GENERALIZED RAMSEY THEORY FOR GRAPHS XII: BIPARTITE RAMSEY SETS

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Dedicated to Gerhard Ringel on his 60th birthday

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0. Introduction. Following the notation in Faudree and Schelp [**3**], we write $G \to (F, H)$ to mean that every 2-coloring of $E(G)$, the edge set of G, contains a green (the first color) F or a red (the second color) H. Then the *Ramsey number* $r(F, H)$ of two graphs F and H with no isolated vertices has been defined as the minimum p such that $K_p \to (F, H)$.

For bipartite graphs B_1 and B_2 without isolated vertices we define the *bipartite Ramsey set* $\beta(B_1, B_2)$ as the set of pairs (m, n), $m \leq n$, such that $K_{m,n} \to (B_1, B_2)$ and neither $K_{m-1,n}$ nor $K_{m,n-1}$ have this property. Thus the set $\beta(B_1, B_2)$ can be interpreted as a variation of the Ramsey number $r(B_1, B_2)$. Instead of 2-colorings of the complete graph we now consider 2-colorings of the complete bipartite graph.

The two bipartite Ramsey numbers $b(B_1, B_2)$ (the minimum p with $K_{p,p} \to (B_1, B_2)$), and $b'(B_1, B_2)$ (the minimum $p = m+n$ such that $K_{m,n} \to (B_1, B_2)$) were defined already in [**5**]. They are easily expressed in terms of the bipartite Ramsey set $\beta(B_1, B_2)$ which we now write in the convenient form:

$$\beta(B_1, B_2) = \{(m_h, n_h);\ m_h < m_{h+1},\ m_h \leq n_h\} \quad \text{for} \quad 1 \leq h \leq k. \tag{1}$$

Then $b(B_1, B_2) = n_k$, the smallest n_h, and $b'(B_1, B_2) = \min(m_h + n_h)$. Similar bipartite Ramsey problems are considered in Beineke and Schwenk [**1**], Faudree and Schelp [**3**], and Irving [**7**] while general results on Ramsey theory are given in the book by Bollobás [**2**].

It is trivial that $\beta(B_1, B_2) = \beta(B_2, B_1)$. From our Algorithmic Lemma it is easily deduced that $\beta(B_1, B_2)$ is a non-empty, finite set for all possible pairs of bipartite graphs B_1, B_2. Faudree and Schelp [**3**] have already determined $\beta(B_1, B_2)$ for paths:

$$\beta(P_s, P_t) = \{([\tfrac{1}{2}s] + [\tfrac{1}{2}t] - 1,\ [\tfrac{1}{2}(s+t)] - \epsilon)\}, \tag{2}$$

where $\epsilon = 0$ for s odd, $s \geq t-1$,
for s even, t odd, $s \leq t+1$,
and for $s = t$ odd,
and $\epsilon = 1$ otherwise.

Our purposes include the determination of the bipartite Ramsey sets $\beta(B_1, B_2)$ for all pairs of bipartite graphs of order at most five, for all pairs of stars, and for the path-star pairs $(P_s, K_{1,t})$ with $s \leq 5$. Notation and terminology not specifically mentioned will follow that in [**4**].

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1. Algorithmic Lemma. If the bipartite graph B has $p=p(B)$ vertices, let $Z(B)$ be the set of natural numbers z such that B is a subgraph of $K_{z,p-z}$ and $z \leq \frac{1}{2}p$. We use the notation

$$Z(B)=\{z_1, z_2, \ldots, z_L\} \quad \text{with} \quad z_1<z_2<\ldots<z_L. \tag{3}$$

Then for connected B we have $L=1$. By $\beta'=\beta'(B_1, B_2)$ we denote the set of all pairs (a, b), $a \leq b$, such that $K_{a,b} \rightarrow (B_1, B_2)$. Thus of course $\beta(B_1, B_2)$ is a subset of β'. The independent sets of a and b vertices of $K_{a,b}$ are denoted by V_1 and V_2. If these vertices are labelled by i, $1 \leq i \leq a$, and j, $1 \leq j \leq b$, then we describe edges of $K_{a,b}$ only by (i, j) with $i \in V_1$ and $j \in V_2$.

From the definitions we deduce

$$(a, b) \in \beta' \Rightarrow (a+i, b+j) \in \beta' \quad \text{for} \quad i, j \geq 0, \tag{4}$$

$$(a, b) \notin \beta' \Rightarrow (a-i, b-j) \notin \beta' \quad \text{for} \quad i, j \geq 0, \tag{5}$$

$$(a, b) \in \beta \Leftrightarrow (a, b) \in \beta', (a-1, b) \notin \beta', \quad \text{and} \quad (a, b-1) \notin \beta'. \tag{6}$$

For $1 \leq i \leq k-1$ we have by definition $(m_{i+1}-1, n_{i+1}) \notin \beta'$. This together with (1) $(m_{i+1}-1 \geq m_i)$ and (5) yields $(m_i, n_{i+1}) \notin \beta'$. Using (5) again and assuming $n_{i+1} \geq n_i$, we conclude that $(m_i, n_i) \notin \beta'$, and this contradiction to the definition proves that

$$n_1>n_2>\ldots>n_k. \tag{7}$$

It is easy to see that

$$\beta(K_2, B)=\{(z, p(B)-z);\ z \in Z(B)\}. \tag{8}$$

We now derive the bipartite Ramsey set of two copies of K_2 with any bipartite graph B.

