STUDY OF CERTAIN SIMILITUDES

MARIA J. WONENBURGER

One important point in the determination of the automorphisms of the classical groups is the study of group-theoretic properties of the elements of order 2, that is, the involutions. The group of similitudes and the projective group of similitudes of a non-degenerate quadratic form Q are extensions of the orthogonal group and the projective orthogonal group, respectively. These extended groups may contain involutions which do not belong to the orthogonal group or the projective orthogonal group. To study the automorphisms of such groups, group-theoretic properties of some of these new involutions should be established.

The aim of this paper is to give some properties of a similitude T of ratio ρ whose square T^2 is equal to the scalar multiplication by $-\rho$ (it is always assumed that we are dealing with a field of characteristic $\neq 2$). If $\rho = -1$, T is an involution in the group of similitudes and for any ρ the coset of T in the projective group of similitudes is an involution. Our method is a generalization of a method used by Dieudonné in **(1)** to study an orthogonal transformation whose square equals -1, that is, the case $\rho = 1$.

In § 1 we start with a hermitian form and relate to it a quadratic form (cf. **4**). The quadratic form so obtained has a similitude of the type described above. Then we prove that if a quadratic form Q possesses a similitude of that type either Q has maximal Witt index or it is related to a hermitian form (Proposition 1). In the latter case, the centralizers of T in the group of similitudes or the projective centralizer in the group of semi-similitudes are the group of unitarian similitudes and the group of unitarian semi-similitudes, respectively, of the hermitian form (Proposition 2 and Corollary 1).

In § 2, we use these relations and some results in **(5)** to determine the double centralizer of T in the group of similitudes and the group of proper similitudes (Proposition 5). We determine also the double centralizers of the coset of T in the projective group of similitudes and the projective group of proper similitudes (Propositions 6, 7, and 8).

1. Let N be a left vector space over a field F of characteristic $\neq 2$ and f a hermitian form on N, that is, f is a map of $N \times N$ into F with the following properties:

(1) it is additive, that is, $\begin{cases} f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y) \\ f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2) \end{cases}$

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(2) it is sesquilinear, that is, if $\lambda \in F$,

$$f(\lambda x, y) = \lambda f(x, y); \qquad f(x, \lambda y) = f(x, y)\lambda^J$$

where J is an automorphism of F which we suppose to be different from the identity.

(3) $f(x, y) = (f(y, x))^J$.

This last property implies that J is of order 2, that is, an involution. Therefore the symmetric elements with respect to J form a subfield $K = \{\alpha | \alpha^J = \alpha, \alpha \in F\}$, and F is a quadratic extension of K obtained adjoining a skew-symmetric element θ. In other words, $\theta^J = -\theta$ and $\theta^2 = \mu \in K$. (3) also implies that $f(x, x) \in K$.

A semi-linear automorphism (S, σ) of N is called a unitarian semi-similitude of ratio ρ of f with respect to the automorphism σ of F if $f(xS, xS) = \rho(f(x, y))^\sigma$. When σ is the identity, S is called a unitarian similitude, and a unitarian similitude of ratio 1 is called a unitarian transformation (cf. **3**).

In any case, if f is not identically 0, σ commutes with the automorphism J, for $f(xS, (\alpha y)S) = \rho(f(x, \alpha y))^\sigma = \rho(f(x, y))^\sigma \alpha^{J\sigma}$ and on the other hand $f(xS, (\alpha y)S) = f(xS, \alpha^\sigma(yS)) = f(xS, yS)\alpha^{\sigma J} = \rho(f(x, y))^\sigma \alpha^{\sigma J}$. Hence σ induces an automorphism on K.

The ratio ρ of any unitarian semi-similitude belongs to K since $f(xS, xS) = (f(xS, xS))^J$ implies $\rho f(x, x)^\sigma = \rho^J f(x, x)^\sigma$ and therefore, if f is not identically 0, $\rho = \rho^J$.

