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CUNTZ–PIMSNER ALGEBRAS ASSOCIATED TO TENSOR PRODUCTS OF C^*-CORRESPONDENCES

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Abstract

Given two C^*-correspondences X and Y over C^*-algebras A and B, we show that (under mild hypotheses) the Cuntz–Pimsner algebra $\mathcal{O}_{X\otimes Y}$ embeds as a certain subalgebra of $\mathcal{O}_X \otimes \mathcal{O}_Y$ and that this subalgebra can be described in a natural way in terms of the gauge actions on $\mathcal{O}_X$ and $\mathcal{O}_Y$. We explore implications for graph algebras, crossed products by $\mathbb{Z}$, crossed products by completely positive maps, and give a new proof of a result of Kaliszewski, Quigg, and Robertson related to coactions on correspondences.

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1. Introduction

In what follows, we will attempt to describe the Cuntz–Pimsner algebra of an external tensor product $X \otimes Y$ of correspondences in terms of the Cuntz–Pimsner algebras $\mathcal{O}_X$ and $\mathcal{O}_Y$. In particular we will show that, under suitable conditions, $\mathcal{O}_{X\otimes Y}$ is isomorphic to a certain subalgebra $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$ of $\mathcal{O}_X \otimes \mathcal{O}_Y$. We call this subalgebra the $\mathbb{T}$-*balanced tensor product* because it has the property that $\gamma_z^X(x) \otimes y = x \otimes \gamma_z^Y(y)$ for all $z \in \mathbb{T}$, $x \in \mathcal{O}_X$, and $y \in \mathcal{O}_Y$, where γ^X and γ^Y are the gauge actions on $\mathcal{O}_X$ and $\mathcal{O}_Y$. This idea is inspired by a result of Kumjian in [8] where it is shown that for a cartesian product $E \times F = (E^0 \times F^0, E^1 \times F^1, r_E \times r_F, s_E \times s_F)$ of two graphs, $C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F)$ where the balancing is over the gauge action of the two graphs. Kumjian's proof uses the groupoid model of graph algebras and is therefore independent of our main result. However, we will be able to recover Kumjian's result for row-finite graphs with no sources by considering the C^*-correspondence model of a graph algebra.

After proving our main result we will explore some examples including a generalization of Kumjian's result to the setting of topological graphs, implications for crossed products by $\mathbb{Z}$, crossed product by a completely positive map, and we will give a new proof of a theorem of Kaliszewski, Quigg, and Robertson which was used in [5] to study Cuntz–Pimsner algebras of coaction crossed products. In what follows we will assume that all tensor products are minimal (spatial) unless otherwise stated.

2. Preliminaries

Here we will review the material we will need on correspondences, Cuntz–Pimsner algebras, actions and coactions. A good general reference on the subject of Hilbert C^*-modules and C^*-correspondences is [10]. There is also an overview given in [1]. For Cuntz–Pimsner algebras, we refer the reader to Katsura [7] and to the overview given in [13]. For actions and coactions, we refer to Appendix A of [2] as a general reference, but since we will only be concerned with coactions of discrete groups we will also use a lot of facts from [12]. We will begin by reviewing the basics of C^*-correspondences.

2.1. Correspondences. Suppose that A is a C^*-algebra and X is a right A-module. By an *A-valued inner product* on X we shall mean a map

$$X \times X \ni (x, y) \mapsto \langle x, y\rangle_A \in A$$

which is linear in the second variable and such that:

(1) $\langle x, x\rangle_A \geq 0$ for all $x \in X$, with equality only when $x = 0$;
(2) $\langle x, y\rangle_A^* = \langle y, x\rangle_A$ for all $x, y \in X$;
(3) $\langle x, y \cdot a\rangle_A = \langle x, y\rangle_A a$ for all $x, y \in X$ and $a \in A$.

Note that this implies that $\langle \cdot, \cdot\rangle_A$ is A-linear in the second variable and conjugate A-linear in the first variable. One can prove a version of the Cauchy–Schwarz inequality for such X, which implies that we can define the following norm on X:

$$\|x\|_A := \|\langle x, x\rangle_A\|^{1/2}.$$

If X is a right A-module with an A-valued inner product, X is called a *right Hilbert A-module* if it is complete under the norm $\| \cdot \|_A$ defined above. Note that if $A = \mathbb{C}$ then X is just a Hilbert space and we can think of general Hilbert modules as Hilbert spaces whose scalars are elements of some C^*-algebra A.

Also note that we can make A itself into a right Hilbert A-module by letting the right action of A be given by multiplication in A and an A-valued inner product given by $\langle a, b\rangle_A = a^*b$. We call this the *trivial* Hilbert A-module and denote it by A_A.

Let A be a C^*-algebra and let X be a right Hilbert A-module. If $T : X \to X$ is an A-module homomorphism, then we call T *adjointable* if there is an A-module homomorphism T^* (called the *adjoint* of T) such that

$$\langle T^* x, y\rangle_A = \langle x, Ty\rangle_A$$

for all $x, y \in X$. The operator norm makes the set of all adjointable operators on X into a C^*-algebra which we denote by $\mathcal{L}(X)$.

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Given C^*-algebras A and B, an *A–B-correspondence* is right Hilbert B-module X together with a homomorphism $\phi : A \to \mathcal{L}(X)$ which is called a *left action of A by adjointable operators*. For $a \in A$ and $x \in X$, we will write $a \cdot x$ for $\phi(a)(x)$. If $A = B$ we call this a *correspondence over A* (or B). We call the left-action *injective* if ϕ is injective and *nondegenerate* if $\phi(A)X = X$. We will sometimes write ${}_AX_B$ to indicate that X is an A–B correspondence. Before we continue, we will give a few examples of correspondences.

Example 2.1 [13, Example 8.6]. Let A be a C^* algebra and let α be an endomorphism of A. We can make the trivial module A_A into a correspondence over A by defining $\phi(a)(x) = \alpha(a)x$.

Example 2.2 [6, Definition 3.9]. Let $E = \{E^0, E^1, r, s\}$ be a directed graph (in the sense of [13]). Consider the vector space $c_c(E^1)$ of finitely supported functions on E^1. We can define a right action of $c_0(E^0)$ and a $c_0(E^0)$-valued inner product as follows:

$$\begin{aligned}(x \cdot a)(e) &= x(e)a(s(e)),\\ \langle x, y\rangle_{c_0(E^0)}(v) &= \sum_{\{e\in E^1 : s(e)=v\}} \overline{x(e)}y(e).\end{aligned}$$

We can use the norm defined by this inner product to complete $c_c(E^1)$ into a right Hilbert module $X(E)$. We can define a left action $\phi : c_0(E^0) \to \mathcal{L}(X(E))$ as follows:

$$\phi(a)(x)(e) = a(r(e))x(e).$$

This makes $X(E)$ into a correspondence which is referred to as the *graph correspondence* of E.

This example has the following natural generalization.

Example 2.3 [6, Definition 3.11]. A *topological graph* is a quadruple $E = \{E^0, E^1, r, s\}$ where E^0 and E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is a continuous function, and $s : E^1 \to E^0$ is a local homeomorphism. Let $A := C_0(E^0)$. We can define left and right actions of A and an A-valued inner product on $C_c(E^1)$ similarly to the way we did for ordinary graphs. For $a \in A$ and $x, y \in C_c(E^1)$, let

$$\begin{aligned}(a \cdot x)(e) &:= a(r(e))x(e),\\ (x \cdot a)(e) &:= x(e)a(s(e)),\\ \langle x, y\rangle_A(v) &:= \sum_{\{e\in E^1 : s(e)=v\}} \overline{x(e)}y(e).\end{aligned}$$

We denote the completion of $C_c(E^1)$ under the norm defined by this inner product by $X(E)$. It can be shown [13, pages 80–81] that the left action is injective if and only if r has dense range and the left action is implemented by compacts if and only if r is proper.

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Let X be a Hilbert module over a C^*-algebra A. Then for any $x, y \in X$, the map $\Theta_{x,y} : z \mapsto x \cdot \langle y, z\rangle_A$ is an adjointable operator called a *rank-one* operator. It can be shown that the closed span of the rank-one operators forms an ideal $\mathcal{K}(X)$ in $\mathcal{L}(X)$ which is referred to as the set of *compact operators*. If X is an A–B-correspondence, we say that the left action ϕ of A is *implemented by compacts* if $\phi(A) \subseteq \mathcal{K}(X)$.

There are two types of tensor products which are usually defined on correspondences, an 'internal' tensor product and an 'external' tensor product. These are defined as follows (see [10, Ch. 4] for more detail). Let ${}_AX_B$ and ${}_CY_D$ be correspondences and let $\Phi : B \to C$ be a completely positive map (see [1] for the basics of completely positive maps). Let $X \odot_\Phi Y$ be the quotient of the algebraic tensor product $X \odot Y$ by the subspace spanned by

$$\{x \cdot b \otimes y - x \otimes \Phi(b) \cdot y : x \in X, y \in Y, b \in B\}.$$

This is a right D-module with right action given by $(x \otimes y) \cdot d = (x \otimes y \cdot d)$. We define a D-valued inner product as follows:

$$\langle x \otimes y, x' \otimes y'\rangle = \langle y, \Phi(\langle x, x'\rangle_B) \cdot y'\rangle_D.$$

For a proof that this is indeed an inner product, we refer the reader to the proof of Proposition 4.5 of [10]. We refer to the completion of $X \odot_\Phi Y$ with respect to the norm defined by this inner product as the *internal tensor product* of X and Y and it is denoted by $X \otimes_\Phi Y$. If $\phi_A : A \to \mathcal{L}(X)$ is the left action on X, then we can define a left action of A on $X \otimes_\Phi Y$ by $\phi(a)(x \otimes y) = (\phi_A(a)x) \otimes y$. This makes $X \otimes_\Phi Y$ an A–D-correspondence. In many situations we will have $B = C$ and $\Phi = \mathrm{id}_B$. In this case we will write the associated internal tensor product as $X \otimes_B Y$.