THEOREM 1. *If B is a bipartite graph with p vertices, and $Z^*(B)=\{z;\ z \in Z(B),\ z \neq \frac{1}{2}(p-1),\ z-1 \notin Z(B),\ z+1 \notin Z(B)\}$, then*

$$\beta(2K_2, B)=\{(z_{i+1}, p-z_i);\ 1 \leq i \leq L-1\} \cup \{(p-z_L, p-z_L);\ z_L \leq \tfrac{1}{2}(p-1)\} \cup \{(z+1, p-z+1);\ z \in Z^*(B)\}. \tag{9}$$

Proof. We write $\beta(2K_2, B)=\beta$ and $\beta'(2K_2, B)=\beta'$ during this proof, and first show that

$$(x, y) \in \beta' \Leftrightarrow B \subset K_{x-1,y} \quad \text{and} \quad B \subset K_{x,y-1}. \tag{10}$$

($\Rightarrow$) In the special 2-colorings of $K_{x,y}$, where all edges of a $K_{x,1}$ (respectively, of a $K_{1,y}$) are colored green, and all others red, there is no green $2K_2$, and thus a red B exists with $B \subset K_{x-1,y}$ (respectively, $B \subset K_{x,y-1}$).

($\Leftarrow$) In every 2-coloring of $K_{x,y}$ either a green $2K_2$ exists, or the green edges form a star. In the last case $K_{x-1,y}$ or $K_{x,y-1}$ exist with red edges only, so that a red B is guaranteed.

Now $(x, y) \in \beta$ with $x \leq y$ implies $B \subset K_{x-1,y}$ by (6) and (10). Then a number $z \in Z(B)$

exists with $z \leqslant x-1$, $p-z \leqslant y$, that is, $x = z+1+f$, $y = p-z+g$, $f, g \geqslant 0$. From (10) we find $(z+1, p-z+1) \in \beta'$, and this together with (6) shows $g \geqslant 2$, so as $g = 1$, $f \geqslant 1$ must be impossible. Thus the two following conditions are necessary for $(x, y) \in \beta$:

$$x = z+1, \qquad y = p-z+1, \tag{11}$$

$$x = z+1+f, \qquad y = p-z, \qquad 0 \leqslant f \leqslant p-2z-1. \tag{12}$$

For (11) we observe that $(z+1, p-z+1) \in \beta'$ as above, and use (6) and (10) to get

$$(z+1, p-z+1) \in \beta \Leftrightarrow (z, p-z+1) \notin \beta',$$

and the equivalences

$$(z+1, p-z) \notin \beta' \Leftrightarrow B \not\subset K_{z-1,p-z+1}$$

and

$$B \not\subset K_{z+1,p-z+1} \Leftrightarrow z-1 \notin Z(B),\ z+1 \notin Z(B),\ z \neq \tfrac{1}{2}(p-1) \Leftrightarrow z \in Z^*(B).$$

The latter follows since $z = \frac{1}{2}(p-1)$ would give $B \subset K_{z+1,p-z+1}$.

If in (12) $z \neq Z_L$, we use $B \subset K_{z_{i+1}-1,p-z_i}$, $B \subset K_{z_{i+1},p-z_i-1}$, and (10) to get $(z_{i+1}, p-z_i) \in \beta'$. From $B \not\subset K_{z_{i+1}-1,p-z_i-1}$ and (10), it follows that $(z_{i+1}-1, p-z_i) \notin \beta'$ and $(z_{i+1}, p-z_i-1) \notin \beta'$. We then note that (6) enables us to conclude $(z_{i+1}, p-z_i) \in \beta$, and $(x, y) \notin \beta$ for $f \neq z_{i+1}-z_i-1$ in (12). It remains to consider $z = z_L$ in (12). Here we use $B \subset K_{p-z_L-1,p-z_L}$ (as $z \leqslant \frac{1}{2}(p-1)$ in (12)), and $B \not\subset K_{p-z_L-1,p-z_L-1}$ to deduce, as before, from (10) and (6) that $(x, y) \in \beta$ holds only for $x = y = p-z_L$, $z_L \leqslant \frac{1}{2}(p-1)$. □

In the following we denote by $b_i(B) = b_i$ the maximum of all line independence numbers of the complements of B with regard to $K_{z_i,p-z_i}$. We now find the bipartite Ramsey set of the 3-point path with any bipartite graph.