Let us consider the underlying additive group of the vector space N and define K as a field of operators of such group taking $\alpha \cdot x$, where $\alpha \in K$ and x belongs to the additive group, equal to the element αx of N. We have then a vector space over K which will be denoted by M; M and N contain the same elements. Now we define a bilinear form on M in the following way

$$(x, y) = f(x, y) + f(y, x) = f(x, y) + (f(x, y))^J.$$

It is immediate that this is a symmetric bilinear form on M with values in K. The quadratic form Q associated to this bilinear form will be called the quadratic form on M related to the hermitian form f on N.

f is non-degenerate if and only if Q is non-degenerate, for, if there exists an x such that $f(x, y) = 0$ for every $y \in N$, $(x, y) = 0$ for every $y \in M$, and conversely, it is easily seen that $(x, y) = 0$ for every y implies $f(x, y) = 0$.

If we define now the linear transformation T on M as $xT = \theta x$, T is a similitude of ratio $\rho = -\theta^2 = -\mu$ with respect to Q, for

$$\begin{aligned}(xT, yT) = (\theta x, \theta y) = f(\theta x, \theta y) + f(\theta y, \theta x) &= -\theta^2(f(x, y) + f(y, x)) \\ &= -\mu(x, y).\end{aligned}$$

If $x_1, x_2, \ldots, x_n$ is an orthogonal basis for N with respect to f, $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, where $y_i = \theta x_i$, is an orthogonal basis for M with respect to Q. It is clear that this is a basis for M and it is orthogonal since $(x_i, x_j) = f(x_i, x_j) + f(x_j, x_i) = 0$; $(y_i, y_j) = -\mu(x_i, x_j) = 0$ for $i \neq j$; and

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$(x_i, y_j) = f(x_i, \theta x_j) + f(\theta x_j, x_i) = -\theta(f(x_i, x_j) - f(x_j, x_i)) = 0$ for all i and j. In general $(x, xT) = (x, \theta x) = f(x, \theta x) + f(\theta x, x) = -\theta(f(x, x) - f(x, x)) = 0$.

Now we are going to study the quadratic forms Q which possess a similitude T of ratio ρ such that T^2 is equal to the scalar multiplication by $-\rho$. When the scalar multiplication by an element $\alpha \in K$, F is considered as a linear transformation, which we write on the right, it will be denoted by α_L.

PROPOSITION 1. *Let M be a vector space over K and Q a quadratic form on M. Suppose that Q possesses a similitude T of ratio ρ such that $T^2 = -\rho_L \neq 0$. Then*

(i) *if $-\rho$ is not a square in K, Q is a quadratic form related to a hermitian form f on a vector space N over the field $F = K(\theta)$, $\theta^2 = -\rho$, with respect to the automorphism J of F taking $\alpha + \beta\theta$ into $\alpha - \beta\theta$, $\alpha, \beta \in K$. Therefore* $\dim M = 2 \dim N$.

(ii) *if $-\rho = \alpha^2$, $\alpha \in K$, and Q is non-degenerate M is even dimensional and Q has maximal index.*

Proof. (i) $-\rho$ is not a square. We define M as a left module over the ring of polynomials $K[X]$ in the following way,

$$g(X)x = \left(\sum_{i=0}^{n} \alpha_i X^i\right)x = x\left(\sum_i \alpha_{iL} T^i\right) = x g_L(T);$$

where $\alpha_i \in K$, $x \in M$.

Since $T^2 = -\rho_L$ this defines M as a left $K[X]/(X^2 + \rho) \approx F$ module, where $(X^2 + \rho)$ is the principal ideal generated by $X^2 + \rho$. Then F is the field obtained by the quadratic extension $\theta^2 = -\rho$, that is, $F = K(\theta)$, and therefore M becomes a vector space over F and $\theta x = xT$. When M is considered as a vector space over F it will be denoted by N.

