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LARGE INTERVAL SOLUTION OF THE EMDEN–FOWLER EQUATION USING A MODIFIED ADOMIAN DECOMPOSITION METHOD WITH AN INTEGRATING FACTOR

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Abstract

We propose a new Adomian decomposition method (ADM) using an integrating factor for the Emden–Fowler equation. With this method, we are able to solve certain Emden–Fowler equations for which the traditional ADM fails. Numerical results obtained from testing our linear and nonlinear models are far more reliable and efficient than those from existing methods. We also present a complete error analysis and a convergence criterion for this method. One drawback of the traditional ADM is that the interval of convergence of the Adomian truncated series is very small. Some techniques, such as Pade approximants, can enlarge this interval, but they are too complicated. Here, we use a continuation technique to extend our method to a larger interval.

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1. Introduction

Scientists and engineers have great interest in the Emden–Fowler equation

$$y''(x) + \frac{p}{x}y'(x) + af(x)M(y) = 0 \tag{1.1}$$

because of its important application in many physical and mathematical models. There are several research directions for equations of this type. Some researchers have studied their qualitative properties [1, 5, 9]; for example, Bartolucci and Montefusco [1] investigated the concentration-compactness problem and the mass-quantization properties. Others have established the existence and uniqueness of their

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solutions [7, 13, 20]. For example, Guo et al. [6] used a fixed point theorem to obtain such existence and uniqueness. Recently, much attention has been paid to the computation of their numerical solutions [2, 12, 17–19]. Chowdhury and Hashim [2], for example, used a homotopy asymptotic method to find their approximate solutions.

Adomian developed the concept of Adomian decomposition method (ADM), which has been widely used for nonlinear problems. It is well known in the literature that this algorithm can obtain a rapidly convergent solution. However, there are many variants of the ADM: Lin [10] solved double singular boundary value problems using a modified ADM; Khuri [8] combined the Laplace transform with the ADM to numerically solve Bratu's problem; and Duan and Rach [3] modified this method to get a clever recursion scheme for solving boundary value problems without undetermined coefficients. But the differential equations in article [3] are limited to

$$\frac{d^k u(x)}{dx^k} = Nu(x) + g(x)$$

with k boundary conditions. The purpose of this paper is to introduce a new reliable modification of the ADM, with an integrating factor, to solve the Emden–Fowler equation. This extends our previous work [11] on first-order ordinary differential equations to second-order singular ones. Our new method gives a better approximation to the solution than the traditional ADM. Also, it handles the convergence issue successfully in certain problems, where the standard ADM fails. Moreover, we have a complete error and convergence analysis for this new method along with the existence and uniqueness of its exact solution.

Wazwaz [17, 18] employed the ADM to solve singular differential equations of Lane–Emden type, and Emden–Fowler equation (1.1) with $p \geq 0$ [19]. Our method is the same as the ones in these papers, except that it is for a more general case with $p > -1$. Besides, Wazwaz did not explain the construction of the differential operators

$$L(\cdot) = x^{-p}\frac{d}{dx}\Big(x^p \frac{d}{dx}\Big), \quad L^{-1}(\cdot) = \int_0^x x^{-p}\int_0^x x^p(\cdot)\,dx\,dx. \tag{1.2}$$

Although he later gave a verification of (1.2) in article [19], it was still unknown how to construct them. In this paper, we use an integrating factor to derive the same results. This uncovers the general procedure of getting the operators, L and L^{-1}, which can be applied to other types of differential equations.

The numerical solution obtained by the ADM usually exists only on a small interval. The solution diverges rapidly when its domain increases, which is a fatal problem for the ADM. However, there have been many attempts to extend its solution to a larger interval; we review a few of them below.

Many scholars have used the Laplace–Adomian–Pade technique to compute the analytic solutions of differential equations. Their results were a great improvement on the interval of convergence of the Adomian truncated series. Wu [21] used the Laplace transform to get the Lagrange multiplier needed in the variational iteration method, then combined Adomian–Pade technique to solve an initial value problem

(IVP) of regular differential equations. Similarly, Tsai and Chen [14, 15] solved a second-order IVP and the first-order Riccati differential equations, respectively, and Zeng and Qin [22] solved fractional differential equations. None of them studied the second-order singular differential equations; it is unclear at this stage whether the Laplace–Adomian–Pade technique works for these singular problems.

The Laplace–Adomian–Pade calculation, however, is very complicated. Here, we introduce a new method enlarging the domain. It is a trivial continuation to find a series solution in one interval, then use the information at the end point as initial condition to solve the equation in the next interval. After combining solutions in all intervals, we have a solution in a large domain. Certainly, the accuracy of this numerical solution quickly deteriorates after each continuation. However, our modified ADM solution has very high precision. Hence, even though some error magnifications are inevitable, its final accuracy is still assured.