Example 2.4. If $\Phi : A \to B$ is a completely positive map between two C^*-algebras, then we define the *correspondence associated to* Φ to be the correspondence $X_\Phi := {}_AA_A \otimes_\Phi {}_BB_B$ where ${}_AA_A$ and ${}_BB_B$ are the standard correspondences.

Let ${}_AX_B$ and ${}_CY_D$ be correspondences. We can define a right action of $B \otimes D$ and a $B \otimes D$-valued inner product on the algebraic tensor product $X \odot Y$ as follows:

$$(x \otimes y) \cdot (a \otimes b) = (x \cdot a) \otimes (y \cdot b),$$
$$\langle x \otimes y, x' \otimes y'\rangle = \langle x, x'\rangle_B \otimes \langle y, y'\rangle_D.$$

The completion of $X \odot Y$ with respect to the norm defined by this inner product is called the *external tensor product* of X and Y, which we will denote simply by $X \otimes Y$. We can define a left action of $A \otimes C$ as follows: $\phi(a \otimes c)(x \otimes y) = (\phi_A(a)x) \otimes (\phi_C(c)y)$. This makes $X \otimes Y$ an $A \otimes C$–$B \otimes D$-correspondence.

If $\Phi : A \to C$ and $\Phi' : B \to D$ are completely positive maps, then $\Phi \otimes \Phi' : A \otimes B \to C \otimes D$ will be a completely positive map as well. In fact we have the following lemma.

Lemma 2.5. *Let ${}_AX_{A'}$, ${}_BY_{B'}$, ${}_CZ_{C'}$, and ${}_DW_{D'}$ be correspondences and let $\Phi : A' \to C$ and $\Phi' : B' \to D$ be completely positive maps. Then*

$$(X \otimes_\Phi Z) \otimes (Y \otimes_{\Phi'} W) \cong (X \otimes Y) \otimes_{\Phi \otimes \Phi'} (Z \otimes W).$$

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Proof. Note that the left-hand side is the completion of $X \odot Z \odot Y \odot W$ under a the norm defined by a certain pre-inner product and the right-hand side is the completion of $X \odot Y \odot Z \odot W$ under the norm defined by a certain pre-inner product. We can show that the linear map

$$\begin{aligned}\sigma_{23} : X \odot Z \odot Y \odot Z &\to X \odot Y \odot Z \odot W\\ x \otimes z \otimes y \otimes z &\mapsto x \otimes y \otimes z \otimes w\end{aligned}$$

extends to a correspondence isomorphism

$$(X \otimes_{\Phi} Z) \otimes (Y \otimes_{\Phi'} W) \to (X \otimes Y) \otimes_{\Phi \otimes \Phi'} (Z \otimes W)$$

by showing that σ_{23} preserves the pre-inner products. By linearity it suffices to show this for elementary tensors. Let $\langle \cdot, \cdot \rangle_1$ denote the pre-inner product which gives rise to $(X \otimes_{\Phi} Z) \otimes (Y \otimes_{\Phi'} W)$. Let $\langle \cdot, \cdot \rangle_2$ denote the pre-inner product which gives rise to $(X \otimes Y) \otimes_{\Phi \otimes \Phi'} (Z \otimes W)$. Then

$$\begin{aligned}&\langle \sigma_{23}(x \otimes z \otimes y \otimes w), \sigma_{23}(x \otimes z \otimes y \otimes w)\rangle_2\\ &\quad = \langle x \otimes y \otimes z \otimes w, x' \otimes y' \otimes z' \otimes w'\rangle_2\\ &\quad = \langle z \otimes w, (\Phi \otimes \Phi')(\langle x \otimes y, x' \otimes y'\rangle)(z' \otimes w')\rangle\\ &\quad = \langle z \otimes w, (\Phi \otimes \Phi')(\langle x, x'\rangle \otimes \langle y, y'\rangle)(z' \otimes w')\rangle\\ &\quad = \langle z \otimes w, (\Phi(\langle x, x'\rangle)z') \otimes (\Phi'(\langle y, y'\rangle)w')\rangle\\ &\quad = \langle z, \Phi(\langle x, x'\rangle)z'\rangle \otimes \langle w, \Phi'(\langle y, y'\rangle)w'\rangle\\ &\quad = \langle x \otimes z \otimes y \otimes w, x' \otimes z' \otimes y' \otimes z'\rangle_1.\end{aligned}$$

Thus σ_{23} extends to an isomorphism giving us

$$(X \otimes_{\Phi} Z) \otimes (Y \otimes_{\Phi'} W) \cong (X \otimes Y) \otimes_{\Phi \otimes \Phi'} (W \otimes Z). \qquad \square$$

Example 2.6. Let $\Phi : A \to C$ and $\Phi' : B \to D$ be completely positive maps. Since ${}_A A_A \otimes {}_B B_B = {}_{A\otimes B}(A \otimes B)_{A\otimes B}$ and ${}_C C_C \otimes {}_D D_D = {}_{C\otimes D}(C \otimes D)_{C\otimes D}$, we have

$$\begin{aligned}X_{\Phi\otimes\Phi'} &= ({}_{A\otimes B}(A \otimes B)_{A\otimes B}) \otimes_{\Phi\otimes\Phi'} ({}_{C\otimes D}(C \otimes D)_{C\otimes D})\\ &= ({}_A A_A \otimes {}_B B_B) \otimes_{\Phi\otimes\Phi'} ({}_C C_C \otimes {}_D D_D).\end{aligned}$$

Applying the preceding lemma gives

$$\begin{aligned}&\cong ({}_A A_A \otimes_{\Phi} {}_C C_C) \otimes ({}_B B_B \otimes_{\Phi'} {}_D D_D)\\ &= X_{\Phi} \otimes X_{\Phi'}.\end{aligned}$$

Thus $X_{\Phi\otimes\Phi'} \cong X_{\Phi} \otimes X_{\Phi'}$.

The following facts (see [10, Ch. 4]) will be useful in proving our main result:

Lemma 2.7. *Let X and Y be C^*-correspondences over C^*-algebras A and B, respectively. Then $\mathcal{K}(X \otimes Y) \cong \mathcal{K}(X) \otimes \mathcal{K}(Y)$ via the map κ which takes $S \otimes T \in \mathcal{K}(X) \otimes \mathcal{K}(Y)$ to the linear map $x \otimes y \mapsto Sx \otimes Ty$. Further, if the left actions of A and B are injective and implemented by compacts then so is the left action of $A \otimes B$ on $X \otimes Y$.*

Before we end this section, we will give another example of an external tensor product.

EXAMPLE 2.8. Let $E = \{E^0, E^1, r, s\}$ and $F := \{F^0, F^1, r', s'\}$ be topological graphs. Define

$$E \times F := \{E^0 \times F^0, E^1 \times F^1, r \times r', s \times s'\}$$

(i.e. the topological analog of the product graph in [8]). Since the product of two continuous maps is continuous and the product of two local homeomorphisms is a local homeomorphism, $E \times F$ is a topological graph. Let $\rho : C_0(E^0) \otimes C_0(F^0) \to C_0(E^0 \times F^0)$ and $\sigma : C_0(E^1) \otimes C_0(F^1) \to C_0(E^1 \times F^1)$ be the standard isomorphisms. Note that $\sigma(C_c(E^1) \otimes C_c(F^1)) \subseteq C_c(E^1 \times F^1)$ and that

$$\begin{aligned}\sigma((a \otimes b) \cdot (x \otimes y))(e, f) &= \sigma(a \cdot x \otimes b \cdot y)(e, f)\\ &= a(r(e))x(e)b(r'(f))y(f)\\ &= a(r(e))b(r'(f))x(e)y(f)\\ &= \rho(a \otimes b)(r(e), r'(f))\sigma(x \otimes y)(e, f).\end{aligned}$$

Similarly,

$$\sigma((x \otimes y) \cdot (a \otimes b))(e, f) = \sigma(x \otimes y)(e, f)\rho(a \otimes b)(s(e), s'(f)).$$

Let $A = C_0(E^0)$ and $B = C_0(F^0)$. If $\langle \cdot, \cdot \rangle_1$ is the $A \otimes B$-valued tensor product associated to the external tensor product $X(E) \otimes X(F)$ and $\langle \cdot, \cdot \rangle_2$ is the $C_0(E \times F)$-valued inner product associated to $X(E \times F)$, then

$$\begin{aligned}\rho(\langle x \otimes y, x' \otimes y' \rangle_1)(v, w) &= \rho(\langle x, x' \rangle_A \otimes \langle y, y' \rangle_B)(v, w)\\ &= \langle x, x' \rangle_A(v)\langle y, y' \rangle_B(w)\\ &= \sum_{s(e)=v, s'(f)=w} \overline{x(e)}x'(e)\overline{y(f)}y'(f)\\ &= \sum_{s(e)=v, s'(f)=w} \overline{x(e)y(f)}x'(e)y'(f)\\ &= \sum_{s(e)\times s'(f)=v\times w} \overline{\sigma(x \otimes y)(e, f)}\sigma(x' \otimes y')(e, f)\\ &= \langle \sigma(x \otimes y), \sigma(x' \otimes y') \rangle_2(v, w).\end{aligned}$$

Extending linearly and continuously, we see that (ρ, σ) gives an isomorphism of correspondences: $X(E \times F) \cong X(E) \otimes X(F)$.