THEOREM 2. *If B is a bipartite graph with p vertices, and $\bar{Z}(B) = \{z_i \in Z(B);$ $b_j(B) < b_i(B)$ for $1 \leqslant j \leqslant i-1\}$, then*

$$\beta(P_3, B) = \{(z_i, p-b_i(B));\ z_i \in \bar{Z}(B)\}. \tag{13}$$

Proof. Again it is convenient to write β and β' for $\beta(P_3, B)$ and $\beta'(P_3, B)$. We first determine the set β' by showing that

$$(x, y) \in \beta' \Leftrightarrow z_i \in Z(B) \text{ exists with } x \geqslant z_i,\ y \geqslant p-b_i(B). \tag{14}$$

($\Rightarrow$) If the edges $(1, 1), (2, 2), \ldots, (x, x)$ of $K_{x,y}$ are colored green and all others red, then no green P_3 and thus a red B exists in $K_{x,y}$. The subgraph $K_{z_i,p-z_i}$ of $K_{x,y}$ with the vertices of this red B contains at most b_i of the x independent green edges. Then z_i vertices either belong to V_1 (or to V_2) and at least $p-z_i-b_i$ (or z_i-b_i) of the vertices in V_2 are among the vertices $x+1, x+2, \ldots, y$, that is, $y-x \geqslant p-z_i-b_i$ (or $y-x \geqslant z_i-b_i$). These inequalities yield $y \geqslant p-b_i$ if $x \geqslant z_i$ (or $x \geqslant p-z_i \geqslant z_i$).

($\Leftarrow$) Because of (4) it suffices to show $(z_i, p-b_i) \in \beta'$ for $z_i \in Z(B)$. In any 2-coloring of $K_{z_i,p-b_i}$ we either find a green P_3, or at least $p-b_i-z_i$ vertices in V_2 are incident only with red edges. A subgraph $K_{z_i,p-z_i}$ of $K_{z_i,p-b_i}$, in which these $p-b_i-z_i$ vertices are among the $p-z_i$ vertices, contains at most b_i green edges, which are independent. Thus the complement of these green edges with regard to $K_{z_i,p-z_i}$ contains a red B, and (14) is proved.

Now (14) always guarantees $(z_i, p-b_i)\in\beta'$. Also from (14) we deduce $(z_i-1, p-b_i)\notin\beta'$ and $(z_i, p-b_i-1)\notin\beta'\Leftrightarrow b_j<b_i$ for $1\leqslant j<i$. Then (6) completes the proof of Theorem 2. □

The following lemma describes algorithmic steps for the general determination of $\beta(B_1, B_2)$. We start with m_1 from (a). For $h\geqslant 1$, we then may use (b), (c), and (d) cyclically to find for m_h the corresponding n_h by (b), to ask whether we have finished using (c), and, otherwise, to find the next value m_{h+1} by means of (d).

ALGORITHMIC LEMMA. (a) $m_1=z_1(B_1)+z_1(B_2)-1$;

(b) $(m_h, y)\in\beta'(B_1, B_2)$, *and* $x\geqslant m_h$ *exists with* $(x, y-1)\notin\beta'(B_1, B_2)\Rightarrow y=n_h$;

(c) $(n_h-1, n_h-1)\notin\beta'\Leftrightarrow h=k$;

(d) $(x, n_h-1)\notin\beta'(B_1, B_2)$, *and* $y\leqslant n_h-1$ *exists with* $(x+1, y)\in\beta'(B_1, B_2)\Rightarrow x+1=m_{h+1}$, *if* $h<k$.

Proof. (a) If in $K_{x,y}$ all edges which are incident with $z_1(B_1)-1$ vertices of V_1 (and all edges in case of $x<z_1(B_1)-1$) are colored green and all others red, then for $x-(z_1(B_1)-1)\leqslant z_1(B_2)-1$ neither a green B_1 nor a red B_2 can occur, and hence $m_1\geqslant z_1(B_1)+z_1(B_2)-1$.

For any 2-coloring of $K_{x,y}$ with $x=z_1(B_1)+z_1(B_2)-1$, and $y=1+2^x\max_{i=1,2}\{p(B_i)-z_1(B_i)-1\}$ we consider the (x, y)-matrix M with elements $a_{i,j}=1$ if the edge (i, j) is green, and $a_{i,j}=0$ otherwise. Then in M at least one of the 2^x different columns occurs at least $\max_{i=1,2}\{p(B_i)-z_1(B_i)\}$ times. This column contains $z_1(B_1)$ entries 1, or $z_1(B_2)$ entries 0. Hence M must contain a $(z_1(B_1), p(B_1)-z_1(B_1))$-submatrix only with entries 1, or a $(z_1(B_2), p(B_2)-z_1(B_2))$-submatrix only with entries 0. Thus $K_{x,y}$ contains a green B_1 or a red B_2, and $m_1\leqslant z_1(B_1)+z_1(B_2)-1$ is proved.

(b) If $y>n_h$ then $(m_h, y-1)\in\beta'$, while if $y<n_h$ then $(m_h, y)\notin\beta'$, and either case yields a contradiction.

(c) For $h<k$ we deduce from $(m_{h+1}, n_{h+1})\in\beta'$ and (4) that $(n_h-1, n_h-1)\in\beta'$, as (1) and (7) yield $m_{h+1}\leqslant n_{h+1}\leqslant n_h-1$. The assumption $(n_k-1, n_k-1)\in\beta'$ then implies the existence of n_h with $n_h\leqslant n_k-1$ which contradicts (7).

