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AN INDUCTIVE REARRANGEMENT THEOREM

KONG-MING CHONG

Introduction. In [**2**, Theorem 3.2, p. 429] and [**3**, Theorem 2.1, p. 155], the author established two induction theorems which gave rise to a series of fundamental results in spectral and rearrangement inequalities. In particular, the classical inequalities of Hardy–Littlewood–Pólya [**4**, Theorem 108, p. 89] and Pólya [**6**] were derived and conditions for equalities obtained (see [**2**, Theorem 3.8, p. 433] and [**3**, Theorem 2.6, p. 157]). In [**5**, Theorem 6, p. 651; and Theorem 20, p. 569], Fischer and Holbrook also gave alternative conditions for equalities to hold in the aforesaid inequalities. In this paper, we show that the result of Fischer and Holbrook can be proved by induction using an inductive rearrangement theorem which turns out to be a stronger version of the induction theorems given in [**2**, Theorem 3.2, p. 429] and [**3**, Theorem 2.1, p. 155].

1. Preliminaries. Let $\mathbf{R}^n$ denote the set of all n-tuples of real numbers. For any n-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, we denote by

$$\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$$

the n-tuple the components of which are those of $\mathbf{x}$ arranged in decreasing order of magnitude. If $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}^n$, then $\mathbf{a} \ll \mathbf{b}$ means that

$$(1.1) \quad \sum_{i=1}^{k} a_i^* \leqq \sum_{i=1}^{k} b_i^*$$

for $1 \leqq k \leqq n$, and we write $\mathbf{a} \prec \mathbf{b}$ if, in addition to $\mathbf{a} \ll \mathbf{b}$, there is equality in (1.1) for $k = n$.

As in [**1**], we call expressions of the form $\mathbf{a} \prec \mathbf{b}$ (respectively $\mathbf{a} \ll \mathbf{b}$) *strong* (respectively *weak*) *spectral inequalities*. The spectral inequality $\mathbf{a} \prec \mathbf{b}$ (respectively $\mathbf{a} \ll \mathbf{b}$) is said to be *strictly strongly* (respectively *strictly weak*) if (1.1) *is strict for at least one* $k < n$ (respectively *for* $k = n$). Moreover, the spectral inequality $\mathbf{a} \prec \mathbf{b}$ (respectively $\mathbf{a} \ll \mathbf{b}$) is said to be *absolutely strong* (respectively *absolutely weak*) if (1.1) is strict for every k satisfying $1 \leqq k < n$ (respectively $1 \leqq k \leqq n$).

The following proposition gives a characterization of both absolutely strong and absolutely weak spectral inequalities.

PROPOSITION 1.1. *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ *are*

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n-tuples in $\mathbf{R}^n$, *then the spectral inequality* $\mathbf{a} \prec \mathbf{b}$ [*respectively* $\mathbf{a} \prec \mathbf{b}$] *is absolutely weak* [*respectively absolutely strong*] *if and only if*

$$(1.2) \quad \sum_{i=1}^{n} (a_i - t)^+ < \sum_{i=1}^{n} (b_i - t)^+$$

for all $t < b_1{}^*$ [*respectively for all* $t \in (b_n{}^*, b_1{}^*)$ *and with equality in* (1.2) *for all* $t \leqq b_n{}^*$], *where* $x^+ = \max\{x, 0\}$ *for any* $x \in \mathbf{R}$.

Proof. The proof is similar to that given in [**4**, p. 90].

2. An inductive rearrangement theorem. To give a short inductive proof of the result of Fischer and Holbrook [**5**, Theorem 6, p. 561; and Theorem 20, p. 569] regarding the case of equalities in the inequalities of Hardy, Littlewood and Pólya, we need the following inductive rearrangement theorem which is a stronger version of the induction theorem given either in [**2**, Theorem 3.2, p. 429] or in [**3**, Theorem 2.1, p. 155] as all the results obtained in [**2**] or [**3**] can be derived inductively as a direct consequence of the present theorem.