2.2. Cuntz–Pimsner algebras. In order to describe the Cuntz–Pimsner algebra of a correspondence, we will first need to discuss representations of correspondences. Given a correspondence X over a C^*-algebra A, a *Toeplitz representation* of X in a C^* algebra B is a pair (ψ, π) where $\psi : X \to B$ is a linear map and $\pi : A \to B$ is a $*$-homomorphism such that:

(1) $\psi(a \cdot x) = \pi(a)\psi(x)$ for all $a \in A$ and all $x \in X$;
(2) $\psi(x \cdot a) = \psi(x)\pi(a)$ for all $a \in A$ and all $x \in X$;
(3) $\pi(\langle x, y_A \rangle) = \psi(x)^*\psi(y)$ for all $x, y \in X$.

(Note that (2) is actually implied by (3), but we include it for clarity.) We will write $C^*(\psi, \pi)$ for the C^*-subalgebra of B generated by the images of ψ and π in B. It can be shown that there is a unique (up to isomorphism) C^*-algebra $\mathcal{T}_X$, called the *Toeplitz algebra* of X, which is generated by a representation (i_X, i_A) which is 'universal' in the sense that for any representation (ψ, π) of X in any C^*-algebra B, there is a unique $*$-homomorphism $\psi \times \pi : \mathcal{T}_X \to B$ such that $\psi = (\psi \times \pi) \circ i_A$ and $\pi = (\psi \times \pi) \circ i_X$. This construction if due to Katsura and is discussed in detail in [7], for instance.

Let $X^{\otimes n}$ denote the n-fold internal tensor product of X with itself; by convention we let $X^{\otimes 0} = A$. Given a Toeplitz representation (ψ, π) of X in B, we define a map $\psi^n : X^{\otimes n} \to B$ for each $n \in \mathbb{N}$ as follows: we let $\psi^0 = \pi$ and $\psi^1 = \psi$ and then, for each $n > 1$, we set $\psi^n(x \otimes y) = \psi(x)\psi^{n-1}(y)$ where $x \in X$ and $y \in X^{\otimes n-1}$.

Let X be a correspondence over A and let (ψ, π) be a Toeplitz representation of X. Then, Proposition 2.7 of [7] states that

$$C^*(\psi, \pi) = \overline{\operatorname{span}}\{\psi^n(x)\psi^m(y)^* : x \in X^{\otimes n}, y \in X^{\otimes m}\}.$$

From [7, Lemma 2.4] we get the following result. Let (ψ, π) be a Toeplitz representation of X in B. For each $n \in \mathbb{N}$ there is a homomorphism $\psi^{(n)} : \mathcal{K}(X^{\otimes n}) \to B$ such that:

(1) $\pi(a)\psi^{(n)}(k) = \psi^{(n)}(\phi(a)k)$ for all $a \in A$ and all $k \in \mathcal{K}(X^{\otimes n})$;
(2) $\psi^{(n)}(k)\psi(x) = \psi(kx)$ for all $x \in X$ and all $k \in \mathcal{K}(X^{\otimes n})$.

Let X be a correspondence over a C^*-algebra A. We define the *Katsura ideal* of A to be the ideal

$$J_X = \{a \in A : \phi(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker(\phi)\},$$

where ϕ is the left action. This is often written more compactly as $J_X = \phi^{-1}(\mathcal{K}(X)) \cap (\ker(\phi))^{\perp}$. In many cases of interest, one can consider only correspondences whose left actions are injective and implemented by compacts. In this case we have $J_X = A$.

The Katsura ideal is also sometimes described as the largest ideal of A which maps injectively onto the compacts. This is made precise by the following proposition (see [7]).

Proposition 2.9. *Suppose that X is a correspondence over a C^*-algebra A, ϕ is the left action map, and I is an ideal of A which is mapped injectively into $\mathcal{K}(X)$ by ϕ. Then $I \subseteq J_X$.*

We are now ready to define the Cuntz–Pimsner algebra $\mathcal{O}_X$. A Toeplitz representation is said to be *Cuntz–Pimsner covariant* if $\psi^{(1)}(\phi(a)) = \pi(a)$ for all $a \in J_X$. The *Cuntz–Pimsner algebra $\mathcal{O}_X$* is the quotient of $\mathcal{T}_X$ by the ideal generated by

$$\{i_X^{(1)}(\phi(a)) - i_A(a) : a \in J_X\}.$$

Letting $q : \mathcal{T}_X \to \mathcal{O}_X$ denote the quotient map, it can be shown that $\mathcal{O}_X$ is generated by the Cuntz–Pimsner covariant representation $(k_X, k_A) = (q \circ i_X, q \circ i_A)$ and that this

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representation is universal for Cuntz–Pimsner covariant representations: if (ψ, π) is a Cuntz–Pimsner covariant representation of X in B, then there is a $*$-homomorphism $\psi \times \pi : \mathcal{O}_X \to B$ such that $\psi = (\psi \times \pi) \circ k_X$ and $\pi = (\psi \times \pi) \circ k_A$. This construction is also due to Katsura and is discussed in detail in [7].

Example 2.10. If E is a directed graph and $X(E)$ is the graph correspondence of Example 2.2, then one can show that $\mathcal{O}_{X(E)} \cong C^*(E)$ where $C^*(E)$ is the C^*-algebra of the graph (see [13] for details). For this reason, if F is a topological graph, the graph algebra of F is defined to be $\mathcal{O}_{X(F)}$.

One of the most important results about Cuntz–Pimsner algebras is the so-called 'gauge-invariant uniqueness theorem'. In order to state this theorem we need the following definition. Let (ψ, π) be a Toeplitz representation of a correspondence X. Then we say that $C^*(\psi, \pi)$ *admits a gauge action* if there is an action γ of $\mathbb{T}$ on $C^*(\psi, \pi)$ such that:

(1) $\gamma_z(\pi(a)) = \pi(a)$ for all $z \in \mathbb{T}$ and all $a \in A$;
(2) $\gamma_z(\psi(x)) = z\psi(x)$ for all $z \in \mathbb{T}$ and all $x \in X$.

When such an action exists, it is unique.

Theorem 2.11 (Gauge invariant uniqueness theorem [7, Theorem 6.4]). *Let X be a correspondence over A and let (ψ, π) be a Cuntz–Pimsner covariant representation of X. Then the $*$-homomorphism $\psi \times \pi : \mathcal{O}_X \to C^*(\psi, \pi)$ is an isomorphism if and only if (ψ, π) is injective and admits a gauge action.*

2.3. Actions, coactions and gradings. We will make use of the relationship between the gauge action of $\mathbb{T}$ and the natural $\mathbb{Z}$-grading of Cuntz–Pimsner algebras. This relationship comes from the duality between actions of an abelian group and coactions of the dual group. We will briefly recall the basics of actions and coactions here. By an *action* of a locally compact group G on a C^*-algebra A, we shall mean a strongly continuous group homomorphism $\alpha : G \to Aut(A)$. We will refer to the triple (A, G, α) as a *C^*-dynamical system*. For $s \in G$ we will write α_s for the automorphism $\alpha(s)$. Let G be a discrete group. Then the group C^*-algebra $C^*(G)$ is generated by a unitary representation $G \ni s \mapsto u_s \in \mathcal{U}C^*(G)$. We will abuse notation and write s for u_s. The function $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ defined by $s \mapsto s \otimes s$ for all $s \in G$ is called the *comultiplication map* on $C^*(G)$. There is a comultiplication map defined for any locally compact group, but in general it will map into $M(C^*(G) \otimes C^*(G))$. See [2, appendix] for more details.

A *coaction* of a group G on a C^*-algebra A is a nondegenerate, injective homomorphism $\delta : A \to M(A \otimes C^*(G))$ such that:

(1) $\delta(A)(1 \otimes C^*(G)) \subseteq A \otimes C^*(G)$;
(2) $(\delta \otimes \mathrm{id}_G) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta$ where both sides are viewed as maps $A \to M(A \otimes C^*(G) \otimes C^*(G))$.

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A coaction is called *nondegenerate* if the closed linear span of $\delta(A)(1 \otimes C^*(G))$ is equal to $A \otimes C^*(G)$. We will also refer to the triple (A, G, δ) as a coaction. Note that if G is discrete then $C^*(G)$ is unital (with unit u_e) and thus $\delta(A) \subseteq \delta(A)(1 \otimes C^*(G))$ so by condition (1), δ maps into $A \otimes C^*(G)$. In fact, the results of [12] tell us that (nondegenerate) coactions of discrete groups correspond to group gradings. We will discuss this more formally in a moment, but first we will state precisely what we mean by a grading of a C^*-algebra. Let G be a discrete group and A a C^*-algebra. By a *G-grading* of A we shall mean a collection $\{A_s\}_{s\in G}$ of linearly independent closed subspaces of A such that the following hold:

(1) $A_sA_t \subseteq A_{st}$ for all $s, t \in G$;
(2) $A_s^* = A_{s^{-1}}$;
(3) $A = \overline{\operatorname{span}}_{s\in G}(A_s)$.