2. Description of technique

In order to give a brief description of the standard ADM, we consider the general differential equation

$$Lu + Ru + Nu = g(x),$$

where L is the highest-order linear operator which is invertible, R is an operator of the remaining linear part and N is a nonlinear operator. Applying L^{-1}, the inverse operator of L, we obtain

$$u = L^{-1}g + \varphi(x) - L^{-1}Ru - L^{-1}Nu, \tag{2.1}$$

where $\varphi(x)$ satisfies $L\varphi = 0$ and can be obtained from the given initial conditions.

According to the ADM, we consider the convergent series solution, $u(x)$, of the form

$$u = \sum_{k=0}^{\infty} u_k. \tag{2.2}$$

Suppose that the nonlinear terms are decomposed into the series

$$Nu = \sum_{k=0}^{\infty} A_k(u_0, u_1, \ldots, u_k), \tag{2.3}$$

where the components A_k are called *Adomian polynomials*. Substituting (2.2) and (2.3) into equation (2.1) and defining all the terms by

$$\begin{aligned}
u_0 &= L^{-1}g + \varphi(x),\\
u_1 &= -L^{-1}Ru_0 - L^{-1}A_0,\\
u_2 &= -L^{-1}Ru_1 - L^{-1}A_1,\\
&\vdots\\
u_{i+1} &= -L^{-1}Ru_i - L^{-1}A_i,
\end{aligned}$$

we solve u_i successively for $i = 1, 2, \ldots$.

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There are many choices of Adomian polynomials A_i. Wazwaz [16] gave

$$\begin{aligned} A_0 &= N(u_0), \\ A_1 &= u_1 N'(u_0), \\ A_2 &= u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0), \\ A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0), \\ &\vdots \end{aligned} \tag{2.4}$$

and the other polynomials can be generated similarly. Note that A_0 depends only on u_0, A_1 depends only on u_0 and u_1 and so on. Note that El-Kalla [4] made another choice of such polynomials,

$$A_n = N\Big(\sum_{i=0}^{n} u_i\Big) - \sum_{i=0}^{n-1} A_i. \tag{2.5}$$

We now present our modified ADM with an integrating factor. Consider the IVP of the Emden–Fowler equation,

$$y''(x) + \frac{p}{x} y'(x) + af(x)M(y) + ar(x) = 0, \quad y(0) = q, \quad y'(0) = 0, \tag{2.6}$$

where $p > -1$ and a are constants; $f(x), r(x)$ and $M(x)$ are continuous functions. Assume that this IVP problem has a unique solution in $[0, c]$, where $c > 0$. In order to find this solution, we multiply the integrating factor $\exp\left(\int (p/x)\, dx\right) = x^p$ on both sides of (2.6), which yields

$$\frac{d}{dx}(x^p y') = x^p\Big(y'' + \frac{p}{x} y'\Big) = -ax^p[f(x)M(y) + r(x)].$$

Then integrating and simplifying both sides, we get

$$x^p y'(x) = y'(0) \cdot 0^p - a \int_0^x s^p [f(s)M(y(s)) + r(s)]\, ds, \tag{2.7}$$

$$y'(x) = -ax^{-p} \int_0^x s^p [f(s)M(y(s)) + r(s)]\, ds, \tag{2.8}$$

where $p \geq 0$. We need to be more careful if $-1 < p < 0$, since $s^p \to \infty$ as $s \to 0^+$ and the integral in (2.7) is an improper integral. In this case, we choose $b \in (0, c)$, so that

$$x^p y'(x) = y'(b) \cdot b^p - a \int_b^x s^p [f(s)M(y(s)) + r(s)]\, ds. \tag{2.9}$$

By L'Hôpital's rule,

$$\lim_{b \to 0^+} y'(b) \cdot b^p = \lim_{b \to 0^+} \frac{y'(b)}{b^{-p}} = \lim_{b \to 0^+} \frac{y''(b)}{-pb^{-p-1}} = \lim_{b \to 0^+} \frac{-1}{p} y''(b) b^{p+1} = 0.$$

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Since $f(s)M(y(s)) + r(s)$ is continuous, Lemma 3.1(i) in Section 3 guarantees that the improper integral

$$\int_0^x s^p[f(s)M(y(s)) + r(s)]\,ds = \lim_{b\to 0^+}\int_b^x s^p[f(s)M(y(s)) + r(s)]\,ds$$

converges. So, after taking the limit as $b \to 0^+$ on both sides of (2.9), we have the same result as in equation (2.8).

Integrating (2.8) and using initial conditions $y(0) = q$, we obtain

$$y(x) = q - a\int_0^x t^{-p}\int_0^t s^p[f(s)M(y(s)) + r(s)]\,ds\,dt. \tag{2.10}$$

Again, more care is needed if $p \geq 0$, since $t^{-p} \to \infty$ as $t \to 0^+$. Similarly, we select a $b \in (0, c)$, then

$$y(x) = y(b) - a\int_b^x t^{-p}\int_0^t s^p[f(s)M(y(s)) + r(s)]\,ds\,dt. \tag{2.11}$$

Lemma 3.2 in Section 3 ensures that the improper integral

$$\int_0^x t^{-p}\int_0^t s^p[f(s)M(y(s)) + r(s)]\,ds\,dt = \lim_{b\to 0^+}\int_b^x t^{-p}\int_0^t s^p[f(s)M(y(s)) + r(s)]\,ds\,dt$$

converges. So, after taking the limit of (2.11) as $b \to 0^+$, we have the same result (2.10).