Let J be the automorphism of F taking $\alpha + \beta\theta$ into $\alpha - \beta\theta$, $\alpha, \beta \in K$. Defining

$$f(x, y) = \tfrac{1}{2}[(x, y) - \rho^{-1}\theta(xT, y)] \tag{1}$$

f is a hermitian form relative to J. For, first of all, if (x, y) is the bilinear form associated to Q, $(xT, y) = \rho^{-1}(xT^2, yT) = -(x, yT)$ and hence $f(y, x) = \frac{1}{2}[(y, x) - \rho^{-1}\theta(yT, x)] = \frac{1}{2}[(x, y) + \rho^{-1}\theta(xT, y)] = \frac{1}{2}[(x, y) - \rho^{-1}\theta(xT, y)]^J = (f(x, y))^J$ and $f(\theta x, y) = \frac{1}{2}[(xT, y) - \rho^{-1}\theta(-\rho x, y)] = \theta/2\,[(x, y) - \rho^{-1}\theta(xT, y)] = \theta f(x, y)$.

The quadratic form related to f is Q since

$$f(x, y) + f(y, x) = f(x, y) + (f(x, y))^J = (x, y).$$

If $x_1, x_2, \ldots, x_m$ is an orthogonal basis of N with respect to f, $x_i, y_i = x_i T$, $i = 1, 2, \ldots, m$ is an orthogonal basis of M with respect to Q.

(ii) $-\rho = \alpha^2$. We have again $(xT, y) = -(x, yT)$ and taking $y = x$, $(xT, x) = -(xT, x) = 0$, that is, any vector is taken by T in an element orthogonal to it.

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Let x be a non-isotropic vector with respect to Q. Then x and $y = xT$ are two non-isotropic vectors orthogonal to each other. Therefore the plane P spanned by these two vectors is non-isotropic and is changed into itself by T. Let $Q_{P^\perp}$ be the restriction of Q to the vector space $P^\perp$ orthogonal to the plane P. The space $P^\perp$ is taken into itself by T, which induces on it a similitude with respect to $Q_{P\perp}$ whose square is $-\rho_L$. Now making an induction on the dimension of M the statement is proved, for, if $Q_{P\perp}$ has maximal index, Q has maximal index since $\alpha x + y \in P$ and $Q(\alpha x + y) = \alpha^2(Q(x) - Q(x)) = 0$. As in the preceding case there exist orthogonal bases of the form $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ where $y_i = x_i T$ but now the x_i can be chosen such that $Q(x_i) = 1$.

In the next proposition f is again a hermitian form on the vector space N over $F = K(\theta)$; where $\theta^2 = \mu \in K$, Q is the quadratic form related to f, and $xT = \theta x$.

Proposition 2. *The unitarian semi-similitudes (U, σ) of f are semi-similitudes of Q of the same ratio, relative to the automorphism induced in K by σ and such that if $\theta^\sigma = \gamma\theta$, $\gamma \in K$, $TU = UT\gamma_L$. Conversely, a semi-similitude (S, σ_1) of ratio ρ of Q such that $ST\gamma_L = TS$ can be considered as a unitarian semi-similitude of ratio ρ of f relative to the automorphism σ of F coinciding with σ_1 on K and taking θ into $\theta^\sigma = \gamma\theta$.*

Proof. If (U, σ) is a unitarian semi-similitude of ratio ρ, σ induces an automorphism in K and U is a semi-linear transformation of M relative to this automorphism. Moreover

$$(xU, yU) = f(xU, yU) + f(yU, xU) = \rho(f(x, y) + f(y, x))^\sigma = \rho(x, y)^\sigma$$

and

$$xTU = (\theta x)U = \theta^\sigma(xU) = \gamma\theta(xU) = xUT\gamma_L.$$

As for the converse, let (S, σ_1) be a semi-similitude of Q such that $ST\gamma_L = TS$. Since $(xST\gamma_L, yST\gamma_L) = -\gamma^2\mu\rho(x, y)^{\sigma_1}$ must be equal to $(xTS, yTS) = \rho(xT, yT)^{\sigma_1} = -\rho\mu^{\sigma_1}(x, y)^{\sigma_1}$, $\mu^{\sigma_1} = \gamma^2\mu$.