We will say that a grading is *full* if we have $\overline{A_sA_t} = A_{st}$ for all $s, t \in G$. We summarize [12, Lemmas 1.3 and 1.5] as follows.

Lemma 2.12. *A nondegenerate coaction of a discrete group G on a C^*-algebra A gives a G-grading of A. Specifically, we let $A_s = \{a \in A : \delta(a) = a \otimes s\}$ for each $s \in G$.*

Remark 2.13 [2, Example A.23]. If G is abelian, then every coaction of G corresponds to an action of the dual group $\widehat{G}$ and vice versa. To see this, we first identify $C^*(G)$ and $C_0(\widehat{G})$ using the abstract Fourier transform: $\mathcal{F}(x)(\chi) = \chi(x)$. We also recall that, in this situation, condition (1) from the definition of coactions is equivalent to δ taking values in $C_b(\widehat{G}, A) \in M(A \otimes C^*(G))$ (see [2] for details). With this in mind, if (A, G, δ) is a coaction, then we define an action α^δ of $\widehat{G}$ by setting $\alpha^\delta_\chi(a) = \delta(a)(\chi)$. Conversely, given an action α of $\widehat{G}$ on a C^*-algebra A, we define a coaction δ^α by letting $\delta^\alpha(a)(\chi) = \alpha_\chi(a)$.

Let (A, G, α) be a C^*-dynamical system with G compact and abelian and let $(A, \widehat{G}, \delta^\alpha)$ be the associated coaction of the dual group $\widehat{G}$. Since G is compact, $\widehat{G}$ will be discrete and therefore δ^α will give a $\widehat{G}$-grading of A as in Lemma 2.12. Identifying $A \otimes C^*(\widehat{G})$ with $A \otimes C(G)$ and this with $C(G, A)$, the elementary tensor $a \otimes u_\chi$ corresponds to the map $s \mapsto \chi(s)a$. Therefore if $a \in A_\chi$ then $\alpha_s(a) = \delta^\alpha(a)(s) = \chi(s)a$ and so the sets A_χ can be thought of equivalently as

$$A_\chi = \{a \in A : \alpha_s(a) = \chi(s)a\}.$$

Thus each A_χ coincides with the so-called *spectral subspace* associated to χ.

Just as every action of a compact abelian group determines a grading by the dual group, every grading by a discrete abelian group determines an action of its dual group. To see this, just note that a G-grading of A makes A into a Fell bundle over G and, as in [12], we get a coaction of G associated to this Fell bundle which corresponds to an action of $\widehat{G}$ as follows.

Proposition 2.14. *Let A be a C^*-algebra and suppose that $\{A_s\}_{s\in G}$ is a G-grading of A. Let $\chi \in \widehat{G}$. For each $s \in G$ and each $a \in A_s$ define $\alpha_\chi(a) = \chi(s)a$. The maps α_χ extend to automorphisms of A such that $\alpha : \chi \mapsto \alpha_\chi$ is an action of $\widehat{G}$ on A.*

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3. Tensor products balanced over group actions or group gradings

We wish to show that the Cuntz–Pimsner algebra of an external tensor product $X \otimes Y$ of correspondences is isomorphic to a certain subalgebra $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$ of the tensor product $\mathcal{O}_X \otimes \mathcal{O}_Y$. This subalgebra is called the $\mathbb{T}$-*balanced tensor product* of $\mathcal{O}_X$ and $\mathcal{O}_Y$. Following [8], we define the general construction as follows.

Definition 3.1. Let G be a compact abelian group, and let (A, G, α) and (B, G, β) be C^*-dynamical systems. We define the *G-balanced tensor product* $A \otimes_G B$ to be the fixed point algebra $(A \otimes B)^\lambda$ where $\lambda : G \to A \otimes B$ is the action characterized by $\lambda_s(a \otimes b) = \alpha_s(a) \otimes \beta_{s^{-1}}(b)$.

Proposition 3.2. *If $a \otimes b \in A \otimes_G B$ then $\alpha_s(a) \otimes b, a \otimes \beta_s(b) \in A \otimes_G B$ for all $s \in G$ and $\alpha_s(a) \otimes b = a \otimes \beta_s(b)$.*

Proof. To show that $\alpha_s(a) \otimes b \in A \otimes_G B$ note that for any $t \in G$ we have

$$\begin{aligned}\alpha_t(\alpha_s(a)) \otimes \beta_{t^{-1}}(b) &= \alpha_s(\alpha_t(a)) \otimes \beta_{t^{-1}}(b)\\ &= (\alpha_s \otimes \mathrm{id}_B)(\alpha_t(a) \otimes \beta_{t^{-1}}(b))\\ &= (\alpha_s \otimes \mathrm{id}_B)(a \otimes b)\\ &= \alpha_s(a) \otimes b,\end{aligned}$$

showing that $a \otimes \beta_s(b) \in A \otimes_G B$ is similar. Now that this has been established, the equality follows easily:

$$\begin{aligned}\alpha_s(a) \otimes b &= \alpha_{s^{-1}}(\alpha_s(a)) \otimes \beta_s(b)\\ &= a \otimes \beta_s(b).\end{aligned}$$ □

Thus the actions $\alpha \otimes \iota_B$ and $\iota_A \otimes \beta$ coincide on $A \otimes_G B$ where ι_A and ι_B are the trivial actions. We will refer to the restriction of $\alpha \otimes \iota_B$ to $A \otimes_G B$ (or equivalently the restriction of $\iota_A \otimes \beta$ to $A \otimes_G B$) as the *balanced action* of G, and we will denote it by $\alpha \otimes_G \beta$.

The main result of this paper can be stated roughly as

$$\mathcal{O}_{X \otimes Y} \cong \mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$$

for suitable X and Y where we are balancing over the gauge actions on $\mathcal{O}_X$ and $\mathcal{O}_Y$. This generalizes the following example from [8].

Example 3.3. Let E and F be source-free, row-finite discrete graphs and let $E \times F$ denote the product graph as in Example 2.8. Then $C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F)$.

As we noted above, actions of compact abelian groups correspond to gradings of the dual group. It will be useful to be able to describe $\mathbb{T}$-balanced tensor products in terms of the corresponding $\mathbb{Z}$-gradings. But first we will need a fact which follows from the Peter–Weyl theorem [4, Theorem VII.1.35].

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Lemma 3.4. *Let G be a compact abelian group with a normalized Haar measure and let χ be a character (i.e. a continuous homomorphism $G \to \mathbb{T}$). Then*

$$\int_G \chi(s)\,ds = \begin{cases} 1 & \chi \text{ is the trivial homomorphism} \\ 0 & \text{otherwise.} \end{cases}$$

Now we are ready to describe G-balanced tensor products in terms of gradings of $\widehat{G}$.

Proposition 3.5. *Let (A, G, α) and (B, G, β) be C^*-dynamical systems with G abelian. Then, as discussed in the previous section, the coactions δ^α and δ^β give $\widehat{G}$-gradings of A and B:*

$$A_\chi = \{a \in A : \alpha_s(a) = \chi(s)a\},$$
$$B_\chi = \{b \in B : \beta_s(b) = \chi(s)b\}.$$

Let

$$S_\chi := \{a \otimes b : a \in A_\chi, b \in B_\chi\}$$

and let

$$S := \bigcup_{\chi \in \widehat{G}} S_\chi.$$

Then $A \otimes_G B = \overline{\operatorname{span}}(S)$.

Proof. First, note that if $a \otimes b \in S$, then $a \otimes b \in S_\chi$ for some χ and so, for all $s \in G$,

$$\begin{aligned} \lambda_s(a \otimes b) &= \alpha_s(a) \otimes \beta_{s^{-1}}(b) \\ &= \chi(s)a \otimes \chi(s^{-1})b \\ &= \chi(s)\chi(s^{-1})(a \otimes b) \\ &= a \otimes b \end{aligned}$$

so $a \otimes b \in A \otimes_G B$ and hence $S \subseteq A \otimes_G B$. Since $A \otimes_G B$ is a C^*-algebra, we have that $\overline{\operatorname{span}}(S) \subseteq A \otimes_G B$.

Now, since A is densely spanned by the A_χ and B is densely spanned by the B_χ, the tensor product $A \otimes B$ will be densely spanned by elementary tensors $a \otimes b$ where $a \in A_\chi$ and $b \in B_{\chi'}$ for some $\chi, \chi' \in \widehat{G}$. More precisely, let

$$T_{\chi,\chi'} = \{a \otimes b : a \in A_\chi, b \in B_{\chi'}\}$$

and let

$$T = \bigcup_{\chi,\chi' \in \widehat{G}} T_{\chi,\chi'}.$$

Then we have $A \otimes B = \overline{\operatorname{span}}(T)$. Let $\varepsilon : A \otimes B \to A \otimes_G B$ be the conditional expectation $c \mapsto \int_G \lambda_s(c)\,ds$. Then, since ε is continuous, linear and surjective, $\varepsilon(T)$ densely spans $A \otimes_G B$. Let $a \otimes b \in T$, say $a \otimes b \in T_{\chi,\chi'}$. Then using Lemma 3.4 and the fact that a product of two characters is a character, we have

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$$
\begin{aligned}
\varepsilon(a\otimes b) &= \int_G \lambda_s(a\otimes b)\,ds \\
&= \int_G (\chi(s)a\otimes\chi'(s^{-1})b)\,ds \\
&= \Big(\int_G \chi(s)\overline{\chi'(s)}\,dz\Big)a\otimes b \\
&= \Big(\int_G (\chi\overline{\chi'})(s)\,ds\Big)a\otimes b \\
&= \begin{cases} 1 & \text{if } \chi\overline{\chi'} \text{ is trivial} \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

But $\chi\overline{\chi'}$ will be trivial if and only if $\chi=\chi'$. This means that if $\chi\neq\chi'$ (i.e. $a\otimes b\in T\backslash S$) then $\varepsilon(a\otimes b)=0$ and if $\chi=\chi'$ (i.e. $a\otimes b\in S$) then $\varepsilon(a\otimes b)=a\otimes b$. This implies that $\varepsilon(T)=S$ and thus, since T densely spans $A\otimes B$, the linearity and continuity of ε imply that S densely spans $\varepsilon(A\otimes B)=A\otimes_G B$. Therefore $A\otimes_G B=\overline{\operatorname{span}}(S)$. □

Proposition 2.14 tells us that the $\widehat{G}$-grading of $A\otimes_G B$ just described should give us an action of G on $A\otimes_G B$. We will now show that this action coincides exactly with the balanced action.