Now the solution of (2.10) by the ADM has the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

where

$$y_0(x) = q - a\int_0^x \left[t^{-p}\int_0^t s^p r(s)\,ds\right]dt, \tag{2.12}$$

$$y_m(x) = -a\int_0^x \left[t^{-p}\int_0^t s^p f(s)A_{m-1}(s)\,ds\right]dt, \quad m \geq 1. \tag{2.13}$$

The Adomian polynomials

$$M(y) = \sum_{n=0}^{\infty} A_n(y_0, y_1, \ldots, y_n)$$

can be Wazwaz's representation (2.4), that is,

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}M\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0} \tag{2.14}$$

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in general form; El-Kalla's formula (2.5), that is, $A_0 = M(y_0)$ and

$$A_n = M\Big(\sum_{i=0}^{n} y_i\Big) - M\Big(\sum_{i=0}^{n-1} y_i\Big) \quad \text{for } n \geq 1; \tag{2.15}$$

or others.

This method is the same as that of Wazwaz [19] who only dealt with the case $p \geq 0$ in (2.6). We extend it to $p > -1$ and carefully justify all improper integrals. Besides, Wazwaz did not point out why his ADM used (1.2); this seems clear now from the point view of the integrating factor. Furthermore, we easily extend this method to solve more general differential equations, such as

$$y''(x) + L(x)y'(x) + k(x)N(y(x)) + r(x) = 0.$$

3. Analysis of convergence

We now study the convergence and error analysis of our method. Let $\mathfrak{C}(I)$ be the space of all continuous functions on an interval I, and $\mathfrak{C}^k(I)$ be the set of all k times differentiable functions on I with continuous kth derivatives. For the rest of the paper, the norm we use on $\mathfrak{C}(I)$ is the supremum norm $\|g(x)\| = \max_{x\in I} |g(x)|$, which is always finite for $g \in \mathfrak{C}(I)$ with I compact.

Lemma 3.1. *If* $-1 < p < 0$ *and* $h(x) \in \mathfrak{C}[0, c]$*, then:*

(i) *the improper integral* $\int_0^t s^p h(s)\, ds$ *converges and is in* $\mathfrak{C}[0, c] \cap \mathfrak{C}^1(0, c]$*;*
(ii) $\int_0^x [t^{-p} \int_0^t s^p h(s)\, ds]\, dt$ *exists and is in* $\mathfrak{C}^2[0, c]$.

Proof. (i) For a fixed $t \in (0, c]$, it is well known from calculus that the improper integral $\int_0^t s^p\, ds$ converges. Observe that

$$0 \leq |s^p h(s)| \leq s^p \|h(s)\|, \quad s \in (0, t],$$

where $\|h(s)\|$ is finite for $h(x) \in \mathfrak{C}[0, c]$. By the comparison test, the improper integral $\int_0^t |s^p h(s)|\, ds$ converges too, and so does $\int_0^t s^p h(s)\, ds$. The latter integral is continuous at $t = 0$ since

$$\Big|\int_0^t s^p h(s)\, ds\Big| \leq \int_0^t |s^p h(s)|\, ds \leq \|h(s)\| \int_0^t s^p\, ds$$
$$= \|h(s)\| \frac{t^{p+1}}{p+1} \to 0, \quad \text{as } t \to 0^+. \tag{3.1}$$

As a consequence, $\int_0^{c/2} s^p h(s)\, ds$ also converges, and is a constant. Since $s^p h(s) \in \mathfrak{C}(0, c]$, $\int_{c/2}^t s^p h(s)\, ds \in \mathfrak{C}^1(0, c]$, so does

$$\int_0^t s^p h(s)\, ds = \int_0^{c/2} s^p h(s)\, ds + \int_{c/2}^t s^p h(s)\, ds.$$

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(ii) Since $t^{-p}, \int_0^t s^p h(s)\,ds \in \mathfrak{C}[0,c]$ by (i), it follows that

$$\int_0^x \left[t^{-p}\int_0^t s^p h(s)\,ds\right] dt \in \mathfrak{C}^1[0,c] \text{ exists.}$$

Notice that

$$\int_0^x \left[t^{-p}\int_0^t s^p h(s)\,ds\right] dt = \int_0^{c/2} \left[t^{-p}\int_0^t s^p r(s)\,ds\right] dt + \int_{c/2}^x \left[t^{-p}\int_0^t s^p r(s)\,ds\right] dt. \tag{3.2}$$

On the right-hand side of (3.2), the first integral exists and is a constant, and the second integral is in $\mathfrak{C}^2(0,c]$ due to $t^{-p}, \int_0^t s^p h(s)\,ds \in \mathfrak{C}^1(0,c]$. Then the left-hand side of (3.2) is also in $\mathfrak{C}^2(0,c]$. With more detailed analysis, we can show that the second derivative of (3.2) exists, and is continuous at $x = 0$. This completes the proof. □

