108.15 The Madhava-Leibniz theorem

Introduction

The Leibniz formula for π (see [1]), named after Gottfried Leibniz states that:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ \dots \ = \frac{\pi}{4}.$$

It is also called the Leibniz-Madhava series as it is a special case of more general series expansion for the inverse tangent function, also called the Gregory series (see [2]), first discovered by the Indian Mathematician Madhava of Sangamagrama in the 14th century. The special case was first published by Leibniz around 1676.

In this Note we provide a new proof of the same. The following proof is simpler, does not involve series expansion and can be accessed by anyone with basic knowledge of calculus. Furthermore, the reader can generalise the result by following the same steps that are followed by the author.

The standard proof involves series representation of arctangent function and can be found at [3].

Theorem 1

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ \dots \ = \frac{\pi}{4}.$$

Proof: Consider the following integral defined for $n \geqslant 2$,

$$S(n) = \int_0^{\pi/4} \tan^n x \, dx.$$

This integral is evaluated by establishing a reduction formula (for more details about reduction formulae see [4, pp. 430-495]) as follows :

$$\begin{aligned} S(4n) &= \int_0^{\pi/4} \tan^{4n-2} x \tan^2 x \, dx \\ &= \int_0^{\pi/4} \tan^{4n-2} x \left(\sec^2 x - 1\right) dx \\ &= \int_0^{\pi/4} \tan^{4n-2} x \sec^2 x dx - \int_0^{\pi/4} \tan^{4n-2} x \, dx \\ &= \left[\frac{\tan^{4n-1} x}{4n-1}\right]_0^{\pi/4} - S(4n-2) \end{aligned}$$

$$S(4n) = \frac{1}{4n-1} - S(4n-2). \tag{1}$$

Similarly we have

$$S(4n-2) = \frac{1}{4n-3} - S(4n-4).$$

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Substituting this value in (1) we get,

$$S(4n) = \frac{1}{4n-1} - \frac{1}{4n-3} + S(4n-4).$$

Proceeding in a similar way, we get

$$\begin{aligned} S(4n) &= \frac{1}{4n-1} - \frac{1}{4n-3} + \ldots + \frac{1}{3} - S(2) \\ &= \frac{1}{4n-1} - \frac{1}{4n-3} + \ldots + \frac{1}{3} - \int_0^{\pi/4} \tan^2 x \, dx \\ &= \frac{1}{4n-1} - \frac{1}{4n-3} + \ldots + \frac{1}{3} - \Big[\tan x - x\Big]_0^{\pi/4} \\ S(4n) &= \frac{1}{4n-1} - \frac{1}{4n-3} + \ldots + \frac{1}{3} - 1 + \frac{\pi}{4}. \qquad (2) \end{aligned}$$

Now, we let $t = \tan x$ and obtain:

$$\begin{aligned} 0 \leqslant S(4n) &= \int_0^{\pi/4} \tan^{4n} x \, dx = \int_0^1 \frac{t^{4n}}{1+t^2} \, dt \\ &\leqslant \int_0^1 t^{4n} \, dt = \left[\frac{t^{4n+1}}{4n+1}\right]_0^1 = \frac{1}{4n+1} \end{aligned}$$

$$\Rightarrow 0 \leqslant S(4n) \leqslant \frac{1}{4n+1}. \qquad (3)$$

Therefore using 'squeezing theorem' (see [5]) we get

$$\lim_{n \to \infty} S(4n) = \lim_{n \to \infty} \int_0^{\pi/4} \tan^{4n} x \, dx = 0. \qquad (4)$$

Letting $n \to \infty$ in (2) and using (4) we get

$$\begin{aligned} \lim_{n \to \infty} S(4n) &= \lim_{n \to \infty} \left(\frac{1}{4n-1} - \frac{1}{4n-3} + \ldots + \frac{1}{3} - 1 + \frac{\pi}{4}\right) \\ 0 &= \lim_{n \to \infty} \left(\frac{1}{4n-1} - \frac{1}{4n-3} + \ldots + \frac{1}{3} - 1 + \frac{\pi}{4}\right) \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots . \end{aligned}$$

Hence proved.

Exercise for the reader

We encourage the reader to generalise the above result by redefining the integral $S(n)$ as :

$$S(n, \theta) = \int_0^{\theta} \tan^n x \, dx, \text{ where } n \geqslant 2 \text{ and } 0 \leqslant \theta \leqslant \frac{\pi}{4}.$$

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Now, we let $u = \tan x$ and obtain

$$0 \leqslant S(n, \theta) = \int_0^\theta \tan^n x\, dx = \int_0^{\tan\theta} \frac{u^n}{1 + u^2} du$$

$$\leqslant \int_0^{\tan\theta} u^n\, du = \left[\frac{u^{n+1}}{n+1}\right]_0^{\tan\theta} = \frac{\tan^{n+1}\theta}{n+1} \leqslant \frac{1}{n+1}$$

where the last inequality follows from

$$|\tan\theta| \leqslant 1 \text{ for } |\theta| \leqslant \frac{\pi}{4} \Rightarrow \tan^n\theta \leqslant 1$$

$$\Rightarrow 0 \leqslant S(n, \theta) \leqslant \frac{1}{n+1}.$$

This ensures that $S(n, \theta) \to 0$ as $n \to \infty$.

Using similar method and reasoning as in this Note, it can be proved that

$$\theta = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots \text{ where } t = \tan\theta.$$

Substituting different values will yield different infinite series for π. For example, choosing $\theta = \pi/6$ yields

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} + \ldots\right).$$

Unlike the Madhava-Leibnitz series, this can be used to estimate a value for π efficiently (without a large computer).

One can further extend the results for $-\frac{\pi}{4} < \theta < 0$ by observing that

$$S(n, -\theta) = (-1)^{n+1} S(n, \theta).$$

References

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