Hence the mapping $\sigma: \alpha + \beta\theta \rightarrow \alpha^{\sigma_1} + \gamma\beta^{\sigma_1}\theta$ defines an automorphism in F and $((\alpha + \beta\theta)x)S = (\alpha x)S + (\beta xT)S = \alpha^{\sigma_1}(xS) + \beta^{\sigma_1}\gamma(xS)T = (\alpha^{\sigma_1} + \gamma\beta^{\sigma_1}\theta)(xS) = (\alpha + \beta\theta)^\sigma(xS)$ is a semi-linear transformation of N, with respect to the automorphism σ. Moreover, by (1),

$$f(xS, yS) = \tfrac{1}{2}((xS, yS) + \mu^{-1}\theta(xST, yS)) = \rho/2((x, y)^{\sigma_1} + \mu^{-1}\theta\gamma^{-1}(xT, y)^{\sigma_1}),$$

and taking in account $\mu^\sigma = \mu^{\sigma_1} = \gamma^2\mu$

$$f(xS, yS) = \rho/2((x, y) + \mu^{-1}\theta(xT, y))^\sigma = \rho(f(x, y))^\sigma.$$

Corollary 1. *The unitarian similitudes of f are the similitudes of Q commuting with T.*

Corollary 2. *The unitarian semi-similitudes of f with respect to the automorphism J of F are the similitudes of Q anticommuting with T.*

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2. In this section we will need some results of **(5)** and at one point we will have to use the method there developed. The notation will be the same as the preceding section, and Q is supposed to be non-degenerate.

LEMMA 1. *Let T be a similitude of Q of ratio ρ and $T^2 = -\rho_L$. Then all the linear transformations $\alpha_L + \beta_L T$, where $\alpha^2 + \beta^2\rho \neq 0$, are proper similitudes of ratio $\alpha^2 + \beta^2\rho$.*

Proof. If $-\rho$ is not a square it is immediate that the linear transformation of M, $\alpha_L + \beta_L T$ is a similitude since it is the unitarian similitude of N defined by the scalar multiplication by $\alpha + \beta\theta$ whose ratio is $N(\alpha + \beta\theta) = (\alpha + \beta\theta)(\alpha - \beta\theta) = \alpha^2 + \rho\beta^2$.

In any case $(xT, y) = -(x, yT)$ and therefore

$$(x(\alpha_L + \beta_L T), y(\alpha_L + \beta_L T)) = \alpha^2(x, y) + \alpha\beta[(xT, y) + (x, yT)] + \beta^2(xT, yT) = (\alpha^2 + \rho\beta^2)(x, y).$$

Let $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$ be an orthogonal basis with respect to Q such that $y_i = x_i T$. The matrix A of the transformation $\alpha_L + \beta_L T$ with respect to this basis when it is divided in 2×2 blocks has the diagonal form $A = \text{diag.}\,(B_1, B_2, \ldots, B_m)$ where

$$B_i = \begin{pmatrix} \alpha & \beta \\ -\beta\rho & \alpha \end{pmatrix}, \; i = 1, 2, \ldots, m.$$

Therefore $\det.\,(A) = (\det.\,(B_1))^m = (\alpha^2 + \beta^2\rho)^m$ which proves that $\alpha_L + \beta_L T$ is a proper similitude. In particular, T is proper.

PROPOSITION 3. *Let T be a similitude of Q of ratio ρ such that $T^2 = -\rho_L$. Then the similitudes commuting with T are proper.*

Proof. Since T is proper, if ρ is not a square our assertion is given in **(5,** Proposition 2, Corollary 2**)**.