Lemma 3.2. *If $p \ge 0$ and $h(x) \in \mathfrak{C}[0,c]$, then the improper integral $\int_0^x [t^{-p}\int_0^t s^p h(s)\,ds]\,dt$ converges and is in $\mathfrak{C}^2[0,c]$.*

Proof. Since $s^p, h(s) \in \mathfrak{C}[0,c]$, the regular integral

$$\int_0^t s^p h(s)\,ds \in \mathfrak{C}^1[0,c].$$

For $t \in (0,c]$, from (3.1),

$$0 \le \left|t^{-p}\int_0^t s^p h(s)\,ds\right| \le t^{-p}\|h(s)\|\frac{t^{p+1}}{p+1} = \frac{\|h(s)\|}{p+1}t.$$

So by the comparison test, the improper integral $\int_0^x |t^{-p}\int_0^t s^p h(s)\,ds|dt$ converges and so does $\int_0^x [t^{-p}\int_0^t s^p h(s)\,ds]\,dt$. The latter integral in $\mathfrak{C}^2[0,c]$ follows by the same reasoning as the proof of Lemma 3.1(ii), and this completes the proof. □

Corollary 3.3 ($\mathfrak{C}^2$ solution). *Assume that $p > -1$, $f(x), r(x), M(x) \in \mathfrak{C}[0,c]$ and $y(x) \in \mathfrak{C}[0,c]$ is a solution of (2.10). Then $y(x) \in \mathfrak{C}^2[0,c]$ is nonsingular at $x = 0$ with finite $\|y(x)\|$.*

Proof. Since $f(x), r(x), M(y(x)) \in \mathfrak{C}[0,c]$, by both Lemmas 3.1 and 3.2, the integral in (2.10) is in $\mathfrak{C}^2[0,c]$, and then so is the left-hand side $y(x)$ of (2.10). The proof is complete. □

Proposition 3.4 ($\mathfrak{C}^2$ series). *If $p > -1$, $f(x), r(x)$ and $M(x) \in \mathfrak{C}[0,c]$, then $y_m(x)$ generated from (2.12), (2.13) and (2.15) is in $\mathfrak{C}^2[0,c]$ for $m = 0, 1, 2, \ldots$.*

Proof. According to Lemmas 3.1 and 3.2, we can prove the two cases of p together. Recall that

$$y_0(x) = q - a\int_0^x \left[t^{-p}\int_0^t s^p r(s)\,ds\right] dt.$$

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Since $r(s) \in \mathfrak{C}[0,c]$, by both lemmas the integral exists, and is in $\mathfrak{C}^2[0,c]$, then so is $y_0(x)$.

Suppose, for certain m,

$$y_0(x), y_1(x), \dots, y_m(x) \in \mathfrak{C}^2[0,c].$$

Since $f(s), M(s) \in \mathfrak{C}[0,c]$, from (2.15) we have $f(s)A_m(s) \in \mathfrak{C}[0,c]$. Again from the lemmas it follows that

$$y_{m+1}(x) = -a \int_0^x \left[t^{-p} \int_0^t s^p f(s) A_m(s)\, ds \right] dt \in \mathfrak{C}^2[0,c].$$

Now, by induction, the proposition holds. □

Corollary 3.5 (Nonsingular and bounded). *Under the assumptions of Proposition 3.4, $y_m(x)$ is nonsingular at $x = 0$ with finite $\|y_m(s)\|$ for $m = 0, 1, 2, \dots$.*

Theorem 3.6 (Uniqueness). *Assume that $p > -1$, $f(x), r(x) \in \mathfrak{C}[0,c]$, $M(x)$ is Lipschitz continuous with $|M(y) - M(z)| \le L|y - z|$ and $\alpha = |a|L\|f\|c^2/(2(p+1)) < 1$. Then equation (2.6) has at most one solution in $[0,c]$.*

Proof. Let $y(x)$ and $z(x)$ be two solutions of (2.6), then they satisfy (2.10), and are nonsingular at $x = 0$ by Corollary 3.3.

Moreover,

$$\begin{aligned}
|y(x) - z(x)| &= \left| a \int_0^x t^{-p} \int_0^t s^p f(s)[M(y(s)) - M(z(s))]\, ds\, dt \right| \\
&\le |a| \left[\int_0^x t^{-p} \int_0^t s^p \|f\| L \|y - z\|\, ds\, dt \right] \\
&\le |a| L \|f\| \cdot \|y - z\| \int_0^x t^{-p} \frac{t^{p+1}}{p+1} dt \\
&\le |a| L \|f\| \cdot \|y - z\| \frac{c^2}{2(p+1)} = \alpha \|y - z\|, \\
\|y - z\| &= \max_{x \in [0,c]} |y(x) - z(x)| \le \alpha \|y - z\|.
\end{aligned}$$

So we have $(1 - \alpha)\|y - z\| \le 0$, and then $y = z$ when $\alpha < 1$. This completes the proof. □

Theorem 3.7 (Convergence). *Under the assumptions of Theorem 3.6, the* ADM *solution using (2.12), (2.13) and (2.15) converges to a solution of (2.10).*

Proof. Recall that $y_i \in \mathfrak{C}^2[0,c]$ for all i by Proposition 3.4. Let

$$S_n(x) = \sum_{i=0}^{n} y_i(x) \in \mathfrak{C}([0,c]).$$

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denote the nth partial sum. We now prove that this sequence $\{S_n\}$ is a Cauchy sequence in the Banach space $(\mathfrak{C}([0,c]), \|\cdot\|)$ with supremum norm. Hence $\{S_n\}$ converges to some function in $\mathfrak{C}([0,c])$.