THEOREM 2.1. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}^n$ *be two n-tuples. If* $\mathbf{a} \prec \mathbf{b}$ [*respectively* $\mathbf{a} \ll \mathbf{b}$] *and if* $a_1{}^* < b_1{}^*$ [*respectively* $b_n{}^* < a_1{}^* < b_1{}^*$], *then there exists a smallest integer i,* $1 < i \leqq n$, *such that*

$$(2.1) \quad (a_1{}^*, b_{i-1}{}^* + b_i{}^* - a_1{}^*) \prec (b_{i-1}{}^*, b_i{}^*) \quad \textit{and}$$

$$(2.2) \quad (a_2{}^*, a_3{}^*, \ldots, a_n{}^*) \prec (b_1{}^*, b_2{}^*, \ldots, b_{i-2}{}^*, b_{i-1}{}^* + b_i{}^* - a_1{}^*, b_{i+1}{}^*, \ldots, b_n{}^*)$$

[*respectively*

$$(2.3) \quad (a_2{}^*, a_3{}^*, \ldots, a_n{}^*) \ll (b_1{}^*, b_2{}^*, \ldots, b_{i-2}{}^*, b_{i-1}{}^* + b_i{}^* - a_1{}^*, b_{i+1}{}^*, \ldots, b_n{}^*)].$$

If the spectral inequality $\mathbf{a} \prec \mathbf{b}$ [*respectively* $\mathbf{a} \ll \mathbf{b}$] *is absolutely strong* [*respectively absolutely weak*], *then the spectral inequality* (2.2) [*respectively* (2.3)] *is absolutely strong* [*respectively absolutely weak*].

Proof. Since $a_1{}^* < b_1{}^*$ and

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j$$

(respectively $b_n{}^* < a_1{}^* < b_1{}^*$) it is impossible that $a_1{}^* < b_j{}^*$ for all $j > 1$ and so the existence of a minimum integer i satisfying $1 < i \leqq n$ and

$$(2.4) \quad b_i{}^* \leqq a_1{}^* < b_{i-1}{}^*$$

is ensured.

Finally, (2.1) is a direct consequence of (2.4) while the verification of the remaining assertions is straightforward.

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We can now give inductive proofs for the following theorems ([**4**, Theorem 108, p. 89], [**5**, Theorem 6, p. 561; and Theorem 20, p. 569] and [**6**]).

THEOREM 2.2. *If* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$ *and* $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}^n$ *are such that* $\mathbf{a} \prec \mathbf{b}$, *then*

$$(2.5) \quad \sum_{i=1}^{n} \Phi(a_i) \leqq \sum_{i=1}^{n} \Phi(b_i)$$

for all convex functions $\Phi : [b_n^*, b_1^*] \to \mathbf{R}$.

If $\mathbf{a} \prec \mathbf{b}$ *is absolutely strong and if* $\Phi : [b_n^*, b_1^*] \to \mathbf{R}$ *is convex, then equality occurs in* (2.5) *if and only if* Φ *is affine on* $[b_n^*, b_1^*]$.

If $\mathbf{a} \prec \mathbf{b}$ *is strictly strong such that the inequality* (1.1) *is strict for all* $k < n$ *except* $k_1, k_2, \ldots, k_m$ *where* $k_1 < k_2 < \ldots < k_m$, *and if* $\Phi : [b_n^*, b_1^*] \to \mathbf{R}$ *is convex, then equality occurs in* (2.5) *if and only if* Φ *is affine on each of the intervals*

$$[b_{k_1}^*, b_1^*], [b_{k_2}^*, b_{k_1+1}^*], \ldots, [b_n^*, b_{k_m+1}^*].$$

Proof. The validity of (2.5) for the case that $n = 2$ is easily established (cf. [**2**, Lemma 3.3, p. 430]). In general, suppose by induction that the result is true for $n - 1$. Assume that $\mathbf{a} = (a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n) = \mathbf{b}$. If $a_1^* = b_1^*$, then, obviously,