(d) For $x+1>m_{h+1}$, from $(m_{h+1}, n_{h+1})\in\beta'$ by (4) and (7) we get $(m_{h+1}, n_h-1)\in\beta'$, and then (4) gives the contradiction $(x, n_h-1)\in\beta'$. If now $(m_{h+1}, n_h-1)\in\beta'$ is assumed, then by (4), there exist m_i, n_i with $m_i\leqslant m_{h+1}-1$ and $n_i\leqslant n_h-1$, that is, by (1) and (7) the contradiction $i\leqslant h$ and $i\geqslant h+1$, respectively, follows. Hence $(m_{h+1}-1, n_h-1)\notin\beta'$ which yields for $x+1<m_{h+1}$ by (5) the contradiction $(x+1, y)\notin\beta'$. Thus only $x+1=m_{h+1}$ is possible. □

We are now able to utilize the Algorithmic Lemma to verify easily that the bipartite Ramsey set of a pair of stars is a singleton ordered couple.

THEOREM 3. $\beta(K_{1,s}, K_{1,t})=\{(1, s+t-1)\}$. (15)

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Proof. At first $m_1=1$ follows from (a) of the Lemma. In any 2-coloring of $K_{1,s+t-1}$ the one vertex of V_1 is incident either with s green or t red edges, and hence $(1, s+t-1)\in\beta'(K_{1,s}, K_{1,t})$. If in $K_{s+t-2,s+t-2}$ the edges $(i, i+j)$, $1\leqslant i\leqslant s+t-2$, $0\leqslant j\leqslant s-2$, $i+j \pmod{s+t-2}$, are colored green and all others red, then no green $K_{1,s}$ and no red $K_{1,t}$ can exist, that is, $(s+t-2, s+t-2)\notin\beta'(K_{1,s}, K_{1,t})$. For $s+t\geqslant 3$, this together with (b) and (c) of the Lemma proves Theorem 3, and for $s=t=1$ we use (8). □

We now apply the Algorithmic Lemma further in order to determine the bipartite Ramsey set of a small path P_s, $s\leqslant 5$, and any star $K_{1,t}$. This result will be useful in the next section on bipartite Ramsey sets for small graphs.

THEOREM 4. *Let $t\geqslant 3$, $f_1=2$, and let f_{h+1} be the smallest integer with*

$$[(t-1)/(f_{h+1}-1)]<[(t-1)/(f_h-1)], \quad \text{and} \quad f_{d(t)}=[\tfrac{1}{2}(t+3)]. \tag{16}$$

For $s=4$ and $s=5$ then $k(4,t)=d(t)$; $k(5,t)=d(t)$ if t is even and $t>4$; $k(5,4)=3$; $k(5,t)=d(t)-1$ if t is odd; and

$$\beta(P_s, K_{1,t})=\{(m_h, n_h);\ 1\leqslant h\leqslant k(s,t),\ m_{k(5,t)}=t \text{ if } t \text{ is even},$$
$$m_h=f_h \text{ otherwise},\ n_{k(5,4)-1}=6, \qquad n_h=t+[(t-1)/(m_h-1)] \text{ otherwise}\}. \tag{17}$$

Proof. We use $c=[(t-1)/(a-1)]$, $2\leqslant a\leqslant t$. In $K_{a,t+c-1}$ the edges $(i, (i-1)c+j)$ with $1\leqslant i\leqslant a-1$, $1\leqslant j\leqslant c$, and $i=a$, $1\leqslant j\leqslant t-1-(a-2)c$ are colored green, and all others red. Then every vertex is incident with at most $t-1$ red edges, and neither a red $K_{1,t}$ nor a green P_4 can occur, that is,

$$(a, t+[(t-1)/(a-1)]-1)\notin\beta'(P_s, K_{1,t}), \qquad 2\leqslant a\leqslant t, \qquad 4\leqslant s. \tag{18}$$

In any 2-coloring of $K_{a,t+c}$ either we find a red $K_{1,t}$, or every vertex in V_1 is incident with at least $c+1$ green edges. Because of $a(c+1)>c+t$ at least one vertex in V_2 is incident with two green edges. As $c+1\geqslant 2$, and $c+1\geqslant 3$ for $a\leqslant[\frac{1}{2}(t+1)]$, there exist a green P_4, and a green P_5, respectively, and hence

$$(a, t+[(t-1)/(a-1)])\in\beta'(P_s, K_{1,t}), \qquad 2\leqslant a\leqslant\begin{cases} t, & s=4\\ [\frac{1}{2}(t+1)] & s=5.\end{cases} \tag{19}$$

In a similar way we get

$$(3,6)\in\beta'(P_5, K_{1,4}). \tag{20}$$

Any 2-coloring of $K_{3,6}$ either contains a red $K_{1,4}$, or every vertex of V_1 is incident with at least 3 green edges, and at least one vertex of V_2 with 2 green edges, and thus a green P_5 occurs.