For $x \in [0,c]$, by (2.13) and (2.15),

$$\begin{aligned}
|S_{m+1}(x) - S_m(x)| = |y_{m+1}(x)| &= \left| a \int_0^x t^{-p} \int_0^t s^p f(s) A_m \, ds\, dt \right| \\
&\le |a| \int_0^x t^{-p} \int_0^x s^p |f(s)| |M(S_m) - M(S_{m-1})| \, ds\, dt \\
&\le |a| \cdot \|f\| L \|S_m - S_{m-1}\| \int_0^x t^{-p} \int_0^t s^p dt \\
&\le \alpha \|S_m - S_{m-1}\|, \\
\|S_{m+1} - S_m\| = \max_{x\in[0,c]} |S_{m+1}(x) - S_m(x)| &\le \alpha \|S_m - S_{m-1}\|
\end{aligned}$$

for all m. Therefore,

$$\|S_{m+1} - S_m\| \le \alpha \|S_m - S_{m-1}\| \le \alpha^2 \|S_{m-1} - S_{m-2}\| \le \cdots \le \alpha^m \|S_1 - S_0\|.$$

By Corollary 3.5, $\|y_1(x)\|$ is finite, so for $n > m$,

$$\begin{aligned}
\|S_n - S_m\| &\le \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \cdots + \|S_{m+1} - S_m\| \\
&\le (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m) \|S_1 - S_0\| \\
&= \alpha^m \Big(\frac{1 - \alpha^{n-m}}{1 - \alpha} \Big) \|y_1(x)\| \le \frac{\alpha^m}{1-\alpha} \|y_1(x)\| \to 0,
\end{aligned} \tag{3.3}$$

as $m \to \infty$.

Let the limit of $\{S_n\}$ be

$$y(x) = \lim_{n\to\infty} S_n(x) \in \mathfrak{C}([0,c]). \tag{3.4}$$

In view of (2.15) or (2.5),

$$\sum_{i=0}^{n} A_i = M(S_n).$$

So by (2.12) and (2.13),

$$\begin{aligned}
S_n = \sum_{i=0}^{n} y_i &= q - a \int_0^x t^{-p} \int_0^t s^p \Big[r(s) + f(s) \sum_{i=0}^{n-1} A_i \Big] ds\, dt \\
&= q - a \int_0^x t^{-p} \int_0^t s^p [r(s) + f(s) M(S_{n-1})] \, ds\, dt.
\end{aligned}$$

Taking its limit $n \to \infty$, we see that $y(x)$ satisfies (2.10), which completes the proof. □

THEOREM 3.8 (Existence). *Under the assumptions of Theorem 3.6, (2.6) has a unique solution in* $\mathfrak{C}^2[0,c]$.

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Proof. According to Theorem 3.7, (2.10) has a solution $y(x)$ of (3.4). By Corollary 3.3, $y(x) \in \mathfrak{C}^2[0, c]$. So $y(x)$ is a solution of (2.6). The uniqueness part follows from Theorem 3.6. The proof is complete. □

Theorem 3.9 (Error bound). *Under the assumptions of Theorem 3.6, the* ADM *solution* $\sum_{i=0}^{m} y_i$ *of* (2.6) *using* (2.12), (2.13) *and* (2.15) *has maximal truncation error*

$$\Big|y(x) - \sum_{i=0}^{m} y_i(x)\Big| \le \frac{\alpha^{m+1}}{L(1-\alpha)} \|M(y_0(x))\| \quad \text{for } x \in [0, c].$$

Proof. Observe from (2.13) that

$$\begin{aligned} |y_1(x)| &= \Big|a \int_0^x \Big[t^{-p} \int_0^t s^p f(s) A_0(s)\, ds\Big]\, dt\Big| \\ &\le |a| \cdot \|f\| \Big|\int_0^x t^{-p} \int_0^t s^p M(y_0(s))\, dt\Big| \\ &\le |a| \cdot \|f\| \cdot \|M(y_0(x))\| \frac{c^2}{2(p+1)} \\ &= \frac{\alpha}{L} \|M(y_0(x))\|, \quad x \in [0, c]. \end{aligned}$$

So $\|y_1(x)\|$ has the same upper bound. Let $n \to \infty$ in (3.3), we achieve the desired bound, and so the proof is now complete. □

4. Numerical examples

In the numerical experiments below, we test our new ADM with integrating factors for the following examples. The computations are performed using Mathematica software with 32 working digits. We use the representation $3.7(-2) = 3.7 \cdot 10^{-2}$ for errors.