Proposition 3.6. *Let $\{S_\chi\}_{s\in\widehat{G}}$ be the $\widehat{G}$-grading of $A\otimes_G B$ described in the previous proposition and let γ be the action associated to this grading by Proposition 2.14. Then $\gamma=\alpha\otimes_G\beta$.*

Proof. It suffices to check that these maps coincide on each S_χ. Let $a\otimes b\in S_\chi$. Then for each $s\in\widehat{G}$ we have $\gamma_s(a\otimes b)=\chi(s)(a\otimes b)$ by definition. On the other hand, since $a\in A_\chi$,

$$
\begin{aligned}
(\alpha\otimes_G\beta)_s(a\otimes b) &= \alpha_s(a)\otimes b \\
&= (\chi(s)a)\otimes b \\
&= \chi(s)(a\otimes b)
\end{aligned}
$$

so $\gamma_s(a\otimes b)=(\alpha\otimes_G\beta)_s(a\otimes b)$ for every $s\in G$ and every $a\otimes b$ in S_χ. □

The following lemma will be useful later.

Lemma 3.7. *Let $\{A_n\}_{n\in\mathbb{Z}}$ and $\{B_n\}_{n\in\mathbb{Z}}$ be saturated $\mathbb{Z}$-gradings of C^*-algebras A and B and let $\{S_n\}_{n\in\mathbb{Z}}$ be the $\mathbb{Z}$-grading of $A\otimes_{\mathbb{T}} B$ as in the previous propositions. Then $A\otimes_{\mathbb{T}} B$ is generated by S_1.*

Proof. First, we will show that $S_nS_m=S_{n+m}$. We already have that $S_nS_m\subseteq S_{n+m}$ so it suffices to show the reverse inclusion. Let $a\otimes b\in S_{n+m}$. Then $a\in A_{n+m}$ and $b\in B_{n+m}$ so $a=\sum a_ia_i'$ and $b=\sum b_ib_i'$ with $a_i\in A_n$, $a_i'\in A_m$, $b_i\in B_n$, and $b_i'\in B_m$. Therefore $a\otimes b=\sum_{i,j}(a_i\otimes b_j)(a_i'\otimes b_j')$ where $a_i\otimes b_j\in S_n$ and $a_i'\otimes b_j'\in S_m$ so $S_{n+m}\subseteq S_nS_m$ and hence $S_nS_m=S_{n+m}$.

Since 1 generates $\mathbb{Z}$ as a group, S_1 generates $\overline{\operatorname{span}}\bigcup S_n=A\otimes_{\mathbb{T}} B$ as a C^*-algebra. □

4. Ideal compatibility

In this section we introduce technical conditions which we will need for our main result to hold.

DEFINITION 4.1. Let X and Y be correspondences over C^*-algebras A and B and let J_X, J_Y, $J_{X\otimes Y}$ be the Katsura ideals (i.e.

$$J_X = \{a \in A : \phi(a) \in \mathcal{K}(X) \text{ and } aa' = 0 \text{ if } \phi_X(a') = 0\}$$

and so on). We say that X and Y are *ideal-compatible* if $J_{X\otimes Y} = J_X \otimes J_Y$.

The simplest way for this condition to hold is if the left actions of A and B on X and Y are injective and implemented by compacts. In this case it will also be true that the left action of $A \otimes B$ on $X \otimes Y$ will be injective and implemented by compacts. Thus we will have that $J_X = A$, $J_Y = B$, and $J_{X\otimes Y} = A \otimes B$ so ideal compatibility is automatic. Thus we have established the following proposition.

PROPOSITION 4.2. *Let X_A and Y_B be correspondences such that the left actions of A and B are injective and implemented by compacts. Then X and Y are ideal-compatible.*

The following two lemmas are inspired by [5, Lemma 2.6].

LEMMA 4.3. *Let X and Y be correspondences over A and B. Then $J_X \otimes J_Y \subseteq J_{X\otimes Y}$.*

PROOF. Since ϕ_X maps J_X injectively into $\mathcal{K}(X)$ and ϕ_Y maps J_Y injectively into $\mathcal{K}(Y)$, $\phi_{X\otimes Y} = \phi_X \otimes \phi_Y$ will map $J_X \otimes J_Y$ injectively into $\mathcal{K}(X) \otimes K(Y)$, but $\mathcal{K}(X) \otimes \mathcal{K}(Y) = \mathcal{K}(X \otimes Y)$ so $\phi_{X\otimes Y}$ maps $J_X \otimes J_Y$ injectively into $\mathcal{K}(X \otimes Y)$. Thus by Proposition 2.9, $J_X \otimes J_Y \subseteq J_{X\otimes Y}$. □

Recall that if A and B are C^*-algebras and C is an subalgebra of A, the triple (C, A, B) is said to satisfy the *slice map property* if

$$C \otimes B = \{x \in A \otimes B : (\mathrm{id}_A \otimes \omega)(x) \in C \text{ for all } \omega \in B^*\}.$$

LEMMA 4.4. *Let X and Y be correspondences over C^*-algebras A and B and suppose that Y is an imprimitivity bimodule. If (J_X, A, B) satisfies the slice map property, then X and Y are ideal-compatible.*

PROOF. It suffices to show that $J_{X\otimes Y} \subseteq J_X \otimes J_Y$. Since Y is an imprimitivity bimodule we have that the left action ϕ_Y is an isomorphism $B \cong \mathcal{K}(Y)$ and thus ϕ_Y maps all of B maps injectively into $\mathcal{K}(Y)$, so $J_Y = B$. We must show that $J_{X\otimes Y} \subseteq J_X \otimes B$.

Let $c \in J_{X\otimes Y}$. Since (J_X, A, B) satisfies the slice map property, showing that $c \in J_X \otimes B$ is equivalent to showing that $(\mathrm{id} \otimes \omega)(c) \in J_X$ for all $\omega \in B^*$. Recalling the definition of J_X, this means we must show that $\phi_X((\mathrm{id} \otimes \omega)(c)) \in \mathcal{K}(X)$ and that

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$(\mathrm{id} \otimes \omega)(c)a = 0$ for all $a \in \ker(\phi_X)$. With this in mind, let $\omega \in B^*$. Since ϕ_X is linear, we have that

$$\begin{aligned}\phi_X((\mathrm{id}_A \otimes \omega)(c)) &= (\phi_X \otimes \omega)(c)\\ &= (\phi_X \otimes (\omega \circ \phi_Y^{-1} \circ \phi_Y))(c)\\ &= (\mathrm{id}_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1})) \circ (\phi_X \otimes \phi_Y)(c)\\ &= (\mathrm{id}_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1})) \circ \phi_{X\otimes Y}(c).\end{aligned}$$

To see that this is in $\mathcal{K}(X)$, note that since $c \in J_{X\otimes Y}$ by assumption, we know that $\phi(c) \in \mathcal{K}(X \otimes Y) = \mathcal{K}(X) \otimes \mathcal{K}(Y)$. Note that since Y is an imprimitivity bimodule, ϕ_Y^{-1} is well defined as a map $\mathcal{K}(Y) \to B$. Since $(\mathrm{id}_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1}))$ maps $\mathcal{K}(X) \otimes \mathcal{K}(Y) \to \mathcal{K}(X)$ we have that

$$(\mathrm{id}_{\mathcal{K}(X)} \otimes (\omega \circ \phi_Y^{-1})) \circ \phi_{X\otimes Y}(c) \in \mathcal{K}(X)$$

so $(\mathrm{id} \otimes \omega)(c) \in \mathcal{K}(X)$.