If ρ is a square and $-\rho$ is also a square, say $-\rho = \alpha^2$, -1 is a square in K and $T\alpha_L^{0-1} = T'$ is a similitude of ratio -1, $T'^2 = 1_L$, and a similitude commutes with T' if and only if it commutes with T.

Let $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$ be an orthogonal basis of Q such that $y_i = x_i T'$. We can assume that $Q(x_i) = 1 = -Q(y_i)$. Then, taking $u_i = x_i + y_i$ and $v_i = x_i - y_i$, the u_i, v_i, $i = 1, 2, \ldots, m$ form a basis for M and

$$u_i T' = y_i + x_i = u_i,\; v_i T' = y_i - x_i = -v_i;\; (u_i, v_j) = 2\delta_{ij}, \text{ and } (u_i, u_j) = 0.$$

If the similitude U of ratio ρ_1 commutes with T' it takes the totally isotropic space P spanned by the u_i's and the totally isotropic space R spanned by the v_i's into themselves. Therefore the matrix of U with respect to the basis $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m$ has the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ are $m \times m$ matrices.

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Since

$$(u_iU, v_jU) = \left(\sum_{h=1}^{m} \alpha_{ih}u_h, \sum_{k=1}^{m} \beta_{jk}v_k\right) = 2\sum_{h=1}^{m} \alpha_{ih}\beta_{jh} = \rho_1(u_i, v_j) = 2\rho_1\delta_{ij};$$

$$\sum_{h=1}^{m} \alpha_{ih}\beta_{jh} = \rho_1\delta_{ij}.$$

Hence, if B' stands for the transpose of B,

$$\text{det.}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \text{det. }(A)\text{ det. }(B) = \text{det. }(A)\text{ det. }(B') = \text{det. }(AB')$$

$$= \text{det. }(\rho_1\delta_{ij}) = \rho_1^m$$

which proves that U is proper.

Suppose now that $\rho = \mu^2$ and $-\rho$ is not a square, that is, -1 is not a square. Then $T' = T\mu_L^{-1}$ is an orthogonal transformation. Let g be an element of the Clifford group Γ of the Clifford algebra $C(M, Q)$ which defines an inner automorphism of $C(M, Q)$ which induces on M the transformation T. Since $T'^2 = -1_L$, $g^2 = \gamma e \in M_{[2m]}$.

If S is a similitude commuting with T, and hence with T', let s be an element of the extended Clifford group (cf. **5**, Definition 3) associated to S with the factor $\alpha + \beta e$ (**5**, Definition 4). We are going to show that the assumption that S is improper and therefore $s \in C^-(M, Q)$ leads to contradiction. If $x \in M$,

$$(gsg^{-1}s^{-1})^{-1}xgsg^{-1}s^{-1} = xT'S(T')^{-1}S^{-1} = x,$$

(cf. **5**, Lemma 9), which indicates that $gsg^{-1}s^{-1} = \epsilon \in K$ and $gs = \epsilon sg$.

Multiplying on the left by g and taking into account the value of g^2 given above, we get $g^2s = \gamma es = \epsilon gsg = \epsilon^2 sg^2 = \epsilon^2 s\gamma e = -\gamma\epsilon^2 es$ which implies $\epsilon^2 = -1$ which is a contradiction.

PROPOSITION 4. *Let T be a similitude of Q of ratio ρ such that $T^2 = -\rho_L$. Then*

(i) *if* dim $M = 4r + 2$ *the similitudes anticommuting with T are improper.*

(ii) *if* dim $M = 4r$ *the similitudes anticommuting with T are proper.*

Proof. If S is any similitude anticommuting with T any other similitude anticommuting with T can be written as SU, where U is a similitude which commutes with T. Proposition 3 asserts that U is proper, therefore, if S is proper (improper) all the similitudes anticommuting with T are proper (improper).