Example 4.1. Consider Wazwaz's Emden–Fowler equation [19]

$$y''(x) + \frac{2}{x} y'(x) + a y^n(x) = 0, \quad y'(0) = 0, \quad y(0) = 1. \tag{4.1}$$

Multiplying the differential equation by the integrating factor x^2 and simplifying, we get

$$[x^2 y'(x)]' = x^2 y''(x) + 2x y'(x) = -a x^2 y^n(x),$$

$$x^2 y'(x) = A - a \int_d^x s^2 y^n(s)\, ds, \tag{4.2}$$

$$y'(x) = \frac{A}{x^2} - \frac{a}{x^2} \int_d^x s^2 y^n(s)\, ds, \tag{4.3}$$

where the constant A is obtained by setting $x = d$ in (4.2), that is,

$$A = d^2 y'(d). \tag{4.4}$$

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Now integrating both sides of (4.3) yields

$$y(x) = B - \frac{A}{x} - a\int_d^x \frac{1}{t^2}\int_d^t s^2 y^n(s)\,ds\,dt, \tag{4.5}$$

which on substituting $x = d$ in (4.5) becomes

$$B = y(d) + \frac{A}{d} = y(d) + dy'(d). \tag{4.6}$$

Since the given initial values are $y(0) = 1$ and $y'(0) = 0$, and since $d = 0$, $A = 0$ and $B = 1$, we have

$$y(x) = 1 - a\int_0^x \frac{1}{t^2}\int_0^t s^2 y^n(s)\,ds\,dt. \tag{4.7}$$

For numerical testing, consider the case $n = 1$, when equation (4.1) has the exact solution $y(x) = (\sin\sqrt{a}x)/\sqrt{a}x$ which can be compared to the solution by the ADM. Applying (2.12) and (2.13) to (4.7), our ADM with the integrating factor becomes

$$y_0(x) = 1,$$
$$y_m(x) = -a\int_0^x \frac{1}{t^2}\int_0^t s^2 y_{m-1}(s)\,ds\,dt, \quad m \geq 1,$$

in which case the Adomian polynomials in (2.14) and (2.15) are the same. Choosing $a = 1$ for the experiment, our ADM generates

$$\begin{aligned} y_0 &= 1, \\ y_1 &= -0.166\,667x^2, \\ y_2 &= 0.008\,333\,33x^4, \\ y_3 &= -0.000\,198\,413x^6, \\ &\vdots \end{aligned}$$

Next, we apply our convergence theorems in Section 3 to this example. For $p = 2$, $M(y) = y$ and $f(x) = 1$, we have $L = 1$, $\|f\| = 1$ and $\alpha = c^2/6$ in Theorem 3.6. So, Theorem 3.7 ensures that our ADM solution converges in the interval $[0, c]$ for $c < \sqrt{6}$. In fact, this solution exists in a even bigger interval, for example, $[0, 5]$. If we choose $c = 1$, then $\alpha = 1/6$ and Theorem 3.9 gives the error bound

$$\left| y(x) - \sum_{i=0}^m y_i(x) \right| \leq \frac{1}{5 \cdot 6^{m\cdot}} \quad \text{for } x \in [0, 1].$$

The maximal truncation errors and their error bounds for $0 \leq x \leq 1$ and a few values of m are shown in Table 1.

The traditional ADM for (4.7) computes the sequence

$$y_{m+1} = \int_0^x \int_0^t \left[-\frac{2}{s} y_m'(s) - y_m(s) \right] ds\,dt.$$

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TABLE 1. Maximal errors and bounds of the traditional and our ADM for Example 4.1.

m	Traditional ADM	Our method	Error bound
3	1.4	2.7(−6)	1.0(−3)
4	2.8	2.5(−8)	1.5(−4)
5	5.7	1.6(−10)	2.5(−5)
6	11.3	7.6(−13)	4.2(−6)
7	22.7	2.8(−15)	7.1(−7)

This yields

$$\begin{aligned} y_0 &= 1, \\ y_1 &= -0.5x^2, \\ y_2 &= x^2 + 0.041\,667x^4, \\ y_3 &= -2x^2 - 0.111\,11x^4 - \cdots, \\ y_4 &= 4x^2 + 0.240\,741x^4 + \cdots, \\ &\vdots \end{aligned}$$

Actually, its solution $\sum_{i=0}^{n} y_i(x)$ does not converge since $y_1, y_2, \ldots$ all have an x^2 term; this is verified in Table 1.

One major defect of the traditional ADM is that it cannot be used for a large interval or global solution. Its numerical solution diverges rapidly when the applied domain increases. Here, we employ a trivial continuation to extend its solution domain. We try to solve (4.1) in many small intervals and combine them together. To be more precise, after we find the numerical solution $y(x)$ of (4.1) on $[0, c]$, we use $y(c)$ and $y'(c)$ as the initial conditions to compute $y(x)$ on $[c, c']$, and then approximate $y(x)$ on $[c', c'']$ etc. Repeating this process, we obtain a solution in a large interval.