$$(a_2^*, a_3^*, \ldots, a_n^*) \prec (b_2^*, b_3^*, \ldots, b_n^*)$$

and so, by the induction hypothesis, we have

$$\Phi(a_2^*) + \Phi(a_3^*) + \ldots + \Phi(a_n^*) \leqq \Phi(b_2^*) + \Phi(b_3^*) + \ldots + \Phi(b_n^*)$$

whence (2.5) follows since $\Phi(a_1^*) = \Phi(b_1^*)$. If $a_1^* < b_1^*$, then Theorem 2.1 implies the existence of a smallest integer i, $1 < i \leqq n$, such that both (2.1) and (2.2) hold. By applying the result for the case that $n = 2$ to (2.1) and the induction hypothesis to (2.2), we readily obtain (2.5) as a consequence.

For the second part, the case that $n = 2$ can be derived from [**4**, Theorem 90, pp. 74–75] via [**2**, Lemma 3.3, p. 430]. In general, suppose by induction that the result is true for $n - 1$. If $\mathbf{a} = (a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n) = \mathbf{b}$ is absolutely strong, then, by Theorem 2.1, for a smallest integer i such that $1 < i \leqq n$, both (2.1) and (2.2) hold and the latter is also absolutely strong. If $a_1^* = b_i^*$, then (2.2) is equivalent to

$$(a_2^*, a_3^*, \ldots, a_n^*) \prec (b_1^*, b_2^*, \ldots, b_{i-1}^*, b_{i+1}^*, \ldots, b_n^*)$$

and so equality in (2.5) implies

$$\Phi(a_2^*) + \ldots + \Phi(a_n^*) = \Phi(b_1^*) + \Phi(b_2^*) + \ldots + \Phi(b_{i-1}^*) + \Phi(b_{i+1}^*) + \ldots + \Phi(b_n^*)$$

and so the inductive hypothesis entails the affinity of Φ on $[b_n^*, b_1^*]$. If $a_1^* < b_i^*$, then both (2.1) and (2.2) are absolutely strong and so the result

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for $n = 2$ applied to (2.1) together with the induction hypothesis applied to (2.2) will yield the case for any n.

The last part is a direct consequence of the second part by virtue of the easily proven fact that the last hypothesis is equivalent to

$$\begin{aligned}&(a_{k_j}^*, a_{k_j-1}^*, \dots, a_{k_{j-1}+1}^*) \prec (b_{k_j}^*, b_{k_j-1}^*, \dots, b_{k_{j-1}+1}^*) \quad \text{and}\\ &\Phi(a_{k_j}^*) + \Phi(a_{k_j-1}^*) + \dots + \Phi(a_{k_{j-1}+1}^*) = \Phi(b_{k_j}^*) + \Phi(b_{k_j-1}^*)\\ &\qquad\qquad + \dots + \Phi(b_{k_{j-1}+1}^*)\end{aligned}$$

for $j = 1, 2, \dots, m + 1$, where $k_0 = 0$ and $k_{m+1} = n$.

THEOREM 2.3. *If* $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbf{R}^n$ *and* $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbf{R}^n$ *are such that* $\mathbf{a} \ll \mathbf{b}$, *then*

$$(2.6) \quad \sum_{i=1}^{n} \Phi(a_i) \leqq \sum_{i=1}^{n} \Phi(b_i)$$

for all increasing convex functions $\Phi : [\min (a_n^*, b_n^*), b_1^*] \to \mathbf{R}$.

If $\mathbf{a} \ll \mathbf{b}$ *is absolutely weak and if* Φ *is increasing and convex, then equality holds in* (2.6) *if and only if* Φ *is a constant function on* $[\min (a_n^*, b_n^*), b_1^*]$.

If $\mathbf{a} \ll \mathbf{b}$ *is strictly weak such that the inequality* (1.1) *is strict for all* $k \leqq n$ *except* $k_1, k_2, \dots, k_m$ *where* $k_1 < k_2 < \dots < k_m$ *then equality occurs in* (2.6) *if and only if f is affine on the intervals*

$$(b_{k_1}^*, b_1^*), (b_{k_2}^*, b_{k_1+1}^*), \dots, (b_{k_m}^*, b_{k_{m-1}+1}^*)$$

and is constant on $[\min (a_n^*, b_n^*), b_{k_m+1}^*]$.