We now consider $K_{t+1,t+1}$, t odd, and $K_{t-1,t+1}$, t even. For $1\leqslant i\leqslant\frac{1}{2}(t+1)$, and $1\leqslant i\leqslant\frac{1}{2}(t-2)$, respectively, the edges $(2i-1, 2i-1)$, $(2i-1, 2i)$, $(2i, 2i-1)$, $(2i, 2i)$, and for $K_{t-1,t+1}$ in addition $(t-1, t-j)$, $0\leqslant j\leqslant 2$, are colored green and all others red. Thus there is no green P_5 and no red $K_{1,t}$, and therefore

$$(t+1, t+1) \notin \beta'(P_5, K_{1,t}), \quad t \text{ odd} \tag{21}$$

$$(t-1, t+1) \notin \beta'(P_5, K_{1,t}), \quad t \text{ even.} \tag{22}$$

Next we suppose for a 2-coloring of $K_{t,t+1}$, t even, that there exist neither a green P_5 nor a red $K_{1,t}$. Then every vertex in V_1 is incident with at least two green edges, and every vertex in V_2 with at least one green edge, and as maximal connected subgraphs the only possibilities are $K_{2,2}$ or $K_{1,r}$ with $r \geq 2$ vertices in V_2. If there are g copies of green $K_{2,2}$, and $K_{1,r_1}, K_{1,r_2}, \ldots, K_{1,r_b}$ denote the green stars, then we have $t = b + 2g$ and $t+1 = 2g + \sum_{i \leq b} r_i$ vertices. Together with $r_i \geq 2$ we obtain $b \leq 1$. Since $b = 2g - t$ is even we get $b = 0$, and this implies $t = 2g - 1$, which contradicts t even; hence

$$(t, t+1) \in \beta'(P_5, K_{1,t}), \quad t \text{ even.} \tag{23}$$

We now apply the Algorithmic Lemma to deduce $\beta(P_s, K_{1,t})$ for $s = 4, 5$. From (a) we see that $m_1 = 2 = f_1$. Then from (b) we obtain $n_h = t + [(t-1)/(m_h - 1)]$ if for $m_{k(5,t)} = t$ we use (23) and (21), and if for $m_h = f_h$ with $h \neq k(5,4) - 1$ we use $h = m_h$ in (19) and (18). For $m_{k(5,4)-1} = f_2 = 3$ we use (20), (22) with $a = 3$, and (b) to get $n_{k(5,4)-1} = 6$.

For $s = 4$, or $s = 5$, t even, $t > 4$, we get $n_d = t + 1$, and for $s = 5$, $t = 4$, we find $n_{d+1} = n_3 = t + 1 = 5$. By substituting $a = t$ in (18) and using (c), it follows that $k(4, t) = d(t)$, $k(5, t) = d(t)$, t even, $t > 4$, and $k(5, 4) = d + 1 = 3$. If $s = 5$, t odd, then $n_{d-1} = t + 2$, and we obtain $k(5, t) = d(t) - 1$, t odd, from (21) and (c).

In the cases $s = 4$, $h < d(t)$, and $s = 5$, $h < d(t) - 1$, we consider $a = f_{h+1} - 1$ in (18) together with $[(t-1)/(f_{h+1} - 2)] = [(t-1)/(f_h - 1)]$ from (16), so as $a = f_{h+1}$ in (19) together with (16), to conclude $m_{h+1} = f_{h+1}$ using part (d) of the Lemma. For $s = 5$, t even, $t > 4$, the case $h = d(t) - 1$ yields $n_{d-1} = t + 2$, and from (22), (23), and (d) we obtain $m_{h+1} = m_d = m_{k(5,t)} = t$. For $s = 5$, $t = 4$, there remain two cases. If $h = 1$, then $n_1 = 7$, and $a = 2$ in (18), (20), and (d) show $m_2 = f_2 = 3$. If $h = 2$, then $n_2 = 6$, and (22), (23), and (d) imply $m_3 = t = 4$. □

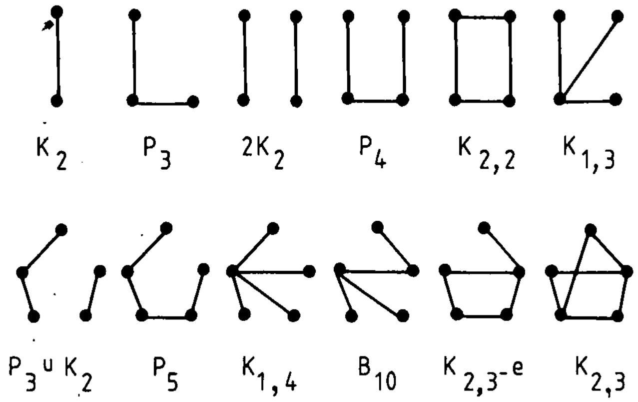

Figure 1. The small bipartite graphs.