The details of this continuing process are as above (see (4.2)–(4.6)). Suppose that we already have the numerical solution $y(x)$ of (4.1) on $[0, d]$. Then the extended solution on $[d, d']$ is

$$y(x) = y(d) + dy'(d)\Big(1 - \frac{d}{x}\Big) - a\int_d^x \frac{1}{t^2}\int_d^t s^2 y^n(s)\,ds\,dt.$$

So for $n = 1$ our ADM solution can be produced from

$$y_0(x) = y(d) + dy'(d)\Big(1 - \frac{d}{x}\Big), \tag{4.8}$$

$$y_m(x) = -a\int_0^x \frac{1}{t^2}\int_0^t s^2 y_{m-1}(s)\,ds\,dt, \quad m \geq 1. \tag{4.9}$$

As a numerical example, let $a = 1$ and $d = 5$. We use the ADM solution previously solved in $[0, 5]$ and apply equations (4.8) and (4.9) to compute the approximate

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TABLE 2. Errors of our ADM in the enlarged interval [0, 10] for Example 4.1.

m	$x = 2$	$x = 3$	$x = 5$	$x = 10$	Max error
8	2.1(−12)	3.1(−9)	3.0(−5)	9.9(−5)	9.9(−5)
9	2.0(−14)	6.7(−11)	1.8(−6)	6.6(−6)	6.6(−6)
10	1.7(−16)	1.2(−12)	8.9(−8)	3.6(−7)	3.6(−7)

solution in [5, 10] as follows:

$$\begin{aligned}
y_0 &= -\frac{2.377\,24}{x} - 0.283\,664,\\
y_1 &= \frac{17.8962}{x} - 8.340\,42 + 1.188\,62x - 0.047\,277\,4x^2,\\
y_2 &= -\frac{32.3591}{x} + 27.3647 - 8.948\,11x + 1.390\,07x^2 + \cdots,\\
&\ \ \vdots
\end{aligned}$$

Its maximal errors and errors at certain middle points in the extended interval [0, 10] are shown in Table 2.

EXAMPLE 4.2. Consider another Emden–Fowler equation by Wazwaz [19]

$$y''(x) + \frac{5}{x}y'(x) + 8a(e^y + 2e^{y/2}) = 0, \quad y'(0) = y(0) = 0. \tag{4.10}$$

This has the exact solution

$$y(x) = -2\ln(1 + ax^2). \tag{4.11}$$

Multiplying both sides of (4.10) by the integrating factor x^5,

$$\begin{aligned}
[x^5y'(x)]' &= x^5y''(x) + 5x^4y'(x) = -8ax^5(e^y + 2e^{y/2}),\\
x^5y'(x) &= A - 8a\int_d^t s^5[e^{y(s)} + 2e^{y(s)/2}]\,ds.
\end{aligned} \tag{4.12}$$

Setting $x = d$ in (4.12), we have the constant

$$A = d^5y'(d).$$

Then, dividing both sides of (4.12) by x^5 and integrating yields

$$y(x) = B - \frac{A}{4x^4} - 8a\int_d^x \frac{1}{t^5}\int_d^t s^5[e^{y(s)} + 2e^{y(s)/2}]\,ds\,dt. \tag{4.13}$$

Again setting $x = d$ in (4.12), we have the constant

$$B = y(d) + \frac{A}{4d^4} = y(d) + \frac{1}{4}dy'(d). \tag{4.14}$$

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Since the given initial values are $y(0) = y'(0) = 0$, we have $d = A = B = 0$ and the integral equation

$$y(x) = -8a \int_0^x \frac{1}{t^5} \int_0^t s^5 [e^{y(s)} + 2e^{y(s)/2}]\, ds\, dt. \tag{4.15}$$

From the power series

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \cdots,$$
$$e^{y/2} = 1 + \frac{y}{2} + \frac{y^2}{8} + \frac{y^3}{48} + \cdots,$$

we obtain

$$e^y + 2e^{y/2} = 3 + 2y + \frac{3y^2}{4} + \frac{5y^3}{24} + \frac{9y^4}{192} + \cdots.$$

Applying (2.12) and (2.13) to (4.15), our ADM with the integrating factor becomes

$$y_0(x) = -8a \int_0^x \frac{1}{t^5} \int_0^t s^5 \cdot 3\, ds\, dt,$$
$$y_m(x) = -8a \int_0^x \frac{1}{t^5} \int_0^t s^5 A_{m-1}(s)\, ds\, dt, \quad m \geq 1.$$

In this case, the Adomian polynomials (2.14) and (2.15) are totally different except for

$$A_0 = 2y_0 + \frac{3y_0^2}{4} + \frac{5y_0^3}{24} + \frac{9y_0^4}{192} + \cdots.$$

If equation (2.15) is used, then we can apply the convergence theorems in Section 3. For this example, we set $p = 5$, $f(x) = 8$ and $M(y) = e^y + 2e^{y/2}$. Assuming $a > 0$, the true solution (4.11) has range $(-\infty, 0]$. By the mean value theorem,

$$M(y) - M(z) = M'(\zeta)(y - z) = (e^\zeta + e^{\zeta/2})(y - z)$$

for ζ between y and z in $(-\infty, 0]$. So the Lipschitz constant $L = 2$ and $\alpha = 4ac^2/3$ in Theorem 3.6. Hence, Theorem 3.7 guarantees that our ADM solution converges in $[0, c]$ for $c < \sqrt{3/a}/2$.

