_m)]^p\Big\} & \text{if } d \in \mathbb{N} \cap [3, 2n+2]. \end{cases}$$

Applying this, (4.21), $l(R_m) < l(R_j)$, and $l(R_m) \in (\frac{1}{2^{k+1}}, \frac{1}{2^k}]$, we obtain

$$\begin{aligned} \bigcup_{\substack{m\in\widetilde{\Omega}_j \\ l(R_m)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]}} R_m^-(\gamma) &= \bigcup_{d=1}^{2n+2} \bigcup_{\substack{m\in\widetilde{\Omega}_j \\ l(R_m)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}],\, R_m^-(\gamma)\cap \widetilde{S}_d\neq\emptyset}} R_m^-(\gamma) \\ &\subset \bigcup_{d=1}^{2n+2} \widetilde{E}_{k,d} =: \widetilde{E}_k \end{aligned}$$

and

$$\left|\widetilde{E}_k\right| \le \sum_{d=1}^{2n+2} \left|\widetilde{E}_{k,d}\right|$$

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$$\begin{aligned}
&\le 2\left[3l(R_j)\right]^n \frac{2(1-\gamma)}{2^{kp}} + 2n\frac{2}{2^k}\left[3l(R_j)\right]^{n-1} 3(1-\gamma)\left[l(R_j)\right]^p \\
&= \frac{2^2 3^n(1-\gamma)}{2^{kp}}[l(R_j)]^n + \frac{2^2 3^n(1-\gamma)n}{2^k}[l(R_j)]^{n-1+p}.
\end{aligned}$$

This finishes the proofs of (4.19) and (4.20). From this, the fact that

$$\widetilde{\Omega}_k = \bigcup_{k=\widetilde{k}}^{\infty}\left\{m \in \widetilde{\Omega}_k :\ l(R_m) \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]\right\},$$

(iv) in Step 2, and $l(R_j) \in (\frac{1}{2^{\widetilde{k}+1}}, \frac{1}{2^{\widetilde{k}}}]$, we infer that

$$\begin{aligned}
\sum_{m\in\widetilde{\Omega}_j} \left|R_m^+(\gamma)\right| &= \sum_{k=\widetilde{k}}^{\infty} \sum_{\{m\in\widetilde{\Omega}_j:\ l(R_m)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]\}} \left|R_m^+(\gamma)\right| \\
&= \sum_{k=\widetilde{k}}^{\infty} \sum_{\{m\in\widetilde{\Omega}_j:\ l(R_m)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]\}} \left|R_m^-(\gamma)\right| \le \sum_{k=\widetilde{k}}^{\infty} \left|\widetilde{E}_k\right| \\
&\le \sum_{k=\widetilde{k}}^{\infty} \left\{\frac{3^n(1-\gamma)}{2^{kp-2}} + \frac{3^n(1-\gamma)n}{2^{k-2}}[l(R_j)]^{p-1}\right\}[l(R_j)]^n \\
&= \left\{\frac{2^{2p+2}3^n}{2^p-1}\frac{1}{2^{(\widetilde{k}+1)p}} + 2^4 3^n n[l(R_j)]^{p-1}\frac{1}{2^{\widetilde{k}+1}}\right\}[l(R_j)]^n \\
&\le \left[\frac{2^{2p+2}3^n(1-\gamma)}{2^p-1} + 2^4 3^n(1-\gamma)n\right][l(R_j)]^{n+p} \\
&= \frac{2^{2p+2}3^n(1-\gamma) + 2^4 3^n(1-\gamma)(2^p-1)n}{2^n(1-\gamma)(2^p-1)}\left|R_j^+(\gamma)\right| \\
&=: \widetilde{C}_3\left|R_j^+(\gamma)\right|.
\end{aligned}$$

Combining this and (4.18), we conclude that

$$\begin{aligned}
\sum_{j\in\Gamma_{i,2}} \left|R_j^+(\gamma)\right| &= \left(\sum_{j\in\Omega} + \sum_{m\in\Gamma_{i,2}\setminus\Omega}\right)\left|R_j^+(\gamma)\right| \\
&\le \sum_{j\in\Omega}\left|R_j^+(\gamma)\right| + \sum_{j\in\Omega}\sum_{m\in\widetilde{\Omega}_j}\left|R_m^+(\gamma)\right| \\
&\le \sum_{j\in\Omega}\left|R_j^+(\gamma)\right| + \widetilde{C}_3\sum_{j\in\Omega}\left|R_j^+(\gamma)\right| = \left(\widetilde{C}_3+1\right)\sum_{j\in\Omega}\left|R_j^+(\gamma)\right| \\
&\le \left(\widetilde{C}_3+1\right)\left|R_i^+(\gamma)\right| =: C_3\left|R_i^+(\gamma)\right|,
\end{aligned}$$

which completes the proof of (4.15) when $h = 2$. This, together with (4.17), further implies that (4.15) holds and hence (4.14).

Step 4. In this step, we prove that there exists a positive constant C_4, depending only on n and p, such that, for any $i \in \mathbb{N} \cap [1, N]$, there exists a measurable set $F_i \subset R_i^+(\gamma)$ such that

$$\frac{1}{|R_i^+(\gamma)|^{1-\beta}}\int_{F_i}|f| \in \left(\frac{\lambda}{2}, \infty\right) \tag{4.22}$$

and

$$\sum_{i=1}^{N}\mathbf{1}_{F_i} \le C_4. \tag{4.23}$$

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In what follows, for any finite subset $A \subset \mathbb{N}$, we denote the cardinality of A by $\sharp A$. For any given $i \in \mathbb{N} \cap [1, N]$, we define F_i by considering the following two cases for $\sharp\Gamma_i$.

Case 1) $\sharp\Gamma_i \le 2^{\frac{1}{1-\beta}}\lceil C_1\rceil$. In this case, let $F_i := R_i^+(\gamma)$.

Case 2) $\sharp\Gamma_i > 2^{\frac{1}{1-\beta}}\lceil C_1\rceil$. In this case, for any $k \in \mathbb{N}$, let

$$E_i^k := R_i^+(\gamma) \cap \left\{\sum_{j\in\Gamma_i} \mathbf{1}_{R_j^+(\gamma)} \in [k, \infty)\right\}.$$

Notice that, for any $k, k' \in \mathbb{N}$ with $k < k'$, we have

$$E_i^{k'} \subset E_i^k \tag{4.24}$$

and, for any $(x, t) \in R_i^+(\gamma)$,

$$\sum_{k=1}^{\sharp\Gamma_i} \mathbf{1}_{E_i^k}(x, t) = \sum_{j\in\Gamma_i} \mathbf{1}_{R_j^+(\gamma)}(x, t). \tag{4.25}$$

Then we define $F_i := R_i^+(\gamma) \setminus E_i^{2\lceil C_1\rceil}$.

We show that, for any $i \in \mathbb{N} \cap [1, N]$, F_i satisfies (4.22). We consider the following two cases for $\sharp\Gamma_i$.

Case 1) $\sharp\Gamma_i \le 2^{\frac{1}{1-\beta}}\lceil C_1\rceil$. In this case, from the definition of $F_i = R_i^+(\gamma)$ and (4.7), it follows that

$$\frac{1}{|R_i^+(\gamma)|^{1-\beta}} \int_{F_i} |f| = \left|R_i^+(\gamma)\right|^\beta \fint_{R_i^+(\gamma)} |f| \in (\lambda, \infty).$$

Case 2) $\sharp\Gamma_i > 2^{\frac{1}{1-\beta}}\lceil C_1\rceil$. In this case, using (4.24), $E_i^k \subset R_i^+(\gamma)$ for any $k \in \mathbb{N} \cap [1, \sharp\Gamma_i]$, (4.25), and (4.14), we obtain

$$\begin{aligned}
&(\sharp\Gamma_i)^{\frac{\beta}{1-\beta}} 2^{\frac{1}{1-\beta}}\lceil C_1\rceil \left(\int_{E_i^{2\lceil C_1\rceil}} |f|\right)^{\frac{1}{1-\beta}}\\
&\le \left(\sum_{k=1}^{\sharp\Gamma_i} \int_{E_i^k} |f|\right)^{\frac{1}{1-\beta}} = \left[\int_{R_i^+(\gamma)} |f| \sum_{k=1}^{\sharp\Gamma_i} \mathbf{1}_{E_i^k}\right]^{\frac{1}{1-\beta}}\\
&= \left[\int_{R_i^+(\gamma)} |f| \sum_{j\in\Gamma_i} \mathbf{1}_{R_j^+(\gamma)}\right]^{\frac{1}{1-\beta}} \le (\sharp\Gamma_i)^{\frac{\beta}{1-\beta}} \sum_{j\in\Gamma_i} \left[\int_{R_j^+(\gamma)} |f|\right]^{\frac{1}{1-\beta}}\\
&\le (\sharp\Gamma_i)^{\frac{\beta}{1-\beta}} \lceil C_1\rceil \left[\int_{R_i^+(\gamma)} |f|\right]^{\frac{1}{1-\beta}},
\end{aligned}$$

which, together with (4.7) and the definition of F_i, further implies that

$$\int_{F_i} |f| = \int_{R_i^+(\gamma)} |f| - \int_{E_i^{2\lceil C_1\rceil}} |f| \ge \frac{1}{2}\int_{R_i^+(\gamma)} |f| > \frac{\lambda}{2}\left|R_i^+(\gamma)\right|^{1-\beta}.$$

Combing the above two cases, we complete the proof of (4.22).

Now, we turn to prove (4.23). To this end, we first show that, for any $(x, t) \in F_i$,

$$\sum_{j\in\Gamma_i} \mathbf{1}_{R_j^+(\gamma)}(x, t) \le 2^{\frac{1}{1-\beta}}\lceil C_1\rceil. \tag{4.26}$$

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Indeed, if $\sharp\Gamma_i \le 2^{\frac{1}{1-\beta}}\lceil C_1\rceil$, then, for any $(x,t)\in F_i$,

$$\sum_{j\in\Gamma_i}\mathbf{1}_{R_j^+(\gamma)}(x,t)\le \sharp\Gamma_i \le 2^{\frac{1}{1-\beta}}\lceil C_1\rceil. \tag{4.27}$$

If $\sharp\Gamma_i > 2^{\frac{1}{1-\beta}}\lceil C_1\rceil$, then, from (4.25), we deduce that, for any $(x,t)\in F_i$,

$$\sum_{j\in\Gamma_i}\mathbf{1}_{R_j^+}(x,t)=\sum_{k=1}^{2\lceil C_1\rceil}\mathbf{1}_{E_i^k}(x,t)\le 2^{\frac{1}{1-\beta}}\lceil C_1\rceil,$$

which, together with (4.27), completes the proof of (4.26).