Let $x_1, y_1, x_2, y_2, \ldots, x_m, y_m$ be an orthogonal basis of Q with the property $y_i = x_iT$, $i = 1, 2, \ldots, m$. The orthogonal involution S defined by $x_iS = -x_i$, $y_iS = y_i$, $i = 1, 2, \ldots, m$ is an improper similitude if $m = \frac{1}{2}\dim M$ is odd and it is proper if m is even and in any case it anticommutes with T which proves the proposition.

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PROPOSITION 5. *Let T be a similitude of Q of ratio ρ such that $T^2 = -\rho_L$. Then the centralizer of the centralizer of T in the group of similitudes or the group of proper similitudes consists of similitudes of the form $\alpha_L + \beta_L T$, where $\alpha^2 + \rho\beta^2 \neq 0$. In particular if $-\rho$ is not a square and $F = K(\theta)$, $\theta^2 = -\rho$, the double centralizer of T is isomorphic to F', the multiplicative group of non-zero elements of F.*

Proof. By Proposition 3 the centralizer $C(T)$ of T in the group of similitudes is contained in the group of proper similitudes.

If $-\rho$ is not a square, Q is related to a hermitian form f and Corollary 1 of Proposition 2 asserts that $C(T)$ is the group of unitarian similitudes of f. Therefore the centralizer of $C(T)$ consists of the scalar multiplications by the non-zero elements of F, that is, of the similitudes of the form $\alpha_L + \beta_L T$.

If $-\rho = \alpha^2$, considering $T\alpha_L^{-1}$ we can assume that $T^2 = 1_L$ and $\rho = -1$. Let us take a basis u_i, v_i, $i = 1, 2, \ldots, m$ as in the proof of Proposition 3. Then the similitudes of Q commuting with T are the linear transformations defined by matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A')^{-1}\rho_1 \end{pmatrix},$$

where A is any non-singular $m \times m$ matrix, $\rho_1 \neq 0$ and A' stands for the transpose of A. The centralizer of this subgroup of similitudes consists of similitudes S of the form

$$\begin{pmatrix} \alpha' I_m & 0 \\ 0 & \beta' I_m \end{pmatrix}$$

where I is the identity matrix of order $m \times m$ and $\alpha', \beta' \neq 0$. Hence $u_i S = \alpha' u_i = \alpha'(x_i + y_i)$, $v_i S = \beta' v_i = \beta'(x_i - y_i)$ and therefore $x_i S = \frac{1}{2}(u_i + v_i)S = \frac{1}{2}(\alpha' + \beta')x_i + \frac{1}{2}(\alpha' - \beta')x_i T$; $y_i S = \frac{1}{2}(u_i - v_i)S = \frac{1}{2}(\alpha' + \beta')y_i + \frac{1}{2}(\alpha' - \beta')y_i T$, that is, $S = \alpha_L + \beta_L T$, where $\alpha = \frac{1}{2}(\alpha' + \beta')$, $\beta = \frac{1}{2}(\alpha' - \beta')$ and $\alpha^2 - \beta^2 = \alpha'\beta' \neq 0$.

PROPOSITION 6. *Let T be a similitude of Q of ratio ρ such that $T^2 = -\rho_L$ and* $\dim M = 4r + 2$. *Then the centralizer of the centralizer of the coset of T in the projective group of proper similitudes consists of the cosets of the similitudes $\alpha_L + \beta_L T$ where $\alpha^2 + \beta^2\rho \neq 0$. In particular, if $-\rho$ is not a square the double centralizer of the coset of T in the projective group of proper similitudes is isomorphic to F'/K'.*

Proof. The centralizer of the coset of T in the projective group of proper similitudes $P\gamma^+$ is the group of cosets of proper similitudes either commuting or anticommuting with T. Since $\dim M = 4r + 2$, by Proposition 4, there are not elements anticommuting with T in the group of proper similitudes; hence, the centralizer of the coset of T in $P\gamma^+$ consists of the cosets of $C(T)$.