2. Bipartite Ramsey sets for small graphs. From the list of all graphs of order $p \leqslant 6$ in [**4**], we show in Fig. 1 those bipartite graphs which have $p \leqslant 5$ vertices and no isolates. We call these twelve graphs the *small bipartite graphs* and list symbolic names for all but the tenth one which is then denoted by B_{10}. (It can also be written as $K_1+K_1+K_1+\bar{K}_2$ but that is too long a symbol.)

THEOREM 5. *For all pairs (B_i, B_j) of small bipartite graphs B_i and B_j from Fig. 1 the bipartite Ramsey sets $\beta(B_i, B_j)$ are gathered in Table 1.*

TABLE 1. $\beta(B_i, B_j)$ FOR ALL SMALL BIPARTITE GRAPHS.

	K_2	P_3	$2K_2$	P_4	$K_{2,2}$	$K_{1,3}$	$P_3 \cup K_2$	P_5	$K_{1,4}$	B_{10}	$K_{2,3}-e$	$K_{2,3}$
K_2	(1, 1)	(1, 2)	(2, 2)	(2, 2)	(2, 2)	(1, 3)	(2, 3)	(2, 3)	(1, 4)	(2, 3)	(2, 3)	(2, 3)
P_3		(1, 3)	(2, 2)	(2, 3)	(2, 4)	(1, 4)	(2, 3)	(2, 3)	(1, 5)	(2, 4)	(2, 4)	(2, 5)
$2K_2$			(3, 3)	(3, 3)	(3, 3)	(2, 4) (3, 3)	(3, 3)	(3, 3)	(2, 5) (4, 4)	(3, 3)	(3, 3)	(3, 3)
P_4				(3, 3)	(3, 4)	(2, 5) (3, 4)	(3, 3)	(3, 4)	(2, 7) (3, 5)	(3, 4)	(3, 4)	(3, 7) (4, 5)
$K_{2,2}$					(3, 7) (5, 5)	(2, 6) (3, 5)	(3, 4)	(3, 4)	(2, 8) (3, 7) (5, 6)	(3, 5)	(3, 7) (5, 5)	(3, 10) (4, 8)
$K_{1,3}$						(1, 5)	(2, 5) (3, 4)	(2, 5)	(1, 6)	(2, 5)	(2, 6) (3, 5)	(2, 7) (4, 6)
$P_3 \cup K_2$							(3, 3)	(3, 4)	(2, 6) (3, 5)	(3, 4)	(3, 4)	(3, 5)
P_5								(3, 5)	(2, 7) (3, 6) (4, 5)	(3, 5)	(3, 5)	(3, 7) (4, 5)
$K_{1,4}$									(1, 7)	(2, 7) (3, 6)	(2, 8) (3, 7) (5, 6)	(2, 9) (3, 8)
B_{10}										(3, 5)	(3, 5)	(3, 7) (4, 6)
$K_{2,3}-e$											(3, 7) (5, 5)	(3, 10) (4, 8)
$K_{2,3}$												(3, 13) (5, 11) (7, 9)

Proof. The first three rows of Table 1 are immediate consequences of (8), (13), and (9). Then $\beta(P_s, P_t)$, $4 \leqslant s, t \leqslant 5$, $\beta(K_{1,s}, K_{1,t})$, $3 \leqslant s, t \leqslant 4$, and $\beta(P_s, K_{1,t})$, $4 \leqslant s \leqslant 5$, $3 \leqslant t \leqslant 4$, can be derived from (2), (15), and (17), respectively. For the remaining pairs (B_i, B_j) (excluding $(K_{2,3}, K_{2,3})$ for the moment) we first prove the validity of $(x, y) \in \beta'(B_i, B_j)$ for all pairs of Table 1. By the *g-degree* and *r-degree* of a vertex v we will mean the number of green and red edges incident with v in a 2-coloring of $K_{x,y}$.

For $B = P_3 \cup K_2$ we prove $(3, 3) \in \beta'(B, P_4)$, $(3, 3) \in \beta'(B, B)$, $(3, 4) \in \beta'(B, K_{2,3} - e)$, $(3, 5) \in \beta'(B, K_{2,3})$, $(2, 6)$ and $(3, 5) \in \beta'(B, K_{1,4})$: All green edges in $K_{x,y}$ without a green $P_3 \cup K_2$ are either part of one star, or of one $K_{2,2}$, or they all are independent, and in any case a red B_j occurs.