Then we prove that there exists a positive constant $\widetilde{C}_4$, depending only on n, p, and γ, such that, for any given $(x,t)\in\mathbb{R}^{n+1}$ and $k\in\mathbb{Z}$,

$$\sum_{i\in\mathbb{N}\cap[1,N],\, l(R_i)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]}\mathbf{1}_{R_i^+(\gamma)}(x,t)\le\widetilde{C}_4. \tag{4.28}$$

To do this, let

$$I^k_{(x,t)} := \left\{i\in\mathbb{N}\cap[1,N]:\ l(R_i)\in\left(\frac{1}{2^{k+1}},\frac{1}{2^k}\right] \text{ and } (x,t)\in R_i^+(\gamma)\right\}$$

and

$$S_{(x,t)} := Q\left(x,\frac{1}{2^{k-1}}\right)\times\left(t-\frac{1}{2^{kp-1}},t+\frac{1}{2^{kp-1}}\right).$$

Then, for any $(x,t)\in\mathbb{R}^{n+1}$,

$$\bigcup_{i\in I^k_{(x,t)}}R_i^-(\gamma)\subset S_{(x,t)}. \tag{4.29}$$

Indeed, by the definition of $I^k_{(x,t)}$, we find that, for any $i\in I^k_{(x,t)}$ and $(y,s)\in R_i^-(\gamma)$,

$$\|y-x\|_\infty\le 2l(R_i)\le\frac{1}{2^{k-1}} \text{ and } |t-s|\le 2[l(R_i)]^p\le\frac{1}{2^{kp-1}}.$$

Therefore, $(y,s)\in S_{(x,t)}$ and hence (4.29) holds. From the definitions of $I^k_{(x,t)}$ and $S_{(x,t)}$, (iv) in Step 2, and (4.29), we infer that

$$\begin{aligned}
\sum_{\substack{i\in\mathbb{N}\cap[1,N]\\ l(R_i)\in(\frac{1}{2^{k+1}},\frac{1}{2^k}]}}\mathbf{1}_{R_i^+(\gamma)}(x,t) &= \sum_{i\in I^k_{(x,t)}}1=\sum_{i\in I^k_{(x,t)}}\frac{|R_i^-(\gamma)|}{2^n(1-\gamma)[l(R_i)]^{n+p}}\\
&\le\frac{2^{(k+1)(n+p)-n}}{1-\gamma}\left|\bigcup_{i\in I^k_{(x,t)}}R_i^-(\gamma)\right|\\
&\le\frac{2^{(k+1)(n+p)-n}}{1-\gamma}\left|S_{(x,t)}\right|\\
&=\frac{2^{(k+1)(n+p)-n}}{1-\gamma}\left(\frac{2}{2^{k-1}}\right)^n\frac{2}{2^{kp-1}}\\
&=\frac{2^{2n+p+2}}{1-\gamma}=:\widetilde{C}_4,
\end{aligned}$$

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which completes the proof of (4.28).

Finally, we show (4.23). For any $(x,t) \in \bigcup_{i=1}^N F_i$, there exists $i_0 \in \mathbb{N}\cap[1,N]$ such that $(x,t) \in F_{i_0}$ and R_{i_0} is the largest parabolic rectangle in the following sense: for any $i \in \mathbb{N}\cap[1,N]$ satisfying $(x,t) \in F_i$, $|R_i| \le |R_{i_0}|$. Let $k_0 \in \mathbb{Z}$ be such that $l(R_{i_0}) \in (\frac{1}{2^{k_0+1}}, \frac{1}{2^{k_0}}]$. Applying the construction of i_0, $F_i \subset R_i^+(\gamma)$ for any $i \in \mathbb{N}\cap[1,N]$, the definition of Γ_{i_0}, (4.26), and (4.28), we obtain

$$\sum_{i=1}^N \mathbf{1}_{F_i} = \left\{ \sum_{\substack{i\in\mathbb{N}\cap[1,N] \\ l(R_i)\in(\frac{1}{2^{k_0+1}},\frac{1}{2^{k_0}}]}} + \sum_{\substack{i\in\mathbb{N}\cap[1,N] \\ l(R_i)\in(\frac{1}{2^{k+1}},\frac{1}{2^{k}}]}} \right\} \mathbf{1}_{F_i}$$
$$\le \left\{ \sum_{\substack{i\in\mathbb{N}\cap[1,N] \\ l(R_i)\in(\frac{1}{2^{k_0+1}},\frac{1}{2^{k_0}}]}} + \sum_{i\in\Gamma_{i_0}} \right\} \mathbf{1}_{R_i^+(\gamma)} \le \widetilde{C}_4 + 2^{\frac{1}{1-\beta}}\lceil C_1 \rceil =: C_4.$$

This finishes the proof of (4.23) and hence Step 4.

Step 5. In this step, we prove (4.5) by considering the following two cases for r.

Case 1) $r \in (1,\infty)$. In this case, from (4.6), (4.22), the Hölder inequality, $F_i \subset R_i^+(\gamma)$, $\alpha = \frac{\gamma}{5^p}$, $P_i = 5R_i$ for any $i \in \mathbb{N}\cap[1,N]$, $\frac{1}{r}-\frac{1}{q}=\beta$, Definition 2.1(i), and (4.23), we deduce that

$$\begin{aligned}
(u^q)(K) &\le 2\sum_{i=1}^N (u^q)\left(P_i^-(\alpha)\right) \\
&\le \frac{2^{q+1}}{\lambda^q}\sum_{i=1}^N (u^q)\left(P_i^-(\alpha)\right)\left[\frac{1}{|R_i^+(\gamma)|^{1-\beta}}\int_{F_i}|f|\right]^q \\
&\le \frac{2^{q+1}}{\lambda^q}\sum_{i=1}^N \frac{1}{|R_i^+(\gamma)|^{q-\beta q}}\int_{P_i^-(\alpha)} u^q\left(\int_{F_i} v^{-r'}\right)^{\frac{q}{r'}}\left(\int_{F_i}|f|^r v^r\right)^{\frac{q}{r}} \\
&\le \frac{2^{q+1}5^{(n+p)(1+\frac{q}{r'})}(1-\alpha)^{1+\frac{q}{r'}}}{(1-\gamma)^{1+\frac{q}{r'}}\lambda^q} \\
&\quad\times\sum_{i=1}^N \fint_{P_i^-(\alpha)} u^q\left[\fint_{P_i^+(\alpha)} v^{-r'}\right]^{\frac{q}{r'}}\left(\int_{F_i}|f|^r v^r\right)^{\frac{q}{r}} \\
&\le \frac{2^{q+1}5^{(n+p)(1+\frac{q}{r'})}(1-\alpha)^{1+\frac{q}{r'}}[u,v]_{TA^+_{r,q}(\alpha)}}{(1-\gamma)^{1+\frac{q}{r'}}\lambda^q} \\
&\quad\times\sum_{i=1}^N\left(\int_{F_i}|f|^r v^r\right)^{\frac{q}{r}} \\
&\le \frac{2^{q+1}5^{(n+p)(1+\frac{q}{r'})}(5^p-\gamma)^{1+\frac{q}{r'}}[u,v]_{TA^+_{r,q}(\alpha)}C_4^{\frac{q}{r}}}{(1-\gamma)^{1+\frac{q}{r'}}5^{p(1+\frac{q}{r'})}\lambda^q} \\
&\quad\times\left(\int_{\mathbb{R}^{n+1}}|f|^r v^r\right)^{\frac{q}{r}}.
\end{aligned}\tag{4.30}$$

This, together with Corollary 3.3, finishes the proof of (4.5) in this case.

Case 2) $r=1$. In this case, by (4.6), (4.22), $\alpha = \frac{\gamma}{5^p}$, $P_i = 5R_i$ for any $i \in \mathbb{N}\cap[1,N]$, $\frac{1}{r}-\frac{1}{q}=\beta$, Definition 2.1(ii), and (4.23), we conclude that

$$(u^q)(K) \le 2\sum_{i=1}^N (u^q)\left(P_i^-(\alpha)\right)$$

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$$\begin{aligned}
&\leq \frac{2^{q+1}}{\lambda^q}\sum_{i=1}^N (u^q)\left(P_i^-(\alpha)\right)\left[\frac{1}{|R_i^+(\gamma)|^{1-\beta}}\int_{F_i}|f|\right]^q\\
&\leq \frac{2^{q+1}5^{n+p}(1-\alpha)}{(1-\gamma)\lambda^q}\sum_{i=1}^N (u^q)_{P_i^-(\alpha)}\left(\int_{F_i}|f|\right)^q\\
&\leq \frac{2^{q+1}5^{n+p}(1-\alpha)[u,v]_{TA_{1,q}^+(\alpha)}}{(1-\gamma)\lambda^q}\sum_{i=1}^N\left(\int_{F_i}|f|v\right)^q\\
&\leq \frac{2^{q+1}5^{n+p}(5^p-\gamma)[u,v]_{TA_{1,q}^+(\alpha)}C_4^q}{5^p\lambda^q}\left(\int_{\mathbb{R}^{n+1}}|f|v\right)^q.
\end{aligned}$$

This, together with Corollary 3.3, finishes the proof of (4.5) in this case. Combining this and (4.30), we obtain (4.5). This finishes the proof of the necessity and hence Theorem 4.1. □

Remark 4.2. Theorem 4.1 when both $r = q$ and $u = v$ coincides with [45, Theorem 6.1].

The proof of Theorem 4.1 also works for the case $\gamma = 0$ and we present this result as follows.