If $-\rho$ is not a square, $C(T)$ is the group of unitarian similitudes of f. Since

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$\dim N = \frac{1}{2} \dim M = 2r + 1$, any unitarian similitude of f has the form $(\alpha + \beta\theta)_L U$, where U is a unitarian transformation. The group of unitarian transformations is generated by quasi-symmetries (cf. **2**; **3**) (transformations leaving invariant elementwise the hyperplane orthogonal to a non-isotropic vector x and taking x into $(\alpha + \beta\theta)x$); therefore, if there exists a transformation $S \in C(T)$ anticommuting with some elements of $C(T)$ and commuting with the rest it must anticommute at least with one quasi-symmetry and either commute or anticommute with the others, which is impossible. Hence the centralizer of the cosets of $C(T)$ consists of the cosets of similitudes commuting with $C(T)$, that is, of the form $\alpha_L + \beta_L T$. The group of these cosets is isomorphic to F'/K'.

If $-\rho$ is a square, $C(T)$ consists of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & \rho_1 (A')^{-1} \end{pmatrix}$$

where A is a non-singular $m \times m$ matrix.

A similitude S anticommuting with some elements of $C(T)$ and commuting with the rest must have the form

$$S = \begin{pmatrix} B & 0 \\ 0 & \rho_1 (B')^{-1} \end{pmatrix}$$

where the $m \times m$ matrix B anticommutes with some of the elementary matrices and commutes with the rest. The only possibility is that B is a diagonal matrix and

$$S = \begin{pmatrix} \alpha' I_m & 0 \\ 0 & \beta' I_m \end{pmatrix}$$

is of the form $(\frac{1}{2}(\alpha' + \beta'))_L + (\frac{1}{2}(\alpha' - \beta'))_L T$, where $\alpha'\beta' \neq 0$.

PROPOSITION 7. *Let T be a similitude of Q of ratio ρ such that $T^2 = -\rho_L$ and* $\dim M = 4r$. *Then the centralizer of the centralizer of the coset of T in the projective group of proper similitudes consists of the coset of the identity and the coset of T.*

Proof. In this case, by Proposition 4, we know that the group of proper similitudes contains all the similitudes which anticommute with T, as well as the ones which commute with it.

If $-\rho$ is not a square, by the corollaries of Proposition 2, the centralizer of the coset of T in the projective group of proper similitudes consists of the cosets of those similitudes which are either unitarian similitudes or unitarian semi-similitudes of automorphism J with respect to the hermitian form f. The centralizer of all those cosets in the group of proper similitudes consists only of the coset of T and the coset of the identity. For now, if $\alpha \neq 0$, $\beta \neq 0$, the similitudes of the form $\alpha_L + \beta_L T$ define cosets which do not commute with the coset of any similitude which anticommutes with T, that is, a similitude defined by a unitarian semi-similitude of f.

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If $-\rho$ is a square, we can assume that $T^2 = 1_L$. With respect to the basis $u_i, v_i, i = 1, 2, \ldots, 2r$, T has the form

$$\begin{pmatrix} I_{2r} & 0 \\ 0 & -I_{2r} \end{pmatrix}.$$

Then the similitudes commuting with it have the form

$$\begin{pmatrix} A & 0 \\ 0 & \rho_1(A')^{-1} \end{pmatrix}$$

and the ones which anticommute are given by the matrices

$$\begin{pmatrix} 0 & A \\ \rho_1(A')^{-1} & 0 \end{pmatrix}$$

where $\rho_1 \neq 0$ and A is any non-singular $2r \times 2r$ matrix. The similitudes which either commute or anticommute with every one of these similitudes are of the form

$$\begin{pmatrix} \alpha I_{2r} & 0 \\ 0 & \pm\alpha I_{2r} \end{pmatrix},$$

that is, they belong to the coset of the identity or the coset of T.

The same argument will establish the following

PROPOSITION 8. *The double centralizer of the coset of T in the projective group of similitudes consists of this coset and the coset of the identity.*

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Queen's University, Kingston
C.S.I.C., Madrid

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