Proof. The first part follows inductively from Theorem 2.1 via [**3**, Lemma 2.2, p. 156] as in Theorem 2.2.

For the second part, while the case that $n = 2$ can be obtained from [**4**, Theorem 90, pp. 74–75] via [**3**, Lemma 2.2, p. 156], the general case again follows inductively from Theorem 2.1.

Finally, using an analysis similar to the one given in the proof of Theorem 2.2, we see that the last part is a direct consequence of the second part above and Theorem 2.2.

Before concluding the present paper, we give a further application of Theorem 2.1 to obtaining the following result which turns out to be a sharpening of [**2**, Theorem 3.9, p. 434] and [**3**, Theorem 2.7, p. 158] regarding the preservation of the strictness or absoluteness of a weak or strong spectral inequality by a certain class of convex functions.

THEOREM 2.4. *If* $\mathbf{a} = (a_1, a_2, \dots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \dots, b_n)$ *are n-tuples in* $\mathbf{R}^n$, *then* $\mathbf{a} \prec \mathbf{b}$ [*respectively* $\mathbf{a} \ll \mathbf{b}$] *if and only if*

$$(2.7) \quad (\Phi(a_1), \Phi(a_2), \dots, \Phi(a_n)) \ll (\Phi(b_1), \Phi(b_2), \dots, \Phi(b_n))$$

for all convex [*respectively increasing convex*] *functions*

$$\Phi : [\min (a_n^*, b_n^*), b_1^*] \to \mathbf{R}.$$

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If the spectral inequality $\mathbf{a} \prec \mathbf{b}$ [*respectively* $\mathbf{a} \ll \mathbf{b}$] *is absolutely strong* [*respectively absolutely weak*] *and if* Φ *is strictly convex* [*respectively strictly increasing and convex*], *then the spectral inequality* (2.7) *is absolutely weak.*

If the spectral inequality $\mathbf{a} \prec \mathbf{b}$ [*respectively* $\mathbf{a} \ll \mathbf{b}$] *is absolutely strong* [*respectively absolutely weak*] *and if* Φ *is convex, then the spectral inequality* (2.7) *is strong if and only if* Φ *is affine* [*respectively constant*].

Proof. The first part follows from Theorem 2.1 as in [**2**, Theorem 3.9, p. 434] and [**3**, Theorem 2.7, p. 158].

For the second part, the case that $n = 2$ is easily seen to hold by virtue of [**2**, Lemma 3.3, p. 430] and [**3**, Lemma 2.2, p. 156]. To prove the result in general, suppose by induction that it is true for $n - 1$. If $\mathbf{a} \prec \mathbf{b}$ is absolutely strong, then Theorem 2.1 implies the existence of a smallest integer i, $1 < i \leqq n$, such that the spectral inequalities (2.1) and (2.2) hold and the latter is absolutely strong. The case that $n = 2$ and the induction hypothesis thus imply that

$$(2.8) \quad (\Phi(a_1^*), \Phi(b_i^* + b_{i-1}^* - a_1^*)) \ll (\Phi(b_{i-1}^*), \Phi(b_i^*))$$

holds and that

$$(2.9) \quad (\Phi(a_2^*), \ldots, \Phi(a_n^*)) \ll (\Phi(b_1^*), \ldots, \Phi(b_{i-2}^*), \Phi(b_{i-1}^* + b_i^* - a_1^*), \Phi(b_{i+1}^*), \ldots, \Phi(b_n^*))$$

is absolutely weak for any strictly convex $\Phi : [b_n^*, b_1^*] \rightarrow \mathbf{R}$. It then follows from the characterization obtained in Proposition 1.1 that (2.8) and the absolutely weak spectral inequality (2.9) give rise to an absolutely weak spectral inequality (2.7). The case $\mathbf{a} \ll \mathbf{b}$ is treated similarly.

The last part is a direct consequence of Theorems 2.2 and 2.3.

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University of Malaya,
Kuala Lumpur, Malaysia

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