Theorem 4.3. *Let $\beta \in [0,1)$, $1 \leq r \leq q < \infty$, $\beta = \frac{1}{r} - \frac{1}{q}$, and (u,v) be a pair of nonnegative functions on $\mathbb{R}^{n+1}$. Then $(u,v) \in TA_{r,q}^+(0)$ if and only M_β^{0+} is bounded from $L^r(\mathbb{R}^{n+1}, v^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$.*

Notice that the condition $u \in A_\infty^+(\gamma)$ is only used to show the necessity of Theorem 4.1. Moreover, using [61, Lemma 2.1 (1)] and [49, Lemma 7.4], we find that, for any $1 \leq r \leq q < \infty$ and any nonnegative function ω on $\mathbb{R}^{n+1}$, $\omega \in A_{r,q}^+(\gamma)$ implies that $\omega \in A_\infty^+(\gamma)$. These, combined with Theorem 4.1 with $u = v$, the self-improving property of $A_{r,q}^+(\gamma)$ with $1 < r \leq q < \infty$ (see [61, Lemma 2.2]), and the Stein–Weiss interpolation theorem (see, for example, [7, Corollary 5.5.2]), further implies the following corollary. We omit the details.

Corollary 4.4. *Let $\gamma \in (0,1)$, $\beta \in [0,1)$, $1 < r \leq q < \infty$, $\beta = \frac{1}{r} - \frac{1}{q}$, and ω be a weight on $\mathbb{R}^{n+1}$. Then $\omega \in A_{r,q}^+(\gamma)$ if and only if $M_\beta^{\gamma+}$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$.*

Remark 4.5. Corollary 4.4 when both $r = q$ and $u = v$ coincides with [45, Theorem 6.2].

The following definition of centered parabolic fractional maximal operators can be found in [61, p. 187].

Definition 4.6. Let $\gamma, \beta \in [0,1)$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, the *centered forward in time parabolic fractional maximal function $\mathcal{M}_\beta^{\gamma+}(f)$ with time lag* and the *centered back in time parabolic fractional maximal function $\mathcal{M}_\beta^{\gamma-}(f)$ with time lag* of f are defined, respectively, by setting, for any $(x,t) \in \mathbb{R}^{n+1}$,

$$\mathcal{M}_\beta^{\gamma+}(f)(x,t) := \sup_{L\in(0,\infty)} \left|R(x,t,L)^+(\gamma)\right|^\beta \fint_{R(x,t,L)^+(\gamma)} |f|$$

and

$$\mathcal{M}_\beta^{\gamma-}(f)(x,t) := \sup_{L\in(0,\infty)} \left|R(x,t,L)^-(\gamma)\right|^\beta \fint_{R(x,t,L)^-(\gamma)} |f|.$$

At the end of this section, we prove the following weak-type parabolic two-weighted boundedness of centered parabolic fractional maximal operators with time lag.

Theorem 4.7. *Let $\gamma, \beta \in [0,1)$, $1 < r \leq q < \infty$, $\beta = \frac{1}{r} - \frac{1}{q}$, and (u,v) be a pair of nonnegative functions on $\mathbb{R}^{n+1}$.*

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(i) *If* $\gamma \in (0,1)$ *and* $u \in A_\infty^+(\gamma)$, *then* $(u,v) \in TA_{r,q}^+(\gamma)$ *if and only if* $\mathcal{M}_\beta^{\gamma+}$ *is bounded from* $L^r(\mathbb{R}^{n+1}, v^r)$ *to* $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$.

(ii) $(u,v) \in TA_{r,q}^+(0)$ *if and only if* $\mathcal{M}_\beta^{0+}$ *is bounded from* $L^r(\mathbb{R}^{n+1}, v^r)$ *to* $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$.

Proof. We first show the sufficiency of both (i) and (ii). Let $\gamma \in [0,1)$. Assume $\mathcal{M}_\beta^{\gamma+}$ is bounded from $L^r(\mathbb{R}^{n+1}, v^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$. Fix $R := R(x,t,L) \in \mathcal{R}_p^{n+1}$ with $(x,t) \in \mathbb{R}^{n+1}$ and $L \in (0,\infty)$. Define $S^+(\gamma) := R^-(\gamma) + (1-\gamma)L^p + 2^p\gamma L^p$ and, for any $\epsilon \in (0,\infty)$, let $f_\epsilon := (v+\epsilon)^{-r'}\mathbf{1}_{S^+(\gamma)}$. Then, for any $(y,s) \in R^-(\gamma)$, $S^+(\gamma) \subset R(y,s,2L)^+(\gamma)$ and $|R(y,s,2L)^+(\gamma)| = 2^{n+p}|S^+(\gamma)|$. From this and Definition 2.4, it follows that, for any $\lambda \in (0, 2^{(n+p)(\beta-1)}|S^+(\gamma)|^\beta (f_\epsilon)_{S^+(\gamma)})$ and $(y,s) \in R^-(\gamma)$,

$$\begin{aligned}\lambda &< 2^{(n+p)(\beta-1)}\left|S^+(\gamma)\right|^\beta (f_\epsilon)_{S^+(\gamma)}\\ &\le \left|R(y,s,2L)^+(\gamma)\right|^\beta \fint_{R(y,s,2L)^+(\gamma)} f_\epsilon \le M_\beta^{\gamma+}(f_\epsilon)(y,s),\end{aligned}$$

which further implies that $R^-(\gamma) \subset \{M_\beta^{\gamma+}(f_\epsilon) > \lambda\}$. Combining this and the assumption that $\mathcal{M}_\beta^{\gamma+}$ is bounded from $L^r(\mathbb{R}^{n+1}, v^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$, we obtain

$$\begin{aligned}\int_{R^-(\gamma)} u^q &= (u^q)(R^-(\gamma)) \le (u^q)\left(\left\{M_\beta^{\gamma+}(f_\epsilon) > \lambda\right\}\right)\\ &\le \frac{C}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} f_\epsilon^r v^r\right)^{\frac{q}{r}} = \frac{C}{\lambda^q}\left[\int_{S^+(\gamma)} (v+\epsilon)^{-r'r} v^r\right]^{\frac{q}{r}}.\end{aligned}$$

Letting $\lambda \to 2^{(n+p)(\beta-1)}|S^+(\gamma)|^\beta (f_\epsilon)_{S^+(\gamma)}$ and $\epsilon \to 0$, dividing both sides of the above inequality by $|R^+(\gamma)|$, using $\frac{1}{r} - \frac{1}{q} = \beta$, and taking the supremum over all $R \in \mathcal{R}_p^{n+1}$, we find that

$$\sup_{R\in\mathcal{R}_p^{n+1}} \fint_{R^-(\gamma)} u^q \left[\fint_{S^+(\gamma)} v^{-r'}\right]^{\frac{q}{r'}} \le \frac{C}{2^{(n+p)(\beta-1)q}}.$$

Thus, if $\gamma = 0$, this and Definition 2.1(i) imply $(u,v) \in TA_{r,q}^+(0)$ and, if $\gamma \in (0,1)$, then applying this and Theorem 3.1 [here we use the assumption that $u \in A_\infty^+(\gamma)$] we also conclude $(u,v) \in TA_{r,q}^+(\gamma)$, which then completes the proof of the sufficiency of both (i) and (ii).

Next, we prove the necessity of (i). Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. From Corollary 3.3(i) and Theorem 4.1, we infer that, to show that $\mathcal{M}_\beta^{\gamma+}$ is bounded from $L^r(\mathbb{R}^{n+1}, v^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$, it remains to prove that there exists a positive constant K, depending only on n, p, γ, and β, such that, for any $(x,t) \in \mathbb{R}^{n+1}$,

$$\mathcal{M}_\beta^{\gamma+}(f)(x,t) \le K M_\beta^{\frac{\gamma}{4}+}(f)(x,t). \tag{4.31}$$

Indeed, fix $L \in (0,\infty)$ and let $P := R(x, t + \frac{\gamma L^p}{2}, L)$. Then we are easy to show that $(x,t) \in P^-(\frac{\gamma}{4})$, $R(x,t,L)^+(\gamma) \subset P^+(\frac{\gamma}{4})$, and $|P^+(\frac{\gamma}{4})| = \frac{1-\frac{\gamma}{4}}{1-\gamma}|R(x,t,L)^+(\gamma)|$. This further implies that

$$\begin{aligned}&\left|R(x,t,L)^+(\gamma)\right|^\beta \fint_{R(x,t,L)^+(\gamma)} |f|\\ &\le \left(\frac{1-\gamma}{1-\frac{\gamma}{4}}\right)^{\beta-1}\left|P^+\left(\frac{\gamma}{4}\right)\right|^\beta \fint_{P^+(\frac{\gamma}{4})} |f| \le \left(\frac{1-\gamma}{1-\frac{\gamma}{4}}\right)^{\beta-1} M_\beta^{\frac{\gamma}{4}+}(f)(x,t).\end{aligned}$$

Taking the supremum over all $L \in (0,\infty)$, we obtain (4.31), which completes the proof of necessity of (i).

The proof of the necessity of (ii) is a slight modification of that of (i) with Theorem 4.1 replaced by Theorem 4.3; we omit the details. This finishes the proof of Theorem 4.7. □

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Remark 4.8. In the case $u = v$, Theorem 4.7 when $r = q$ coincides with [50, Lemma 4.2] and when $r < q$ coincides with [61, Theorem 1.1].

The following is a simple corollary of Theorem 4.7 with $u = v$, the self-improving property of $A^+_{r,q}(\gamma)$ with $1 < r \le q < \infty$, and the Stein–Weiss interpolation theorem, which has been obtained in [50, Theorem 5.4] when $\beta = 0$ and in [61, Theorem 1.3] when $\beta \in (0, 1)$.

Corollary 4.9. *Let $\gamma, \rho \in (0, 1)$, $\beta \in [0, 1)$, $1 < r \le q < \infty$, $\beta = \frac{1}{r} - \frac{1}{q}$, and ω be a weight on $\mathbb{R}^{n+1}$. Then $\omega \in A^+_{r,q}(\gamma)$ if and only if $\mathcal{M}^{\rho+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$.*

5 Characterizations of Weighted Boundedness of Parabolic Fractional Integrals with Time Lag

In this section, we introduce the parabolic forward in time and back in time fractional integral operators with time lag. Then we give the weak-type parabolic two-weighted inequality and the strong-type parabolic weighted inequality for such operators. Recall that the *parabolic distance* d_p on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is defined by setting, for any $(x, t), (y, s) \in \mathbb{R}^{n+1}$,

$$d_p((x,t),(y,s)) := \max\left\{\|x - y\|_\infty,\ |t - s|^{\frac{1}{p}}\right\}.$$

We then introduce the definitions of parabolic forward in time fractional integrals with time lag and parabolic back in time fractional integrals with time lag as follows.