Next, let $a \in \ker(\phi_X)$ and factor ω as $b \cdot \omega'$ for some $b \in B$ and $\omega' \in B^*$ (where $(b \cdot \omega')(b') = \omega'(b'b)$). Then

$$\begin{aligned}(\mathrm{id} \otimes \omega)(c)a &= (\mathrm{id} \otimes \omega)(c(a \otimes 1))\\ &= (\mathrm{id} \otimes b \cdot \omega')(c(a \otimes 1))\\ &= (\mathrm{id} \otimes \omega')(c(a \otimes 1)(1 \otimes b))\\ &= (\mathrm{id} \otimes \omega')(c(a \otimes b))\end{aligned}$$

but

$$\begin{aligned}a \otimes b &\in \ker(\phi_X) \otimes B\\ &\subseteq \ker(\phi_X \otimes \phi_Y)\\ &= \ker(\phi_{X\otimes Y}).\end{aligned}$$

Therefore, since $c \in J_{X\otimes Y}$ we must have $c(a \otimes b) = 0$ and hence

$$(\mathrm{id} \otimes \omega)(c)a = (\mathrm{id} \otimes \omega')(c(a \otimes b)) = 0.$$

Thus, we have established that $(\mathrm{id} \otimes \omega)(c) \in J_X$ for any $c \in J_{X\otimes Y}$ and $\omega \in B^*$ so by the slice map property we have that $J_{X\otimes Y} \subseteq J_X \otimes B = J_X \otimes J_Y$ and thus (by the previous lemma) $J_{X\otimes Y} = J_X \otimes J_Y$. □

In [13, Example 8.13], it is shown that if E is a discrete graph, then

$$J_{X(E)} = \overline{\mathrm{span}}\{\delta_v : 0 < |r^{-1}(v)| < \infty\},$$

where $X(E)$ is the associated correspondence and $\delta_v \in c_0(E^0)$ denotes the characteristic function of the vertex $v \in E^0$. With this in mind, we give the following proposition.

Proposition 4.5. *Let E and F be discrete graphs and let $X = X(E)$ and $Y = X(F)$ be the associated correspondences. Then X and Y are ideal-compatible.*

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Proof. Recall that $X \otimes Y = X(E \times F)$. Thus

$$J_{X\otimes Y} = \overline{\operatorname{span}}\{\delta_{(v,w)} : 0 < |r_{E\times F}^{-1}(v,w)| < \infty\}.$$

By definition, $r_{E\times F} = r_E \times r_F$ so $r_{E\times F}^{-1}(v,w) = r_E^{-1}(v) \times r_F^{-1}(w)$ and thus $|r_{E\times F}^{-1}(v,w)| = |r_E^{-1}(v)| \cdot |r_F^{-1}(w)|$, but $0 < |r_E^{-1}(v)| \cdot |r_F^{-1}(w)| < \infty$ if and only if $0 < |r_E^{-1}(v)| < \infty$ and $0 < |r_F^{-1}(w)| < \infty$. Thus we have that

$$J_{X\otimes Y} = \overline{\operatorname{span}}\{\delta_{(v,w)} : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\}.$$

Since $\delta_{(v,w)} = \delta_v \delta_w$, if we identify $c_0(E^0 \times F^0)$ with $c_0(E^0) \otimes c_0(F^0)$ in the standard way, we see that $\delta_{(v,x)} = \delta_v \otimes \delta_w$. Thus

$$\begin{aligned} J_{X\otimes Y} &= \overline{\operatorname{span}}\{\delta_v \otimes \delta_w : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\} \\ &= \overline{\operatorname{span}}\{f \otimes g : f \in J_X, g \in J_Y\} \\ &= J_X \otimes J_Y. \end{aligned}$$

Therefore, X and Y are ideal-compatible. □

Definition 4.6. Let X be a correspondence over a C^* algebra A. We will call this correspondence *Katsura nondegenerate* if $X \cdot J_X = X$.

Example 4.7. Let X be a correspondence over a C^*-algebra A such that the left action is injective and implemented by compacts. In this case we have that $J_X = A$. Thus,

$$\begin{aligned} X \cdot J_X &= X \cdot A \\ &= X. \end{aligned}$$

Definition 4.8. Recall that a vertex in a directed graph is called a *source* if it receives no edges. We will call such a vertex a *proper source* if it emits at least one edge.

Proposition 4.9. *Let E be a directed graph. Then $X(E)$ is Katsura nondegenerate if and only if E has no proper sources and no infinite receiver emits an edge.*

Proof. Suppose there is $v \in E^0$ such that $|r^{-1}(v)| = \infty$ and $|s^{-1}(v)| > 0$. Then for every $f \in J_X$ we have $f(v) = 0$. Thus for any $g \in C_c(E^1)$, $f \in J_X$, and $e \in s^{-1}(v)$, we have $(g \cdot f)(e) = g(e)f(s(e)) = g(e)f(v) = 0$. Thus $h(e) = 0$ for all $h \in C_c(E^1) \cdot J_X$ and, taking the limit, $x(e) = 0$ for all $x \in X \cdot J_X$. Thus $\delta_e \notin X \cdot J_X$ since $\delta_e(e) = 1 \neq 0$ but $\delta_e \in X$. Therefore $X \neq X \cdot J_X$, that is, X is not Katsura nondegenerate.

Similarly, suppose that E has a proper source v. Then, since $|r^{-1}(v)| = 0$ we must have $f(v) = 0$ for all $f \in J_X$. Then for any $g \in C_c(E^1)$ and $e \in s^{-1}(v)$, we have that $(g \cdot f)(e) = g(e)f(v) = 0$ for $f \in J_X$. Thus by similar reasoning to that above we have that $x(e) = 0$ for all $x \in X \cdot J_X$ and so $\delta_e \notin X \cdot J_X$ but $\delta_e \in X$, and we can again conclude that $X \neq X \cdot J_X$ so X is not Katsura nondegenerate.

On the other hand, suppose that E has no proper sources and no infinite receiver in E emits an edge. Let $e \in E^1$ and let $v = s(e)$. Then $|r^{-1}(v)| < \infty$ and $|r^{-1}(v)| > 0$ by assumption, so a function in J_X can be supported on v. In particular, $\delta_v \in J_X$. Since $\delta_e \cdot \delta_v = \delta_e$ we know that $\delta_e \in X \cdot J_X$. Since e was arbitrary, we have that all such characteristic functions are contained in $X \cdot J_X$. But these functions densely span $C_c(E^1)$ and thus densely span X, so we have that $X \subseteq X \cdot J_X$ and therefore $X = X \cdot J_X$ so X is Katsura nondegenerate. □

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5. Main result

We will begin with a few lemmas.

LEMMA 5.1. *Let X and Y be correspondences over C^*-algebras A and B, respectively. Suppose that (π_X, ψ_X) and (π_Y, ψ_Y) are Toeplitz representations of X and Y in C^*-algebras C and D. Let $\pi := \pi_X \otimes \pi_Y$ and $\psi := \psi_X \otimes \psi_Y$. Then (π, ψ) is a Toeplitz representation of $X \otimes Y$ in $C \otimes D$.*

PROOF. This follows from the following computations:

$$\begin{aligned}
\psi((x \otimes y) \cdot (a \otimes b)) &= \psi_X(x \cdot a) \otimes \psi_Y(y \cdot b) \\
&= \psi_X(x)\pi_X(a) \otimes \psi_Y(y)\pi_Y(b) \\
&= \psi(x \otimes y)\pi(a \otimes b), \\
\psi((a \otimes b) \cdot (x \otimes y)) &= \psi_X(a \cdot x) \otimes \psi_Y(b \cdot y) \\
&= \pi_X(a)\psi_X(x) \otimes \pi_Y(b)\psi_Y(y) \\
&= \pi(a \otimes b)\psi(x \otimes y), \\
\psi(x \otimes y)^*\psi(x' \otimes y') &= \psi_X(x)^*\psi_X(x') \otimes \psi_Y(y)^*\psi_Y(y') \\
&= \pi_X(\langle x, x'\rangle_A) \otimes \pi_Y(\langle y, y'\rangle_B) \\
&= \pi(\langle x, x'\rangle_A \otimes \langle y, y'\rangle_B) \\
&= \pi(\langle x \otimes y, x' \otimes y'\rangle_{A\otimes B}).
\end{aligned}$$

□

LEMMA 5.2. *If (π, ψ) is the Toeplitz representation of $X \otimes Y$ in the previous lemma, then*

$$\psi^{(1)}(\kappa(S \otimes T)) = \psi_X^{(1)}(S) \otimes \psi_Y^{(1)}(T)$$

where κ is as in Lemma 2.7.

PROOF. Let $x, x' \in X$ and $y, y' \in Y$. Then

$$\begin{aligned}
\psi^{(1)}(\kappa(\Theta_{x,x'} \otimes \Theta_{y,y'})) &= \psi^{(1)}(\Theta_{x\otimes y, x'\otimes y'}) \\
&= \psi(x \otimes y)\psi(x' \otimes y')^* \\
&= (\psi_X(x) \otimes \psi_Y(y))(\psi_X(x') \otimes \psi_Y(y'))^* \\
&= \psi_X(x)\psi_X(x')^* \otimes \psi_Y(y)\psi_Y(y')^* \\
&= \psi_X^{(1)}(\Theta_{x,x'}) \otimes \psi_Y^{(1)}(\Theta_{y,y'}).
\end{aligned}$$

Since the rank-ones have dense span in the compacts, this result extends to any $S \in \mathcal{K}(X)$ and $T \in \mathcal{K}(Y)$. □

LEMMA 5.3. *Let X and Y be ideal-compatible correspondences over A and B. Then if (π_X, ψ_X) and (π_Y, ψ_Y) are Cuntz–Pimsner covariant, then so is (π, ψ).*

PROOF. Let $c \in J_{X\otimes Y}$. Since X and Y are ideal-compatible we have that $J_{X\otimes Y} = J_X \otimes J_Y$ so we can approximate c by elements of $J_X \odot J_Y$, that is, by finite sums $\sum_i a_i \otimes b_i$ with $a_i \in J_X$ and $b_i \in J_Y$ for each i.