For $t = 3, 4, 4, 3, 3, 3, 4, 4, 4, 4, 4, 3, 3, 3, 4, 3, 5$ in this sequence we obtain $(2, 5) \in \beta'(K_{1,3}, B_{10})$, $(2, 7)$ and $(3, 6) \in \beta'(K_{1,4}, B_{10})$, $(3, 4) \in \beta'(K_{2,3} - e, P_4)$, $(2, 6) \in \beta'(K_{1,3}, K_{2,3} - e)$, $(2, 7) \in \beta'(K_{1,3}, K_{2,3})$, $(2, 8)$, $(3, 7)$, and $(5, 6) \in \beta'(K_{1,4}, K_{2,3} - e)$, $(2, 9)$ and $(3, 8) \in \beta'(K_{1,4}, K_{2,3})$, $(3, 5) \in \beta'(B_{10}, K_{2,3} - e)$, $(3, 7)$ and $(4, 6) \in \beta'(B_{10}, K_{2,3})$, $(3, 7)$ and $(5, 5) \in \beta'(K_{2,3} - e, K_{2,3} - e)$, $(3, 10) \in \beta'(K_{2,3} - e, K_{2,3})$, if we check for $\beta'(B_i, B_j)$ that $K_{x,t}$ $(t \leqslant y)$ with a green star $K_{1,t}$ contains either a green B_i (if one vertex of V_1 has g-degree $\geqslant 2$ for $B_i \neq K_{1,t}$), or a red B_j, and that $K_{x,y}$ with r-degree $\geqslant y - t + 1$ for all vertices of V_1 contains a red B_j.

$(3, 4) \in \beta'(K_{2,2}, P_5)$: In $K_{3,4}$ two vertices of V_1 with sum of g-degrees $\geqslant 6$ guarantee a green $K_{2,2}$. If otherwise the sum of r-degrees for all pairs of vertices in V_1 is $\geqslant 3$, then either one vertex of V_1 has r-degree $\geqslant 3$, and another r-degree $\geqslant 2$, so that a red P_5 exists, or two vertices of V_1 have r-degree 2, and the third has r-degree 1 or 2, and always a green $K_{2,2}$ or a red P_5 must exist.

$(3, 5) \in \beta'(P_5, K_{2,3} - e)$: If in $K_{3,5}$ two vertices of V_1 have g-degree $\geqslant 3$, then a green P_5 exists. Otherwise two vertices in V_1 have r-degree $\geqslant 3$. If all vertices in V_1 have r-degree $\geqslant 3$, or two vertices have the sum of their r-degrees $\geqslant 7$, then a red $K_{2,3} - e$ exists. If otherwise two vertices of V_1 have r-degree 3, and the third $\leqslant 2$, then either a green P_5 or a red $K_{2,3} - e$ exists.

$(3, 7)$ and $(4, 5) \in \beta'(P_5, K_{2,3})$: If all vertices of V_2 in $K_{3,7}$ or $K_{4,5}$ have g-degree $\leqslant 1$, then there are at least 14 or 15 red edges, and two vertices of V_1 have the sum of their r-degrees $\geqslant 10$ or $\geqslant 8$, respectively, and thus a red $K_{2,3}$ exists. Otherwise V_2 contains a vertex v with g-degree $\geqslant 2$. If g-degree $\geqslant 2$ for two vertices w_1, w_2 of green edges (w_i, v), then either a green P_5 exists, or both have g-degree 2, and are adjacent by green edges to the same vertex of V_2 $(\neq v)$, and thus a red $K_{2,3}$ exists. Otherwise w_1, for instance, has g-degree 1, and either a red $K_{2,3}$ exists, or w_2 has g-degree $\geqslant 5$ or $\geqslant 3$, respectively. If no green P_5 exists, then at least 4 or 2 vertices $j \neq v$ of V_2 with green edges (w_2, j) have r-degree 2 or 3, respectively, and a red $K_{2,3}$ occurs.

$(4, 8) \in \beta'(K_{2,3} - e, K_{2,3})$: If in $K_{4,8}$ three vertices of V_1 have r-degree $\geqslant 5$, then a red $K_{2,3}$ exists. Otherwise, there are two vertices in V_1 with g-degree $\geqslant 4$. If no green $K_{2,3} - e$ exists, then the remaining two vertices of V_1 have r-degree $\geqslant 6$, and we get a red $K_{2,3}$.

For the pairs still missing in Table 1 (excluding $(K_{2,3}, K_{2,3})$) we use the fact that $(x, y) \in \beta'(B_i, B_j)$ implies $(x, y) \in \beta'(B_a, B_b)$ if $B_a \subset B_i$ and $B_b \subset B_j$.

In a second step we collect in Table 2 certain pairs (x, y) with $(x, y) \notin \beta'(B_i, B_j)$. From

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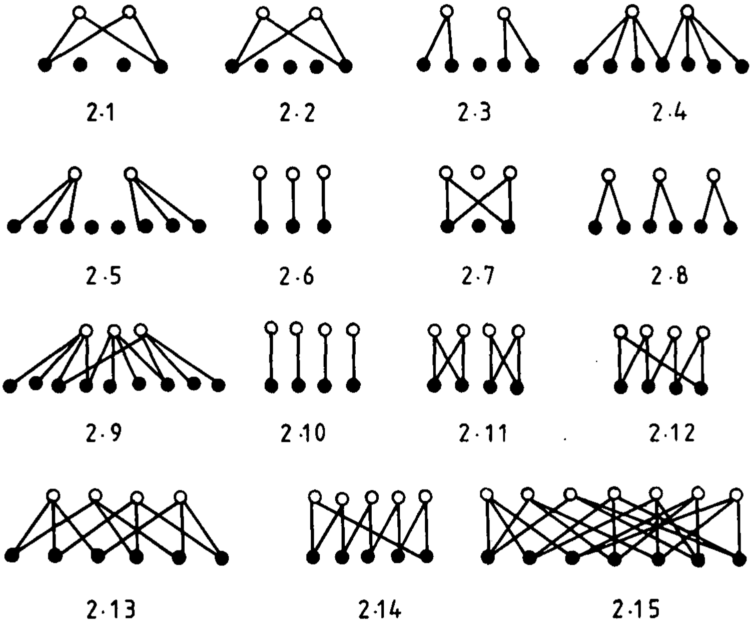

Figure 2. Bipartite graphs with green edges used for Table 2.