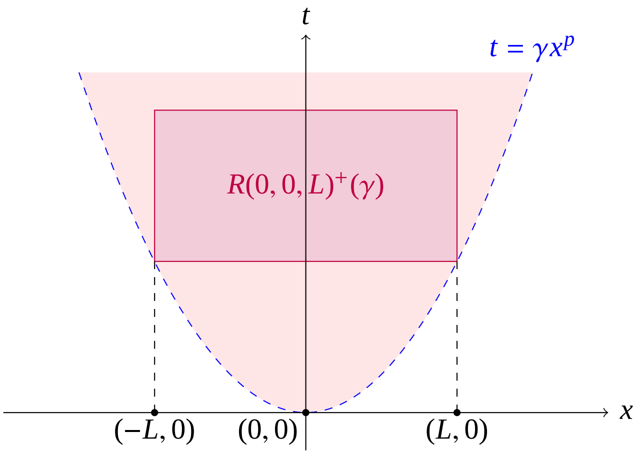
Definition 5.1. Let $\gamma, \beta \in [0, 1)$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, the *parabolic forward in time fractional integral $I^{\gamma+}_\beta(f)$ with time lag* and the *parabolic back in time fractional integral $I^{\gamma-}_\beta(f)$ with time lag* of f are defined, respectively, by setting, for any $(x, t) \in \mathbb{R}^{n+1}$,

$$I^{\gamma+}_\beta(f)(x,t) := \int_{\bigcup_{L\in(0,\infty)} R(\mathbf{0},0,L)^+(\gamma)} \frac{f(x-y, t-s)}{[d_p((y,s),(\mathbf{0},0))]^{(n+p)(1-\beta)}}\,dy\,ds$$

and

$$I^{\gamma-}_\beta(f)(x,t) := \int_{\bigcup_{L\in(0,\infty)} R(\mathbf{0},0,L)^-(\gamma)} \frac{f(x-y, t-s)}{[d_p((y,s),(\mathbf{0},0))]^{(n+p)(1-\beta)}}\,dy\,ds.$$

Remark 5.2. The integral domain $\bigcup_{L\in(0,\infty)} R(\mathbf{0}, 0, L)^+(\gamma)$ in Definition 5.1 is the area above the surface $t = \gamma\|x\|^p_\infty$. In particular, when $n = 1$, the integral domain is exactly the area above the parabola $t = \gamma|x|^p$; see the following figure.



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We have the following relation between the parabolic fractional integral with time lag and the centered parabolic fractional maximal operator with time lag.

Lemma 5.3. *Let $\gamma,\beta \in [0,1)$. Then there exists a positive constant C, depending only on n, γ, and β, such that, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $(x,t) \in \mathbb{R}^{n+1}$,*

$$\mathcal{M}^{\gamma+}_{\beta}(f)(x,t) \le C I^{\gamma+}_{\beta}(|f|)(x,t). \tag{5.1}$$

Proof. Let $(x,t) \in \mathbb{R}^{n+1}$ and $L_0 \in (0,\infty)$. Simply denote $R(x,t,L_0)$ by R. Observe that, for any $(y,s) \in R^+(\gamma)$, $d_p((x,t),(y,s)) \le L$. From this, the fact that $|R^+(\gamma)| = 2^n(1-\gamma)L^{n+p}$, and Definition 5.1, we deduce that

$$\begin{aligned}\left|R^+(\gamma)\right|^{\beta} \fint_{R^+(\gamma)} |f| &\le 2^{n(\beta-1)}(1-\gamma)^{\beta-1} \\ &\quad\times \int_{R^+(\gamma)} \frac{|f(y,s)|}{[d_p((x,t),(y,s))]^{(n+p)(1-\beta)}}\,dy\,ds \\ &\le 2^{n(\beta-1)}(1-\gamma)^{\beta-1} I^{\gamma+}_{\beta}(|f|)(x,t).\end{aligned}$$

Taking the supremum over all $L_0 \in (0,\infty)$, we conclude that (5.1) with $C := 2^{n(\beta-1)}(1-\gamma)^{\beta-1}$ holds, which completes the proof of Lemma 5.3. □

The following lemma is a Welland type inequality in the parabolic setting. For the Welland inequality in the elliptic setting, see [92, (2.3)] and [6, (1.2)].

Lemma 5.4. *Let $\gamma,\beta \in (0,1)$ and $\epsilon \in (0,\min\{\beta,\, 1-\beta\})$. Then there exists a positive constant C, depending only on n, p, γ, β, and ϵ, such that, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $(x,t) \in \mathbb{R}^{n+1}$,*

$$I^{\gamma+}_{\beta}(|f|)(x,t) \le C\left[\mathcal{M}^{\gamma^2+}_{\beta-\epsilon}(f)(x,t)\mathcal{M}^{\gamma^2+}_{\beta+\epsilon}(f)(x,t)\right]^{\frac{1}{2}}. \tag{5.2}$$

Proof. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, $(x,t) \in \mathbb{R}^{n+1}$, and $\Omega^{\gamma+}_{(x,t)} := \bigcup_{L\in(0,\infty)} R(x,t,L)^+(\gamma)$. Without loss of generality, we may assume that $\mathcal{M}^{\gamma^2+}_{\beta-\epsilon}(f)(x,t)\mathcal{M}^{\gamma^2+}_{\beta+\epsilon}(f)(x,t) \in (0,\infty)$; otherwise, (5.2) holds automatically. Fix $\delta \in (0,\infty)$ (which will be determined later) and define

$$\Omega_1 := \left\{(y,s) \in \Omega^{\gamma+}_{(x,t)} : \ d_p((x,t),(y,s)) \in [0,\delta)\right\}$$

and $\Omega_2 := \Omega^{\gamma+}_{(x,t)} \setminus \Omega_1$. Then

$$\begin{aligned}I^{\gamma+}_{\beta}(|f|)(x,t) &= \int_{\Omega_1} \frac{|f(y,s)|\,dy\,ds}{[d_p((x,t),(y,s))]^{(n+p)(1-\beta)}} + \int_{\Omega_2} \cdots \\ &=: I(x,t) + II(x,t).\end{aligned} \tag{5.3}$$

We first estimate $I(x,t)$. Let $\eta := (\frac{1}{\gamma})^{\frac{1}{p}}$ and, for any $i \in \mathbb{N}$,

$$V^+_i := Q\left(x, \eta^{-i+2}\delta\right) \times \left(t+\gamma^i\delta^p, t+\gamma^{i-1}\delta^p\right).$$

Then the following four statements hold obviously:

(i) For any $i \in \mathbb{N}$ and $(y,s) \in V^+_i$, $d_p((x,t),(y,s)) \ge (\gamma^i\delta^p)^{\frac{1}{p}} = \eta^{-i}\delta$.

(ii) For any $i, j \in \mathbb{N}$ with $i \ne j$, $V^+_i \cap V^+_j = \emptyset$.

(iii) For any $i \in \mathbb{N}$, the bottom of V^+_i contains the top of V^+_{i+1}.

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(iv) $\Omega_1 \subset \bigcup_{i\in\mathbb{N}} V_i^+$.

For any $i \in \mathbb{N}$, let $R_i := R(x, t, \eta^{-i+2}\delta)$. Then $V_i^+ \subset R_i^+(\gamma^2)$ and $|V_i^+| = \frac{\gamma}{1+\gamma}|R_i^+(\gamma^2)|$. From this, the monotone convergence theorem, (i) through (iv), and the fact that

$$\left|R_i^+\left(\gamma^2\right)\right| = 2^n\left(1-\gamma^2\right)\eta^{2(n+p)}\left(\eta^{-i}\delta\right)^{n+p},$$

it follows that

$$\begin{aligned}
I(x,t) &\le \sum_{i\in\mathbb{N}} \int_{V_i^+} \frac{|f(y,s)|}{[d_p((x,t),(y,s))]^{(n+p)(1-\beta)}}\,dy\,ds \\
&\le \sum_{i\in\mathbb{N}} \left(\eta^{-i}\delta\right)^{(n+p)(\beta-1)} \int_{V_i^+} |f| \\
&\le 2^{n(1+\epsilon-\beta)}\left(1-\gamma^2\right)^{1+\epsilon-\beta}\eta^{2(n+p)(1+\epsilon-\beta)} \\
&\quad \times \sum_{i\in\mathbb{N}} \left(\eta^{-i}\delta\right)^{(n+p)\epsilon} \left|R_i^+(\gamma^2)\right|^{\beta-\epsilon-1} \int_{R_i^+(\gamma^2)} |f| \\
&\le \frac{2^{n(1+\epsilon-\beta)}(1-\gamma^2)^{1+\epsilon-\beta}\eta^{2(n+p)(1+\epsilon-\beta)}}{\eta^{(n+p)\epsilon}-1}\delta^{(n+p)\epsilon}\mathcal{M}^{\gamma^2+}_{\beta-\epsilon}(f)(x,t).
\end{aligned} \tag{5.4}$$

Now, we estimate $II(x,t)$ similarly. For any $i \in \mathbb{N}$, let

$$U_j^+ := Q\left(x, \eta^j\delta\right) \times \left(t + \gamma^{-j+2}\delta^p, t + \gamma^{-j+1}\delta^p\right).$$

Then the following other four statements hold obviously:

(v) For any $j \in \mathbb{N}$ and $(y,s) \in U_j^+$, $d_p((x,t),(y,s)) \ge (\gamma^{-j+2}\delta^p)^{\frac{1}{p}} = \eta^{j-2}\delta$.

(vi) For any $j, k \in \mathbb{N}$ with $j \ne k$, $U_j^+ \cap U_k^+ = \emptyset$.

(vii) For any $j \in \mathbb{N}$, the bottom of U_{j+1}^+ contains the top of U_j^+.

(viii) $\Omega_2 \subset \bigcup_{j\in\mathbb{N}} U_j^+$.