Unfortunately, the ADM with (2.15) takes too much time to compute. So we use (2.14) instead, in the following experiment with $a = 1$. We compare the maximal errors of the traditional and our ADM for $0 \leq x \leq 1$ in Table 3. It is easy to see that the former solution diverges.

We now try the previously stated trivial continuation to extend the domain to a large interval. Suppose that we have the numerical solution $y(x)$ of (4.10) on $[0, d]$. Then as in (4.12)–(4.14) above the extended solution on $[d, d']$ is

$$y(x) = y(d) + \frac{d}{4} y'(d)\Big(1 - \frac{d^4}{x^4}\Big) - 8a \int_d^x \frac{1}{t^5} \int_d^t s^5 [e^{y(s)} + 2e^{y(s)/2}]\, ds\, dt. \tag{4.16}$$

As a numerical example, let $a = 0.1$, $d = 4$, and $d' = 4.5$. We first solve the ADM solution in $[0, 4]$, and then compute the approximate solution of (4.16) in $[4, 4.5]$. The results are shown in Table 4.

TABLE 3. Maximal errors of the traditional and our ADM for Example 4.2.

m	Traditional ADM	Our ADM
2	0.7	1.0(−2)
3	1.4	1.1(−3)
4	2.8	2.8(−4)
5	5.7	3.5(−5)

TABLE 4. Maximal errors in the enlarged interval $[0, 4.5]$ for Example 4.2.

m	Max error $[0, 4]$	m	Max error $[4, 4.5]$
4	3.8(−2)	1	4.2(−2)
6	7.1(−3)	2	4.0(−2)
8	3.1(−4)	3	3.9(−2)

EXAMPLE 4.3. Consider the final Emden–Fowler equation

$$y''(x) + \frac{2}{x}y'(x) + axy^2(x) = 0, \quad y'(0) = 0, \quad y(0) = 1, \tag{4.17}$$

which has no exact solution. A residual function is introduced to measure the accuracy of our numerical solution. Multiplying both sides of equation (4.17) by x,

$$xy''(x) + 2y'(x) + x^2y^2(x) = 0.$$

We use its left-hand side as our residual function

$$R(z) = xz''(x) + 2z'(x) + x^2z^2(x) \tag{4.18}$$

to estimate the error of the numerical solution to

$$z = z_0 + z_1 + \cdots + z_m.$$

If $z(x)$ is the exact solution of (4.17), then the residual in (4.18) becomes a zero function. Normally for numerical solution $z(x)$, $R(x)$ is not identically zero, and $\|R(x)\|$ shows how close $z(x)$ is to the exact solution $y(x)$. So all methods try to minimize the residual function in (4.18).

Now we show our ADM solution of (4.17) with Adomian polynomials (2.14) and extended intervals. Again (2.15) is not used due to its long computing time. We choose $a = 0.1$ in this case, and first solve the solution in $[0, 6]$, then extend it to the intervals $[6, 10]$ and $[10, 12]$. The maximal errors of residual $R(x)$ in these three intervals are listed in Table 5.

5. Conclusion

We recommend a new reliable and efficient numerical method, the ADM with an integrating factor, for solving Emden–Fowler equation. Because of the singularity, this

TABLE 5. Maximal errors of residuals $R(x)$ in $[0, 12]$ for Example 4.3.

m	$\|R\|$ in $[0, 6]$	m	$\|R\|$ in $[6, 10]$	m	$\|R\|$ in $[10, 12]$
50	1.2(−5)	4	9.1(−4)	5	7.8(−2)
51	8.2(−6)	5	1.7(−4)	6	2.7(−2)
52	5.6(−6)	6	3.1(−5)	7	3.5(−3)

type of equation cannot be solved by the traditional ADM. Our numerical results show that only a few terms of a series solution are enough to achieve very high accuracy. We demonstrate how an integrating factor can be used in the ADM, so that it may be applied to equations more general than the Emden–Fowler equation. For example, Wazwaz's methods [17, 18] can be regarded as special cases of our method. An even wider range of problems can be solved by Duan and Rach's method [3] using our integrating factor technique. One can say the same thing about the Laplace–Adomian–Pade method.

We present a complete error and convergence analysis for our ADM method using El-Kalla's Adomian polynomials (2.15). This underlined theory supports the validity of our method. On the other hand, the analysis of the ADM with Wazwaz's representation (2.14) still remains open.

Since our method yields very high precision, we extend the domain of our solution simply by means of a trivial continuation. This process is repeatedly applied to solve Emden–Fowler equations in a large interval. As for the Pade technique to extend the interval, so far it is unknown whether this works for these singular problems, and it deserves further investigation in future.

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