Figures 2.N, $1 \leqslant N \leqslant 15$, (where only green edges are reproduced) we can deduce the pairs $(x, y)_N$ of Table 2. From Table 1 follow $(3, 3) \notin \beta'(P_4, K_{1,3})$ and $(2, 6) \notin \beta'(P_4, K_{1,4})$. As $(x, y) \notin \beta'(B_i, B_j)$ implies $(x, y) \notin \beta'(B_a, B_b)$ if $B_i \subset B_a$ and $B_j \subset B_b$, all other pairs of Table 2 are checked easily.

By use of the Algorithmic Lemma we now can determine the sets $\beta(B_i, B_j)$ of Table 1.

The only remaining case, $\beta(K_{2,3}, K_{2,3})$, is a consequence of a result in Irving [**7**]. Here sets $C_{s,t}$, $s \leqslant t$, are considered, which contain all pairs (a, b), such that every 2-colored $K_{a,b}$ has a monochromatic $K_{s,t}$ with the s and t vertices chosen from the a and b vertices, respectively, and such that 2-colorings of $K_{a-1,b}$ and $K_{a,b-1}$ exist without a monochromatic $K_{s,t}$. From this we deduce

$$\beta(K_{s,t}, K_{s,t}) = \{(a, b);\ a \leqslant b,\ (a, b) \text{ or } (b, a) \in C_{s,t},\ (a-i, b-j) \text{ and } (b-i, a-j) \notin C_{s,t} \text{ for } i, j \geqslant 0,\ i+j \geqslant 1\}.$$

Now $C_{2,3} = \{(3, 13), (5, 11), (7, 9), (15, 7), (21, 5)\}$ is proved in [**7**], and we obtain $\beta(K_{2,3}, K_{2,3}) = \{(3, 13), (5, 11), (7, 9)\}$, which completes the proof of Theorem 5. □

As the conjecture in [**7**] that $K_{13,17} \to (K_{3,3}, K_{3,3})$ was recently proved in [**6**], we obtain from [**7**] and [**6**] the set

$$C_{3,3} = \{(5, 41), (7, 29), (9, 23), (13, 17), (17, 13), (23, 9), (29, 7), (41, 5)\},$$

TABLE 2. PAIRS $(x, y) \notin \beta'(B_i, B_j)$.

	$K_{2,2}$	$K_{1,3}$	$P_3 \cup K_2$	P_5	$K_{1,4}$	B_{10}	$K_{2,3}-e$	$K_{2,3}$
P_4	$(3, 3)_6$	$(3, 3)$			$(2, 6)$	$(3, 3)$	$(3, 3)$	$(3, 6)_8$ $(4, 4)_{10}$
$K_{2,2}$	$(4, 6)_{13}$	$(2, 5)$ $(4, 4)$	$(3, 3)$	$(3, 3)$	$(2, 7)_4$ $(4, 6)_{13}$ $(5, 5)_{14}$	$(4, 4)$	$(4, 6)$	$(3, 9)_9$ $(7, 7)_{15}$
$K_{1,3}$	$(2, 5)_3$ $(4, 4)_{12}$		$(2, 4)$ $(3, 3)$			$(2, 4)$ $(4, 4)_{12}$	$(2, 5)$ $(4, 4)$	$(3, 6)_8$ $(5, 5)_{14}$
$P_3 \cup K_2$	$(3, 3)_6$	$(2, 4)_1$ $(3, 3)_6$		$(3, 3)_7$	$(2, 5)_2$ $(4, 4)_{10}$	$(3, 3)$	$(3, 3)$	$(4, 4)_{10}$
P_5						$(4, 4)_{11}$	$(4, 4)$	$(3, 6)$ $(4, 4)$
$K_{1,4}$						$(2, 6)$ $(5, 5)$	$(2, 7)$ $(4, 6)$ $(5, 5)$	$(2, 8)_5$ $(7, 7)_{15}$
B_{10}					$(5, 5)_{14}$	$(4, 4)$	$(4, 4)$	$(3, 6)$ $(5, 5)$
$K_{2,3}-e$							$(4, 6)$	$(3, 9)$ $(7, 7)$

and in addition to Table 1 we conclude with the bipartite Ramsey set of the most famous nonplanar bipartite graph.

THEOREM 6. $\beta(K_{3,3}, K_{3,3}) = \{(5, 41), (7, 29), (9, 23), (13, 17)\}$. □

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