For any $j \in \mathbb{N}$, let $P_i := R(x, t, \eta^j\delta)$. Then $U_j^+ \subset P_j^+(\gamma^2)$ and $|U_j^+| = \frac{\gamma}{1+\gamma}|P_j^+(\gamma^2)|$. From this, the monotone convergence theorem, (v) through (viii), and the fact that

$$\left|P_j^+\left(\gamma^2\right)\right| = 2^n\left(1-\gamma^2\right)\eta^{2(n+p)}\left(\eta^{j-2}\delta\right)^{n+p},$$

we deduce that

$$\begin{aligned}
II(x,t) &\le \sum_{j\in\mathbb{N}} \int_{U_j^+} \frac{|f(y,s)|}{[d_p((x,t),(y,s))]^{(n+p)(1-\beta)}}\,dy\,ds \\
&\le \sum_{j\in\mathbb{N}} \left(\eta^{j-2}\delta\right)^{(n+p)(\beta-1)} \int_{U_j^+} |f| \\
&\le 2^{n(1-\epsilon-\beta)}\left(1-\gamma^2\right)^{1-\epsilon-\beta}\eta^{2(n+p)(1-\epsilon-\beta)} \\
&\quad \times \sum_{j\in\mathbb{N}} \left(\eta^{j-2}\delta\right)^{-(n+p)\epsilon} \left|P_j^+(\gamma^2)\right|^{\beta+\epsilon-1} \int_{P_j^+(\gamma^2)} |f| \\
&\le \frac{2^{n(1-\epsilon-\beta)}(1-\gamma^2)^{1-\epsilon-\beta}\eta^{2(n+p)(1-\epsilon-\beta)}}{\eta^{(n+p)\epsilon}-1}\delta^{-(n+p)\epsilon}\mathcal{M}^{\gamma^2+}_{\beta+\epsilon}(f)(x,t).
\end{aligned}$$

Combining this, (5.3), and (5.4) and choosing

$$\delta := \left[\frac{\mathcal{M}^{\gamma^2+}_{\beta+\epsilon}(f)(x,t)}{\mathcal{M}^{\gamma^2+}_{\beta-\epsilon}(f)(x,t)}\right]^{\frac{1}{2(n+p)\epsilon}},$$

we then obtain (5.2) and hence finish the proof of Lemma 5.4. □

Next, we are ready to present the first main result of this section.

Theorem 5.5. *Let $\gamma, \beta \in (0,1)$, $1 \le r < q < \infty$, $\beta = \frac{1}{r} - \frac{1}{q}$, and (u,v) be a pair of nonnegative functions on $\mathbb{R}^{n+1}$. If $u \in A^+_\infty(\gamma)$, then $(u,v) \in TA^+_{r,q}(\gamma)$ if and only if there exists a positive constant C such that, for any $f \in L^r(\mathbb{R}^{n+1}, v^r)$,*

$$\left\|I^{\gamma+}_\beta(f)\right\|_{L^{q,\infty}(\mathbb{R}^{n+1},u^q)} \le C\|f\|_{L^r(\mathbb{R}^{n+1},v^r)}. \tag{5.5}$$

Proof. We first prove the sufficiency. Assume that (5.5) holds. By this, Lemma 5.3, and Theorem 4.7(i), we conclude that $(u,v) \in TA^+_{r,q}(\gamma)$, which completes the proof of the sufficiency.

Then we show the necessity. Assume that $(u,v) \in TA^+_{r,q}(\gamma)$. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $\lambda \in (0,\infty)$. From Corollary 3.5, we infer that there exists $\delta_0 \in (0,\infty)$ such that $(u,v) \in A^+_{r,q+\delta}(\gamma)$ for any $\delta \in (0,\delta_0)$. Choose $\epsilon \in (0, \min\{\beta, 1-\beta\})$ such that

$$\frac{1}{\frac{1}{r} - (\beta+\epsilon)} - q \in (0,\delta_0) \tag{5.6}$$

and let $q_1, q_2 \in (1,\infty)$ satisfy

$$\frac{1}{q_1} = \frac{1}{r} - (\beta-\epsilon) \quad\text{and}\quad \frac{1}{q_2} = \frac{1}{r} - (\beta+\epsilon). \tag{5.7}$$

Then $1 < q_1 < q < q_2 < q+\delta_0 < \infty$. Applying Lemma 5.4 and (4.31), we find that there exists a positive constant C, depending only on n, p, γ, β, and ϵ, such that

$$\begin{aligned}\left\{\left|I^{\gamma+}_\beta(f)\right| > \lambda\right\} &\subset \left\{I^{\gamma+}_\beta(|f|) > \lambda\right\} \subset \left\{C\left[M^{\frac{\gamma^2}{4}+}_{\beta-\epsilon}(f) M^{\frac{\gamma^2}{4}+}_{\beta+\epsilon}(f)\right]^{\frac12} > \lambda\right\}\\ &\subset \left\{M^{\frac{\gamma^2}{4}+}_{\beta-\epsilon}(f) > \frac{\lambda}{C}\right\} \cup \left\{M^{\frac{\gamma^2}{4}+}_{\beta+\epsilon}(f) > \frac{\lambda}{C}\right\}\end{aligned}$$

and hence

$$\begin{aligned}(u^q)\left(\left\{\left|I^{\gamma+}_\beta(f)\right| > \lambda\right\}\right) &\le (u^q)\left(\left\{M^{\frac{\gamma^2}{4}+}_{\beta-\epsilon}(f) > \frac{\lambda}{C}\right\}\right)\\ &\quad + (u^q)\left(\left\{M^{\frac{\gamma^2}{4}+}_{\beta+\epsilon}(f) > \frac{\lambda}{C}\right\}\right).\end{aligned} \tag{5.8}$$

On the one hand, according to the proven conclusion that $q_1 < q$, Proposition 2.3(ii), and Corollary 3.3(i), we obtain $(u,v) \in TA^+_{r,q_1}(\frac{\gamma^2}{4})$, which, together with (5.7) and Theorem 4.1, further implies that there exists a positive constant K_1, depending only on n, p, r, q, β, ϵ, and $[u,v]_{TA^+_{r,q_1}(\frac{\gamma^2}{4})}$, such that

$$(u^q)\left(\left\{M^{\frac{\gamma^2}{4}+}_{\beta-\epsilon}(f) > \frac{\lambda}{C}\right\}\right) \le \frac{K_1}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v^r\right)^{\frac{q}{r}}. \tag{5.9}$$

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On the other hand, from the proven conclusions that $q < q_2 < q + \delta_0$ and $(u, v) \in TA^+_{r,q+\delta}(\gamma)$ for any $\delta \in (0, \delta_0)$ and from Corollary 3.3(i), it follows that $(u, v) \in TA^+_{r,q_2}(\frac{\gamma^2}{4})$, which, combined with (5.7) and Theorem 4.1, further implies that there exists a positive constant K_2, depending only on n, p, r, q, β, ϵ, and $[u, v]_{TA^+_{r,q_1}(\frac{\gamma^2}{4})}$, such that

$$(u^q)\left(\left\{M^{\frac{\gamma^2}{4}+}_{\beta+\epsilon}(f) > \frac{\lambda}{C}\right\}\right) \le \frac{K_2}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v^r\right)^{\frac{q}{r}}.$$

By this, (5.8), and (5.9), we find that

$$(u^q)\left(\left\{\left|I^{\gamma+}_\beta(f)\right| > \lambda\right\}\right) \le \frac{K_1 + K_2}{\lambda^q}\left(\int_{\mathbb{R}^{n+1}} |f|^r v^r\right)^{\frac{q}{r}}.$$

Taking the supremum over all $\lambda \in (0, \infty)$, we then conclude that (5.5) holds. This finishes the proof of the necessity and hence Theorem 5.5. □

For any given $q \in [1, \infty)$, let $A^+_{1,q}(\gamma)$ be the set of all nonnegative functions ω on $\mathbb{R}^{n+1}$ such that $[\omega]_{A^+_{1,q}(\gamma)} := [\omega^{\frac{1}{q}}, \omega^{\frac{1}{q}}]_{TA^+_{1,q}(\gamma)} < \infty$. The following is a direct consequence of Theorem 5.5 when both $u = v$ and $r = 1$; we omit the details.

Corollary 5.6. *Let $\gamma \in (0, 1)$, $q \in (1, \infty)$, $\beta := 1 - \frac{1}{q}$, and ω be a weight on $\mathbb{R}^{n+1}$. Then $\omega \in A^+_{1,q}(\gamma)$ if and only $I^{\gamma+}_\beta$ is bounded from $L^1(\mathbb{R}^{n+1}, \omega)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, \omega^q)$.*

The second main result of this section is the following strong type parabolic weighted inequalities for parabolic fractional integrals with time lag.

Theorem 5.7. *Let $\gamma, \beta \in (0, 1)$, $1 < r < q < \infty$, $\beta = \frac{1}{r} - \frac{1}{q}$, and ω be a weight on $\mathbb{R}^{n+1}$. Then $\omega \in A^+_{r,q}(\gamma)$ if and only if $I^{\gamma+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$.*

Proof. To show the sufficiency, assume that $I^{\gamma+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$. Then, by Lemma 5.3, we find that $\mathcal{M}^{\gamma+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$, which, together with Corollary 4.9, implies that $\omega \in A^+_{r,q}(\gamma)$. This finishes the proof of the sufficiency.

Now, we prove the necessity. Assume that $\omega \in A^+_{r,q}(\gamma)$. Using Corollary 3.5, we conclude that there exists $\delta_0 \in (0, \infty)$ such that $\omega \in A^+_{r,q+\delta}(\gamma)$ for any $\delta \in (0, \delta_0)$. Fix $\epsilon \in (0, \min\{\beta, 1 - \beta\})$ such that (5.6) holds and let $q_1, q_2 \in (1, \infty)$ satisfy (5.7). Then $1 < q_1 < q < q_2 < q + \delta_0 < \infty$ and

$$\frac{1}{2q_1} + \frac{1}{2q_2} = \frac{1}{q}.$$

From this, Lemma 5.4, and the Hölder inequality, we infer that, for any $f \in L^r(\mathbb{R}^{n+1}, \omega^r)$,

$$\begin{aligned}(5.10)\qquad &\left[\int_{\mathbb{R}^{n+1}} \left|I^{\gamma+}_\beta(f)\right|^q \omega^q\right]^{\frac{1}{q}}\\
&\lesssim \left\{\int_{\mathbb{R}^{n+1}} \left[\mathcal{M}^{\gamma^2+}_{\beta-\epsilon}(f)\right]^{\frac{q}{2}} \omega^{\frac{q}{2}} \left[\mathcal{M}^{\gamma^2+}_{\beta+\epsilon}(f)\right]^{\frac{q}{2}} \omega^{\frac{q}{2}}\right\}^{\frac{1}{q}}\\
&\le \left\{\int_{\mathbb{R}^{n+1}} \left[\mathcal{M}^{\gamma^2+}_{\beta-\epsilon}(f)\right]^{q_1} \omega^{q_1}\right\}^{\frac{1}{2q_1}} \left\{\int_{\mathbb{R}^{n+1}} \left[\mathcal{M}^{\gamma^2+}_{\beta+\epsilon}(f)\right]^{q_2} \omega^{q_2}\right\}^{\frac{1}{2q_2}}\\
&=: \mathrm{I} \times \mathrm{II}.\end{aligned}$$

To estimate I, by the fact that $q_1 < q$ and Proposition 2.3(ii), we find that $\omega \in A^+_{r,q_1}(\gamma)$. From this, (5.7), and Corollary 4.9, we deduce that

$$\mathrm{I} \lesssim \left(\int_{\mathbb{R}^{n+1}} |f|^r \omega^r\right)^{\frac{1}{2r}}. \tag{5.11}$$

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To estimate II, by the proven conclusions that $q_2 - q \in (0, \delta_0)$ and $\omega \in A^+_{r,q+\delta}(\gamma)$ for any $\delta \in (0, \delta_0)$, we obtain $\omega \in A^+_{r,q_2}(\gamma)$. From this, (5.7), and Corollary 4.9, it follows that

$$\mathrm{II} \lesssim \left(\int_{\mathbb{R}^{n+1}} |f|^r \omega^r \right)^{\frac{1}{2r}}.$$

Combining this, (5.11), and (5.10), we find that $I^{\gamma+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$, which completes the proof of the necessity and hence Theorem 5.7. □

The following theorem is a direct consequence of Corollary 3.3(i) and Theorems 4.1 and 5.5; we omit the details.