Notice that

$$\begin{aligned}\psi^{(1)}\Big(\phi\Big(\sum_i a_i\otimes b_i\Big)\Big) &= \psi^{(1)}\Big(\sum_i \phi(a_i\otimes b_i)\Big)\\ &= \psi^{(1)}\Big(\sum_i \kappa(\phi_X(a_i)\otimes\phi_Y(b_i))\Big)\\ &= \sum_i \psi_X^{(1)}(\phi_X(a_i))\otimes\psi_Y^{(1)}(\phi_Y(b_i))\\ &= \sum_i \pi_X(a_i)\otimes\pi_Y(b_i)\\ &= \sum_i \pi(a_i\otimes b_i),\end{aligned}$$

where we have used the Cuntz–Pimsner covariance of (π_X,ψ_X) and (π_Y,ψ_Y). Thus if (c_j) is a sequence in $J_X\odot J_Y$ converging to c, we have $\psi^{(1)}(\phi(c_j))=\pi(c_j)$ for all j. Since $\psi^{(1)}$, ϕ, and π are all continuous, we have that $\psi^{(1)}(\phi(c))=\pi(c)$. This establishes that (π,ψ) is Cuntz–Pimsner covariant. □

We are now ready to prove the main result of this paper.

Theorem 5.4. *Let X and Y be ideal-compatible correspondences over C^*-algebras A and B. Then $\mathcal{O}_{X\otimes Y}$ can be faithfully embedded in $\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$. If X and Y are Katsura nondegenerate, then $\mathcal{O}_{X\otimes Y}\cong\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$.*

Proof. We will begin by showing the existence of a homomorphism $\mathcal{O}_{X\otimes Y}\to\mathcal{O}_X\otimes\mathcal{O}_Y$. To show this, we will construct a Cuntz–Pimsner covariant representation of $X\otimes Y$ in $\mathcal{O}_X\otimes\mathcal{O}_Y$ and then apply the universal property of Cuntz–Pimsner algebras.

Let (k_X,k_A) and (k_Y,k_B) be the Cuntz–Pimsner covariant representations of X and Y in $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively. Let $\psi=k_X\otimes k_Y$ and $\pi=k_A\otimes k_B$. Then by Lemma 5.3, (ψ,π) is Cuntz–Pimsner covariant and so we have a homomorphism $F:\mathcal{O}_{X\otimes Y}\to\mathcal{O}_X\otimes\mathcal{O}_Y$ such that

$$(\psi,\pi)=(F\circ k_{X\otimes Y},F\circ k_{A\otimes B}).$$

In particular,

$$F(k_{A\otimes B}(A\otimes B))=\pi(A\otimes B)=(k_A\otimes k_B)(A\otimes B),\tag{5.1}$$

$$F(k_{X\otimes Y}(X\otimes Y))=\psi(X\otimes Y)=(k_X\otimes k_Y)(X\otimes Y).\tag{5.2}$$

Let $\{\mathcal{O}_X^n\}_{n\in\mathbb{Z}}$ and $\{\mathcal{O}_Y^n\}_{n\in\mathbb{Z}}$ denote the $\mathbb{Z}$-gradings of $\mathcal{O}_X$ and $\mathcal{O}_Y$ associated to the standard gauge actions γ_X and γ_Y. Then by Proposition 3.5, the subspaces

$$S_n:=\{x\otimes y: x\in\mathcal{O}_X^n, y\in\mathcal{O}_Y^n\}$$

give a $\mathbb{Z}$-grading of $\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$. Since (5.1) shows that $\pi(A\otimes B)\subseteq S_0$ and (5.2) shows that $\psi(X\otimes Y)\subseteq S_1$, we can see that $C^*(\psi,\pi)\subseteq\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$ and that the action of $\mathbb{T}$ on $\mathcal{O}_X\otimes_{\mathbb{T}}\mathcal{O}_Y$ guaranteed by Lemma 2.14 is a gauge action:

$$\begin{aligned}\gamma_z(\pi(c))&=\pi(c), && c\in A\otimes B,\\ \gamma_z(\psi(w))&=z\psi(w), && w\in X\otimes Y.\end{aligned}$$

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Also, since k_A, k_B, k_X and k_Y are injective, $\pi = k_A \otimes k_B$ and $\psi = k_X \otimes k_Y$ are injective too. Hence by the gauge invariant uniqueness theorem, F is injective. Thus we have established the first part of the theorem.

Now suppose that X and Y are Katsura nondegenerate. We will show that (ψ, π) generates $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$ by showing that (ψ, π) generates S_1 and applying Lemma 3.7. Since $\mathcal{O}_X^1$ is densely spanned by elements of the form $k_X^{n+1}(x)k_X^n(x')^*$, and $\mathcal{O}_Y^1$ is densely spanned by elements of the form $k_Y^{n+1}(y)k_Y^n(y')^*$, we have that S_1 is densely spanned by elements of the form

$$k_X^{n+1}(x)k_X^n(x')^* \otimes k_Y^{m+1}(y)k_Y^m(y')^*. \tag{5.3}$$

By symmetry, we may assume $m = n + l$ for some nonnegative l. Then we may assume that $y = y_1 \otimes y_2$ and $y' = y_1' \otimes y_2'$ with $y_1, y_1' \in Y^{\otimes n}$ and $y_2, y_2' \in Y^{\otimes l}$. Further, since X is Katsura nondegenerate, we can factor $x = x_0 a$ and $x' = x_0' a'$ with $x_0, x_0' \in X$ and $a, a' \in J_X$. Now we can factor (5.3) as follows:

$$\begin{aligned}
&k_X^{n+1}(x)k_X^n(x')^* \otimes k_Y^{m+1}(y)k_Y^m(y')^* \\
&\quad = (k_X^{n+1}(x) \otimes k_Y^{m+1}(y))(k_X^n(x')^* \otimes k_Y^m(y')^*) \\
&\quad = (k_X^{n+1}(x_0)k_A(a) \otimes k_Y^{n+1}(y_1)k_Y^l(y_2))(k_A(a')^*k_X^n(x_0')^* \otimes k_Y^l(y_2')^*k_Y^n(y_1')^*) \\
&\quad = (k_X^{n+1}(x_0) \otimes k_Y^{n+1}(y_1))(k_A(a) \otimes k_Y^l(y_2))(k_A(a')^* \otimes k_Y^l(y_2')^*)(k_X^n(x_0')^* \otimes k_Y^n(y_1')^*) \\
&\quad = (k_X^{n+1}(x_0) \otimes k_Y^{n+1}(y_1))(k_A(aa'^*) \otimes k_Y^l(y_2)k_Y^l(y_2')^*)(k_X^n(x_0')^* \otimes k_Y^n(y_1')^*) \\
&\quad = (k_X^{n+1}(x_0) \otimes k_Y^{n+1}(y_1))(k_X^{(1)}(\phi_X(aa'^*)) \otimes k_Y^{(1)}(\Theta_{y_2,y_2'}))(k_X^n(x')^* \otimes k_Y^n(y_1')^*) \\
&\quad = \psi^{n+1}(x_0 \otimes y_1)(\psi^{(1)}(\phi_X(aa') \otimes \Theta_{y_2,y_2'}))\psi^n(x_0' \otimes y_1')^*.
\end{aligned}$$

Since $\psi^{n+1}(x_0 \otimes y_1)$, $\psi^{(1)}(\phi_X(aa'^*) \otimes \Theta_{y_2,y_2'})$, and $\psi^n(x_0' \otimes y_1')$ are in the algebra generated by (ψ, π), we now know that (ψ, π) generates S_1 and so by Lemma 3.7 (ψ, π) generates all of $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$. Therefore F is surjective hence an isomorphism $\mathcal{O}_{X \otimes Y} \cong \mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$. □

6. Examples

We will now give some examples.

EXAMPLE 6.1. Let $(A, \mathbb{Z}, \alpha)$ and $(B, \mathbb{Z}, \beta)$ be C^*-dynamical systems. Let X be the C^*-correspondence A_A with left action given by $a \cdot x = \alpha_1(a)x$ and let Y be the correspondence B_B with left action given by $b \cdot y = \beta_1(b)y$. This action is injective and implemented by compacts, and we have that $\mathcal{O}_X \cong A \rtimes_\alpha \mathbb{Z}$ and $\mathcal{O}_Y \cong B \rtimes_\beta \mathbb{Z}$ by isomorphisms which carry the gauge action of $\mathbb{T}$ to the dual action of $\mathbb{T}$ (see [11]).

Consider the external tensor product $X \otimes Y$. As a right Hilbert module this is $A_A \otimes B_B$, it carries a right action of $A \otimes B$ characterized by $(x \otimes y) \cdot (a \otimes b) = x \cdot a \otimes y \cdot b$, but since the right actions on X and Y are given by multiplication in A and B, this action of $A \otimes B$ on $X \otimes Y$ is just multiplication in $A \otimes B$. Further, the $A \otimes B$-valued inner product

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on $X \otimes Y$ is given by

$$\begin{aligned}\langle x \otimes y, x' \otimes y' \rangle_{A \otimes B} &= \langle x, x' \rangle_A \otimes \langle y, y' \rangle_B \\ &= x^* x' \otimes y^* y' \\ &= (x \otimes y)^* (x' \otimes y'),\end{aligned}$$

but this is precisely the inner product on $(A \otimes B)_{A \otimes B}$. Thus $\mathrm{id}_A \otimes \mathrm{id}_B$ gives a right Hilbert module isomorphism $X \otimes Y \cong (A \otimes B)_{A \otimes B}$. The left action of $A \otimes B$ on $X \otimes Y$ will be the tensor product of the action of A on X and the action of B on Y. Thus

$$\begin{aligned}(a \otimes b) \cdot (x \otimes y) &= \alpha_1(a) x \otimes \beta_1(b) y \\ &= (\alpha_1(a) \otimes \beta_1(b))(x \otimes y).\end{aligned}$$

Thus, as an $A \otimes B$ correspondence, $X \otimes Y$ can be identified with the correspondence associated to the automorphism $\alpha_1 \otimes \beta_1$ on $A \otimes B$. Since $(\alpha_1 \otimes \beta_1)^{\circ n} = (\alpha_n \otimes \beta_n)$ and $(\alpha_1 \otimes \beta_1)^{-1} = (\alpha_{-1} \otimes \beta_{-1})$, the action of $\mathbb{Z}$ generated by $\alpha_1 \otimes \beta_1$ will be the diagonal action $\alpha \otimes \beta$ of $\mathbb{Z}$ on $A \otimes B$. Thus we have that $\mathcal{O}_{X \otimes Y} \cong (A \otimes B) \rtimes_{\alpha \otimes \beta} \mathbb{Z}$.