Theorem 5.8. *Let $\beta \in (0, 1)$ and $1 \le r < q < \infty$ satisfy $\beta = \frac{1}{r} - \frac{1}{q}$. Let $\{\gamma_i\}_{i=1}^3$ be a sequence of $(0, 1)$ and (u, v) be a pair of nonnegative functions on $\mathbb{R}^{n+1}$. Assume that $u \in A^+_\infty(\gamma)$. Then the following statements are mutually equivalent.*

(i) $(u, v) \in TA^+_{r,q}(\gamma_1)$.

(ii) *$M^{\gamma_2+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, v^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$.*

(iii) *$I^{\gamma_3+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, v^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, u^q)$.*

Remark 5.9. When $r \in (1, \infty)$, if we replace the uncentered fractional maximal operator $M^{\gamma_2+}_\beta$ in Theorem 5.8(ii) by the centered fractional maximal operator $\mathcal{M}^{\gamma_2+}_\beta$, then Theorem 5.8 still holds.

The following theorem is a direct consequence of Theorems 4.1, 4.7(i), 5.5, and 5.7 and Corollaries 3.3(i), 4.4, and 4.9; we omit the details.

Theorem 5.10. *Let $\beta \in (0, 1)$ and $1 < r < q < \infty$ satisfy $\beta = \frac{1}{r} - \frac{1}{q}$. Let $\{\gamma_i\}_{i=1}^7$ be a sequence of $(0, 1)$ and ω be a weight on $\mathbb{R}^{n+1}$. Then the following statements are mutually equivalent.*

(i) $\omega \in A^+_{r,q}(\gamma_1)$.

(ii) *$M^{\gamma_2+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, \omega^q)$.*

(iii) *$M^{\gamma_3+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$.*

(iv) *$\mathcal{M}^{\gamma_4+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, \omega^q)$.*

(v) *$\mathcal{M}^{\gamma_5+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$.*

(vi) *$I^{\gamma_6+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^{q,\infty}(\mathbb{R}^{n+1}, \omega^q)$.*

(vii) *$I^{\gamma_7+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1}, \omega^r)$ to $L^q(\mathbb{R}^{n+1}, \omega^q)$.*

6 Applications to Parabolic Weighted Sobolev Embeddings

In this section, we establish the weighted boundedness of the parabolic Riesz potentials and the parabolic Bessel potentials in [4, 5, 33] for some special parabolic Muckenhoupt weights. Then we apply the results to show the corresponding parabolic Sobolev embeddings.

Let $p \in [2, \infty)$ and $\beta \in (0, n + p)$. For any $(x, t) \in \mathbb{R}^{n+1}$, let

$$h_\beta(x, t) := t^{\frac{\beta-n-p}{p(p-1)}} e^{-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}} \mathbf{1}_{(0,\infty)}(t).$$

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As noted in [47], h_p is a solution of the doubly nonlinear parabolic partial differential equation (1.3) in $\mathbb{R}^{n+1}_+$. In particular, if $p = \beta = 2$, then h_2 is the fundamental solution of the heat equation $\frac{\partial u}{\partial t} - \Delta u = 0$ in $\mathbb{R}^{n+1}_+$. Notice that, for any given $\gamma \in (0,1)$ and for any $(y,s) \in \bigcup_{L\in(0,\infty)} R(\mathbf{0},0,L)^+(\gamma)$,

$$\left|h_\beta(y,s)\right| \sim \frac{1}{[d_p((y,s),(\mathbf{0},0))]^{(n+p)(1-\widetilde{\beta})}} \tag{6.1}$$

with the positive equivalence constants depending only on n, p, γ, and β, where

$$\widetilde{\beta} := 1 - \frac{n+p-\beta}{(p-1)(n+p)} \in (0,1).$$

Let $\gamma \in [0,1)$. The *parabolic Riesz potential* $\mathcal{I}^{\gamma+}_\beta$ *with time lag* is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $(x,t) \in \mathbb{R}^{n+1}$,

$$\mathcal{I}^{\gamma+}_\beta(f)(x,t) := \int_{\bigcup_{L\in(0,\infty)} R(\mathbf{0},0,L)^+(\gamma)} f(x-y,t-s)h_\beta(y,s)\,dy\,ds.$$

According to (6.1) and Theorem 5.7, we easily obtain the following proposition.

Proposition 6.1. *Let $p \in [2,\infty)$, $\gamma \in (0,1)$, $\beta \in (0,n+p)$, $1 < r < q < \infty$ with*

$$\frac{1}{r} - \frac{1}{q} = 1 - \frac{n+p-\beta}{(p-1)(n+p)}, \tag{6.2}$$

and ω be a weight on $\mathbb{R}^{n+1}$. Then $\omega \in A^+_{r,q}(\gamma)$ if and only if $\mathcal{I}^{\gamma+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1},\omega^r)$ to $L^q(\mathbb{R}^{n+1},\omega^q)$.

For any measurable functions f, g on $\mathbb{R}^{n+1}$, the *convolution* $f*g$ of f and g is defined by setting, for any $(x,t) \in \mathbb{R}^{n+1}$,

$$(f*g)(x,t) := \int_{\mathbb{R}^{n+1}} f(x-y,t-s)g(y,s)\,dy\,ds.$$

Recall that, for any $\beta \in (0,n+p)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, the parabolic Riesz potential $h_\beta * f$ of f coincides with $\mathcal{I}^{0+}_\beta(f)$. It was proved in [33] (see also [4, 5]) that, for any $1 \le r < q < \infty$ satisfying (6.2), $\mathcal{I}^{0+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1})$ to $L^q(\mathbb{R}^{n+1})$ if $r \in (1,\infty)$ and is bounded from $L^r(\mathbb{R}^{n+1})$ to $L^{q,\infty}(\mathbb{R}^{n+1})$ if $r = 1$.

Next, we consider the parabolic weighted boundedness of the parabolic Riesz potential operator $\mathcal{I}^{0+}_\beta$. Observe that, for any $\gamma \in (0,1)$, $\beta \in (0,n+p)$, and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, $\mathcal{I}^{0+}_\beta(|f|) \ge \mathcal{I}^{\gamma+}_\beta(|f|)$, which, together with Proposition 6.1, further implies the following proposition.

Proposition 6.2. *Let $p \in [2,\infty)$, $\gamma \in (0,1)$, $\beta \in (0,n+p)$, $1 < r < q < \infty$ satisfy* (6.2), *and ω be a weight on $\mathbb{R}^{n+1}$. If the parabolic Riesz potential operator $\mathcal{I}^{0+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1},\omega^r)$ to $L^q(\mathbb{R}^{n+1},\omega^q)$, then $\omega \in A^+_{r,q}(\gamma)$.*

Remark 6.3. Let p, γ, β, r, q, and ω be the same as in Proposition 6.2. An interesting question is whether or not the converse of Proposition 6.2 holds, that is, whether or not $\omega \in A^+_{r,q}(\gamma)$ implies that $\mathcal{I}^{0+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1},\omega^r)$ to $L^q(\mathbb{R}^{n+1},\omega^q)$.

Let $1 < r \le q < \infty$. Recall that, for any given $E \subset \mathbb{R}$ the *one-sided off-diagonal Muckenhoupt class* $A^+_{r,q}(E)$ is defined to be the set of all nonnegative locally integrable functions ω on E such that

$$[\omega]_{A^+_{r,q}(E)} := \sup_{\substack{x\in\mathbb{R},\, h\in(0,\infty)\\ [x-h,x+h]\in E}} \frac{1}{h}\int_{x-h}^{x} [\omega(y)]^q\,dy \left\{\frac{1}{h}\int_x^{x+h} [\omega(y)]^{-r'}\,dy\right\}^{\frac{q}{r'}}$$

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is finite; see, for example, [6, (1.5)] when $E := (0,\infty)$. It can be easily verified that, if $\omega \in A^+_{r,q}(\mathbb{R})$, then, for any $\gamma \in (0,1)$,

$$\sup_{\substack{x\in\mathbb{R}\\ h\in(0,\infty)}} \frac{1}{1-\gamma h}\int_{x-h}^{x-\gamma h}[\omega(y)]^q\,dy\left\{\frac{1}{1-\gamma h}\int_{x+\gamma h}^{x+h}[\omega(y)]^{-r'}\,dy\right\}^{\frac{q}{r'}} \lesssim [\omega]_{A^+_{r,q}(\mathbb{R})} \tag{6.3}$$

with the implicit positive constant depending only on γ, r, and q. Note that the off-diagonal Muckenhoupt class $A_{r,q}(\mathbb{R}^n)$ is defined in (1.5). We provide a partial answer to the question in Remark 6.3 as follows.

Theorem 6.4. *Let $p \in [2,\infty)$, $\beta \in (0, n+p)$, $1 < r < q < \infty$ with*

$$\frac{1}{r}-\frac{1}{q} = 1-\frac{n+p-\beta}{(p-1)(n+p)} =: \widetilde{\beta},$$

and ω be a weight on $\mathbb{R}^{n+1}$. Let $u \in A_{r,q}(\mathbb{R}^n)$, $v \in A^+_{r,q}(\mathbb{R})$, and, for any $(x,t)\in\mathbb{R}^{n+1}$, $\omega(x,t) = u(x)v(t)$. Then the parabolic Riesz potential $\mathcal{I}^{0+}_\beta$ is bounded from $L^r(\mathbb{R}^{n+1},\omega^r)$ to $L^q(\mathbb{R}^{n+1},\omega^q)$.

Proof. Fix $\gamma\in(0,1)$ and $f\in L^r(\mathbb{R}^{n+1},\omega^r)$. From the self-improving property of $A_{r,q}(\mathbb{R}^n)$ (see, for example, [60, Lemma 3.4.2]), we deduce that there exists $\delta_0\in(0,\infty)$, depending only on n, r, q, and $[u]_{A_{r,q}(\mathbb{R}^n)}$, such that, for any $\delta\in(0,\delta_0)$, $u\in A_{r,q+\delta}(\mathbb{R}^n)$. Choose $\epsilon\in(0,\min\{\widetilde{\beta},\,1-\widetilde{\beta}\})$ such that

$$\frac{1}{\frac{1}{r}-(\widetilde{\beta}+\epsilon)} - q \in (0,\delta_0)$$

and let $q_1, q_2 \in (0,\infty)$ satisfy

$$\frac{1}{q_1} = \frac{1}{r}-(\widetilde{\beta}-\epsilon) \quad\text{and}\quad \frac{1}{q_2} = \frac{1}{r}-(\widetilde{\beta}+\epsilon).$$

Then $1 < q_1 < q < q_2 < q+\delta_0 < \infty$. Therefore, $u\in A_{r,q_2}(\mathbb{R}^n)$. On the other hand, by the Hölder inequality, we conclude that $u \in A_{r,q_1}(\mathbb{R}^n)$. Combining these and (6.3), we find that, for any $\rho\in(0,\gamma]$,

$$\omega \in A^+_{r,q_1}(\rho)\cap A^+_{r,q}(\rho)\cap A^+_{r,q_2}(\rho). \tag{6.4}$$

For any $j\in\mathbb{Z}_+$, define

$$\Omega_j := \bigcup_{L\in(0,\infty)} R(\mathbf{0},0,L)^+\left(\frac{\gamma}{2^j}\right).$$

From Remark 5.2, it follows that, for any $j\in\mathbb{N}$ and $(y,s)\in\Omega_j\setminus\Omega_{j-1}$,

$$\frac{\gamma}{n^{\frac{p}{2}}2^j}|y|^p \le \frac{\gamma}{2^j}\|y\|_\infty^p < s \le \frac{\gamma}{2^{j-1}}\|y\|^p_\infty \le \frac{\gamma}{2^{j-1}}|y|^p$$

and hence

$$\begin{aligned} h_\beta(y,s) &= s^{\frac{\beta-n-p}{p(p-1)}}e^{-\frac{p-1}{p}\left(\frac{|y|^p}{ps}\right)^{\frac{1}{p-1}}} \\ &\le \left[\left(\frac{2^j}{\gamma}\right)^{\frac{1}{p}}+1\right]^{\frac{n+p-\beta}{p-1}} \frac{1}{[d_p((y,s),(\mathbf{0},0))]^{\frac{n+p-\beta}{p-1}}}e^{-\frac{p-1}{p}\left(\frac{2^{j-1}}{p\gamma}\right)^{\frac{1}{p-1}}}, \end{aligned}$$

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which, together with the monotone convergence theorem and (6.1), further implies that, for any $(x,t)\in\mathbb{R}^{n+1}$,

$$\begin{aligned}\mathcal{I}_\beta^{0+}(|f|)(x,t) &= \mathcal{I}_\beta^{\gamma+}(|f|)(x,t) \\ &\quad+\sum_{j\in\mathbb{N}}\int_{\Omega_j\setminus\Omega_{j-1}}|f(y,s)|h_\beta(x-y,t-s)\,dy\,ds \\ &\le \left[\left(\frac{1}{\gamma}\right)^{\frac1p}+1\right]^{\frac{n+p-\beta}{p-1}} I_{\widetilde{\beta}}^{\gamma+}(|f|)(x,t) \\ &\quad+\sum_{j\in\mathbb{N}}\left[\left(\frac{2^j}{\gamma}\right)^{\frac1p}+1\right]^{\frac{n+p-\beta}{p-1}} e^{-\frac{p-1}{p}(\frac{2^j}{2p\gamma})^{\frac{1}{p-1}}} I_{\widetilde{\beta}}^{\frac{\gamma}{2^j}+}(|f|)(x,t).\end{aligned}\tag{6.5}$$

Combining this, (6.4), and Theorem 5.7, we conclude that, to show that $\mathcal{I}_\beta^{0+}$ is bounded from $L^r(\mathbb{R}^{n+1},\omega^r)$ to $L^q(\mathbb{R}^{n+1},\omega^q)$, it suffices to prove that there exists a positive constant C, independent of f, such that

$$\begin{aligned}&\sum_{j\in\mathbb{N}}\left[\left(\frac{2^j}{\gamma}\right)^{\frac1p}+1\right]^{\frac{n+p-\beta}{p-1}} e^{-\frac{p-1}{p}(\frac{2^j}{2p\gamma})^{\frac{1}{p-1}}} \left\|I_{\widetilde{\beta}}^{\frac{\gamma}{2^j}+}(|f|)\right\|_{L^q(\mathbb{R}^{n+1},\omega^q)} \\ &\quad\le C\|f\|_{L^r(\mathbb{R}^{n+1},\omega^r)}.\end{aligned}\tag{6.6}$$

To show (6.6), fix $j\in\mathbb{N}$. From an argument similar to that used in the proof of Lemma 5.4 with γ therein replaced by $\frac{\gamma}{2^j}$, we infer that, for any $(x,t)\in\mathbb{R}^{n+1}$,

$$\begin{aligned}I_{\widetilde{\beta}}^{\frac{\gamma}{2^j}+}(|f|)(x,t) &\le \frac{2^{n(1+\epsilon-\widetilde{\beta})+1}}{(\frac{\gamma}{2^j})^{\frac{2(n+p)(1+\epsilon-\widetilde{\beta})}{p}}[(\frac{\gamma}{2^j})^{-\frac{(n+p)\epsilon}{p}}-1]} \\ &\quad\times \mathcal{M}_{\widetilde{\beta}-\epsilon}^{(\frac{\gamma}{2^j})^2+}(f)(x,t)\mathcal{M}_{\widetilde{\beta}+\epsilon}^{(\frac{\gamma}{2^j})^2+}(f)(x,t),\end{aligned}$$

which, together with the Hölder inequality and an argument similar to that used in the proof of (4.31), further implies that

$$\begin{aligned}&\left\|I_{\widetilde{\beta}}^{\frac{\gamma}{2^j}+}(|f|)\right\|_{L^q(\mathbb{R}^{n+1},\omega^q)} \\ &\quad\le 2^{n(1+\epsilon-\widetilde{\beta})+1}\frac{(\frac{2^j}{\gamma})^{\frac{2(n+p)(1+\epsilon-\widetilde{\beta})}{p}}}{(\frac{2^j}{\gamma})^{\frac{(n+p)\epsilon}{p}}-1}\left\|\mathcal{M}_{\widetilde{\beta}-\epsilon}^{(\frac{\gamma}{2^j})^2+}(f)\mathcal{M}_{\widetilde{\beta}+\epsilon}^{(\frac{\gamma}{2^j})^2+}(f)\right\|_{L^q(\mathbb{R}^{n+1},\omega^q)} \\ &\le 2^{n(1+\epsilon-\widetilde{\beta})+1}\frac{(\frac{2^j}{\gamma})^{\frac{2(n+p)(1+\epsilon-\widetilde{\beta})}{p}}}{(\frac{2^j}{\gamma})^{\frac{(n+p)\epsilon}{p}}-1} \\ &\quad\times\left\|\mathcal{M}_{\widetilde{\beta}-\epsilon}^{(\frac{\gamma}{2^j})^2+}(f)\right\|_{L^{q_1}(\mathbb{R}^{n+1},\omega^{q_1})}^{\frac12}\left\|\mathcal{M}_{\widetilde{\beta}+\epsilon}^{(\frac{\gamma}{2^j})^2+}(f)\right\|_{L^{q_2}(\mathbb{R}^{n+1},\omega^{q_2})}^{\frac12} \\ &\le 2^{n(1+\epsilon-\widetilde{\beta})+1}\left[\frac{1-\frac{(\frac{\gamma}{2^j})^2}{4}}{1-(\frac{\gamma}{2^j})^2}\right]^{[1-(\widetilde{\beta}-\epsilon)]+[1-(\widetilde{\beta}+\epsilon)]}\frac{(\frac{2^j}{\gamma})^{\frac{2(n+p)(1+\epsilon-\widetilde{\beta})}{p}}}{(\frac{2^j}{\gamma})^{\frac{(n+p)\epsilon}{p}}-1} \\ &\quad\times\left\|M_{\widetilde{\beta}-\epsilon}^{\frac{(\frac{\gamma}{2^j})^2}{4}+}(f)\right\|_{L^{q_1}(\mathbb{R}^{n+1},\omega^{q_1})}^{\frac12}\left\|M_{\widetilde{\beta}+\epsilon}^{\frac{(\frac{\gamma}{2^j})^2}{4}+}(f)\right\|_{L^{q_2}(\mathbb{R}^{n+1},\omega^{q_2})}^{\frac12}\end{aligned}\tag{6.7}$$

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$$\leq 2^{n(1+\epsilon-\widetilde{\beta})+1}\left[\frac{4-\gamma^2}{4-4\gamma^2}\right]^2 \frac{(\frac{2^j}{\gamma})^{\frac{2(n+p)(1+\epsilon-\widetilde{\beta})}{p}}}{(\frac{2^j}{\gamma})^{\frac{(n+p)\epsilon}{p}}-1}$$
$$\times \left\| M_{\widetilde{\beta}-\epsilon}^{\frac{(\frac{\gamma}{2^j})^2}{4}+}(f)\right\|_{L^{q_1}(\mathbb{R}^{n+1},\omega^{q_1})}^{\frac{1}{2}} \left\| M_{\widetilde{\beta}+\epsilon}^{\frac{(\frac{\gamma}{2^j})^2}{4}+}(f)\right\|_{L^{q_2}(\mathbb{R}^{n+1},\omega^{q_2})}^{\frac{1}{2}}.$$