Therefore, in this context our main theorem says that

$$(A \otimes B) \rtimes_{\alpha \otimes \beta} \mathbb{Z} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes_{\mathbb{T}} (B \rtimes_\beta \mathbb{Z}).$$

In later work, we hope to investigate whether this result generalizes to groups other than $\mathbb{Z}$.

Example 6.2 (Products of topological graphs). Let $E = (E^0, E^1, r, s)$ and $F = (F^0, F^1, r', s')$ be source-free topological graphs with r and r' proper. Then the left actions of $X(E)$ and $X(F)$ will be injective and implemented by compacts. Recall from Example 2.8 that $X(E) \otimes X(F) \cong X(E \times F)$ where $E \times F$ is the product graph. Our main result says that $\mathcal{O}_{X(E \times F)} \cong \mathcal{O}_{X(E)} \otimes_{\mathbb{T}} \mathcal{O}_{X(F)}$, which translates to

$$C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F).$$

Note that if E and F are discrete graphs, this coincides with Kumjian's result in [8].

Example 6.3 (Products of discrete graphs). Let E and F be discrete graphs with no proper sources and such that no infinite receiver emits an edge. From the discussion in Section 4 we know that the graph correspondences $X(E)$ and $X(F)$ are ideal-compatible and Katsura nondegenerate. By the same reasoning as in the previous example we have that

$$C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F).$$

Note that this stronger than the result in [8] where the graphs are required to be source-free and row-finite.

Example 6.4. Let A and B be C^*-algebras, and let X be a correspondence over A. Viewing B as the correspondence ${}_B B_B$, we can form the $A \otimes B$ correspondence $X \otimes B$. Suppose that X and B are ideal-compatible and Katsura nondegenerate (in fact B will automatically be Katsura nondegenerate). Recall that $\mathcal{O}_B \cong B \otimes C(\mathbb{T})$ with gauge

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action $\iota \otimes \lambda$ where ι is the trivial action and λ is left translation. Thus our main result says that $\mathcal{O}_{X \otimes B} \cong \mathcal{O}_X \otimes_{\mathbb{T}} (B \otimes C(\mathbb{T}))$. Identifying $C(\mathbb{T})$ with $C^*(\mathbb{Z})$, we have $\mathcal{O}_{X \otimes B} \cong \mathcal{O}_X \otimes_{\mathbb{T}} (B \otimes C^*(\mathbb{Z}))$, and characterizing the $\mathbb{T}$-balanced tensor product in terms of the $\mathbb{Z}$-gradings as we have been, we see that

$$\mathcal{O}_{X \otimes B} \cong \overline{\operatorname{span}}\{x \otimes b \otimes w \in \mathcal{O}_X^n \otimes B \otimes C^*(\mathbb{Z})^n : n \in \mathbb{Z}\}.$$

But since $C^*(\mathbb{Z})^n = \operatorname{span}(u_n)$ (where u_n denotes the unitary in $C^*(Z)$ associated to n) we can rephrase this as

$$\overline{\operatorname{span}}\{x \otimes b \otimes u_n \in \mathcal{O}_X \otimes B \otimes C^*(\mathbb{Z}) : x \in \mathcal{O}_X^n\}. \tag{6.1}$$

Now let γ denote the gauge action of $\mathbb{T}$ on $\mathcal{O}_X$ and let δ^γ be the dual coaction of $\mathbb{Z}$. Recall that this coaction can be characterized by the property that $\delta^\gamma(x) = x \otimes u_n$ whenever $x \in \mathcal{O}_X^n$. Since the subspaces $\mathcal{O}_X^n$ densely span $\mathcal{O}_X$, their images under δ^γ will densely span $\delta^\gamma(\mathcal{O}_X)$. Therefore,

$$\begin{aligned}
\delta^\gamma(\mathcal{O}_X) &= \overline{\operatorname{span}}\{x \otimes u_n : x \in \mathcal{O}_X^n\},\\
(\delta^\gamma \otimes \mathrm{id}_B)(\mathcal{O}_X \otimes B) &= \overline{\operatorname{span}}\{x \otimes u_n \otimes b : x \in \mathcal{O}_X^n, b \in B\},\\
\sigma_{23} \circ (\delta^\gamma \otimes \mathrm{id}_B)(\mathcal{O}_X \otimes B) &= \overline{\operatorname{span}}\{x \otimes b \otimes u_n : x \in \mathcal{O}_X^n, b \in B\}.
\end{aligned}$$

Noting that id_B, and σ_{23} are isomorphisms (where σ_{23} is the map which exchanges the second and third tensor factors) and δ^γ is an injective $*$-homomorphism, we see that $\sigma_{23} \circ (\delta^\gamma \otimes \mathrm{id}_B)$ is an injective $*$-homomorphism and is thus an isomorphism onto its image. But its image is $\overline{\operatorname{span}}\{x \otimes b \otimes u_n : x \in \mathcal{O}_X^n, b \in B\}$ and by (6.1) this is isomorphic to $\mathcal{O}_{X \otimes B}$. Therefore we have shown that

$$\mathcal{O}_{X \otimes B} \cong \mathcal{O}_X \otimes B.$$

This result is already known, and was used in [5] to prove facts about coactions on Cuntz–Pimsner algebras.

Example 6.5. Given a C^*-algebra A and a completely positive map Φ, Kwasniewski defines [9, Definition 3.5] a crossed product of A by Φ denoted by $C^*(A, \Phi)$. In [9, Theorem 3.13], it is shown that $C^*(A, \Phi) \cong \mathcal{O}_{X_\Phi}$ where X_Φ is the correspondence associated with Φ as in Definition 2.4. If Φ is an endomorphism this reduces to the Exel crossed product [3].

Suppose that A and B are C^*-algebras and $\Phi : A \to A$ and $\Psi : B \to B$ are completely positive maps. Furthermore, suppose that the associated correspondences X_Φ and X_Ψ are ideal-compatible and Katsura nondegenerate. Then our main result states that $\mathcal{O}_{X_\Phi \otimes X_\Psi} \cong \mathcal{O}_{X_\Phi} \otimes_{\mathbb{T}} \mathcal{O}_{X_\Psi}$. Recalling that $X_\Phi \otimes X_\Psi \cong X_{\Phi \otimes \Psi}$ and using the crossed product notation, we get

$$C^*(A \otimes B, \Phi \otimes \Psi) \cong C^*(A, \Phi) \otimes_{\mathbb{T}} C^*(B, \Psi).$$

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References

[1] B. Blackadar, *Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras* (Springer, Berlin, 2006).

[2] S. Echterhoff, S. Kaliszewski, J. Quigg and I. Raeburn, 'A categorical approach to imprimitivity theorems for C^*-dynamical systems', *Dyn. Syst.* **180** (2002).

[3] R. Exel, 'A new look at the crossed product of a C^*-algebra', *Ergodic Theory Dynam. Systems* **28**(3) (2008), 749–789.

[4] S. Gaal, *Linear Analysis and Representation Theory*, Grundlehren der mathematischen Wissenschaften, 198 (Springer, New York, 1973).

[5] S. Kaliszewski, J. Quigg and D. Robertson, 'Coactions on Cuntz–Pimsner algebras', Preprint, 2012, arXiv:1204.5822.

[6] T. Katsura, 'A construction of C^*-algebras from C^*-correspondences', *Contemp. Math.* **335** (2003), 173–182.

[7] T. Katsura, 'On C^*-algebras associated with C^*-correspondences', *J. Funct. Anal.* **217**(2) (2004), 366–401.

[8] A. Kumjian, 'Notes on C^*-algebras of graphs', *Contemp. Math.* **228** (1998), 189–200.

[9] B. K. Kwaśniewski, 'Exel's crossed product and crossed products by completely positive maps', Preprint, 2014, arXiv:1404.4929.

[10] E. C. Lance, *Hilbert C^*-Modules: A Toolkit for Operator Algebraists*, London Mathematical Society Lecture Note Series, 210 (Cambridge University Press, Cambridge, 1995).

[11] M. V. Pimsner, 'A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by Z', *Fields Inst. Commun.* **12** (2001), 189–212.

[12] J. C. Quigg, 'Discrete C^*-coactions and C^*-algebraic bundles', *J. Aust. Math. Soc. (Ser. A)* **60**(02) (1996), 204–221.

[13] I. Raeburn, *Graph Algebras*, Regional Conference Series in Mathematics, 103 (American Mathematical Society, Providence, RI, 2005).

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