Applying (6.4) and an argument similar to that used in the proof of Corollary 4.4, we find that there exists a positive constant C_1, depending only on n, p, γ, r, q, $[u]_{A_{r,q}(\mathbb{R}^n)}$, and $[v]_{A^+_{r,q}(\mathbb{R})}$, such that

$$\max\left\{\left\| M_{\widetilde{\beta}-\epsilon}^{\frac{(\frac{\gamma}{2^j})^2}{4}+}(f)\right\|_{L^{q_1}(\mathbb{R}^{n+1},\omega^{q_1})}, \left\| M_{\widetilde{\beta}+\epsilon}^{\frac{(\frac{\gamma}{2^j})^2}{4}+}(f)\right\|_{L^{q_2}(\mathbb{R}^{n+1},\omega^{q_2})}\right\}$$
$$\leq C_1\|f\|_{L^r(\mathbb{R}^{n+1},\omega^r)}.$$

Combining this and (6.7), we obtain

$$\left\| I_{\widetilde{\beta}}^{\frac{\gamma}{2^j}+}(|f|)\right\|_{L^q(\mathbb{R}^{n+1},\omega^q)} \lesssim \frac{(\frac{2^j}{\gamma})^{\frac{2(n+p)(1+\epsilon-\widetilde{\beta})}{p}}}{(\frac{2^j}{\gamma})^{\frac{(n+p)\epsilon}{p}}-1}\|f\|_{L^r(\mathbb{R}^{n+1},\omega^r)},$$

where the implicit positive constant is independent of f and j. From this and (6.7), we deduce that

$$\sum_{j\in\mathbb{N}}\left[\left(\frac{2^j}{\gamma}\right)^{\frac{1}{p}}+1\right]^{\frac{n+p-\beta}{p-1}} e^{-\frac{p-1}{p}(\frac{2^j}{2p\gamma})^{\frac{1}{p-1}}}\left\| I_{\widetilde{\beta}}^{\frac{\gamma}{2^j}+}(|f|)\right\|_{L^q(\mathbb{R}^{n+1},\omega^q)}$$
$$\lesssim \sum_{j\in\mathbb{N}}\left[\left(\frac{2^j}{\gamma}\right)^{\frac{1}{p}}+1\right]^{\frac{n+p-\beta}{p-1}} \frac{(\frac{2^j}{\gamma})^{\frac{2(n+p)(1+\epsilon-\widetilde{\beta})}{p}}}{(\frac{2^j}{\gamma})^{\frac{(n+p)\epsilon}{p}}-1} e^{-\frac{p-1}{p}(\frac{2^j}{2p\gamma})^{\frac{1}{p-1}}}\|f\|_{L^r(\mathbb{R}^{n+1},\omega^r)}$$
$$\lesssim \|f\|_{L^r(\mathbb{R}^{n+1},\omega^r)},$$

and hence (6.6) holds, which completes the proof of Theorem 6.4. □

Remark 6.5. (i) Theorem 6.4 when $\omega \equiv 1$ coincides with [33, Theorem 3.1] .

(ii) For any given $E \subset \mathbb{R}^{n+1}$ and $1 < r < q < \infty$, define $\mathcal{A}_{r,q}(E)$ to be the set of all nonnegative locally integrable functions ω on E such that

$$[\omega]_{\mathcal{A}_{r,q}(E)} := \sup_{\substack{R\in\mathcal{R}_p^{n+1}\\ R\subset E}} \fint_R \omega \left(\fint_R \omega^{-r'}\right)^{\frac{q}{r'}} < \infty.$$

Obviously, $\mathcal{A}_{r,q}(\mathbb{R}^{n+1}) \subset A^+_{r,q}(\mathbb{R}^{n+1})$. By (6.5), the fact that $(\mathbb{R}^{n+1}, d_p, |\cdot|)$ is a space of homogeneous type, and the weighted boundedness of fractional integrals on spaces of homogeneous type (see, for example, [43, Theorem 3.3]), we conclude that, if we replace the condition that there exist $u \in A_{r,q}(\mathbb{R}^n)$ and $v \in A^+_{r,q}(\mathbb{R})$ satisfying $\omega(x,t) = u(x)v(t)$ by $\omega \in \mathcal{A}_{r,q}(\mathbb{R}^{n+1})$, then the conclusion of Theorem 6.4 still holds, that is, both $\mathcal{I}^{0+}_\beta$ and $\mathcal{G}^{0+}_\beta$ are bounded from $L^r(\mathbb{R}^{n+1},\omega^r)$ to $L^q(\mathbb{R}^{n+1},\omega^q)$.

In what follows, we fix $p = 2$. Let $\beta \in (0, n+2)$, $q \in [1,\infty)$, and ω be a weight on $\mathbb{R}^{n+1}$. The *weighted parabolic Sobolev space* $W^{\beta,q}(\mathbb{R}^{n+1},\omega)$ is defined by setting

$$W^{\beta,q}(\mathbb{R}^{n+1},\omega) := \left\{h_\beta * g :\ g \in L^q(\mathbb{R}^{n+1},\omega)\right\}.$$

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For any $f \in W^{\beta,q}(\mathbb{R}^{n+1},\omega)$, define $\|f\|_{W^{\beta,q}(\mathbb{R}^{n+1},\omega)} := \|g\|_{L^q(\mathbb{R}^{n+1},\omega)}$, where $g \in L^q(\mathbb{R}^{n+1},\omega)$ satisfying that $f = h_\beta * g$. Theorem 6.4 and Remark 6.5(ii) immediately imply the following parabolic weighted Sobolev embedding result and we omit the details.

Corollary 6.6. *Let $\beta \in (0, n+2)$, $1 < r < q < \infty$ with $\frac{1}{r} - \frac{1}{q} = \frac{\beta}{n+2}$, and ω be a weight on $\mathbb{R}^{n+1}$. If either of the following two conditions holds:*

(i) *there exist $u \in A_{r,q}(\mathbb{R}^n)$ and $v \in A^+_{r,q}(\mathbb{R})$ such that $\omega(x,t) = u(x)v(t)$;*

(ii) $\omega \in \mathcal{A}_{r,q}(\mathbb{R}^{n+1})$,

then $W^{\beta,r}(\mathbb{R}^{n+1},\omega^r) \subset L^q(\mathbb{R}^{n+1},\omega^q)$. Moreover, there exists a positive constant C such that, for any $f \in W^{\beta,r}(\mathbb{R}^{n+1},\omega^r)$,

$$\|f\|_{L^q(\mathbb{R}^{n+1},\omega^q)} \le C\|f\|_{W^{\beta,r}(\mathbb{R}^{n+1},\omega^r)}.$$

We denote by $\mathcal{S}(\mathbb{R}^{n+1})$ the space of all *Schwartz functions* on $\mathbb{R}^{n+1}$ equipped with a well-known topology determined by a countable family of norms and by $\mathcal{S}'(\mathbb{R}^{n+1})$ the space of all *tempered distributions* endowed with the weak-$*$ topology. In addition, let $\mathcal{S}'(\mathbb{R}^{n+1}_+) := \mathcal{S}'(\mathbb{R}^{n+1})|_{\mathbb{R}^{n+1}_+}$, that is, the restriction of $\mathcal{S}'(\mathbb{R}^{n+1})$ on $\mathbb{R}^{n+1}_+$. Using both the Fourier transform formula of h_β with $\beta \in (0, n+2)$ (see [33, (2.4)]) and several elementary properties of the Fourier transform and replacing ω and f, respectively, by $\omega(x,t)\mathbf{1}_{(0,\infty)}(t)$ and $f(x,t)\mathbf{1}_{(0,\infty)}(t)$ in Theorem 6.4 and Remark 6.5(ii), we obtain the following application of Theorem 6.4, which presents a priori estimate for the nonhomogeneous heat equations. We omit the details.

Corollary 6.7. *Let $1 < r < q < \infty$ with $\frac{1}{r} - \frac{1}{q} = \frac{2}{n+2}$, ω be a weight on $\mathbb{R}^{n+1}_+$, and $f \in L^r(\mathbb{R}^{n+1}_+,\omega^r)$. If either of the following two conditions holds:*

(i) *there exist $u \in A_{r,q}(\mathbb{R}^n)$ and $v \in A^+_{r,q}(\mathbb{R}_+)$ such that $\omega(x,t) = u(x)v(t)$,*

(ii) $\omega \in \mathcal{A}_{r,q}(\mathbb{R}^{n+1}_+)$

and if $g \in \mathcal{S}'(\mathbb{R}^{n+1}_+) \cap L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$ satisfies

$$\begin{cases} \dfrac{\partial g}{\partial t}(x,t) - \Delta g(x,t) = f(x,t), & \forall\,(x,t) \in \mathbb{R}^{n+1}_+, \\ \lim\limits_{t\to 0^+} g(x,t) = 0, & \forall\, x \in \mathbb{R}^n, \end{cases}$$

then $g \in L^q(\mathbb{R}^{n+1}_+,\omega^q)$. Moreover, there exists a positive constant C such that

$$\|g\|_{L^q(\mathbb{R}^{n+1}_+,\omega^q)} \le C\|f\|_{L^r(\mathbb{R}^{n+1}_+,\omega^r)}.$$

Acknowledgements After this article was finished, we found that Cruz-Uribe and Myyryläinen [24] simultaneously also introduced the off-diagonal parabolic Muckenhoupt class with time lag and studied the two-weighted boundedness of the centered parabolic fractional maximal operator with time lag. Except the definition and its two basic properties of the off-diagonal parabolic Muckenhoupt class with time lag, both articles have no substantial overlap.

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Weiyi Kong, Dachun Yang (Corresponding author) and Wen Yuan

Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, The People's Republic of China

E-mails: weiyikong@mail.bnu.edu.cn (W. Kong)
dcyang@bnu.edu.cn (D. Yang)
wenyuan@bnu.edu.cn (W. Yuan)

Chenfeng Zhu

School of Mathematical Sciences, Zhejiang University of Technology, Hangzhou 310023, The People's Republic of China

E-mails: chenfengzhu@zjut.edu.cn