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EXAMPLES AND QUESTIONS IN THE THEORY OF FIXED POINT SETS

JOHN R. MARTIN AND SAM B. NADLER, JR.

1. Introduction. All spaces considered in this paper will be metric spaces. A subset A of a space X is called a *fixed point set of* X if there is a *map* (i.e., continuous function) $f: X \to X$ such that $f(x) = x$ if and only if $x \in A$. In [**22**] L. E. Ward, Jr. defines a space X to have the *complete invariance property* (CIP) provided that each of the nonempty closed subsets of X is a fixed point set of X. The problem of determining fixed point sets of spaces has been investigated in [**14**] through [**20**] and [**22**]. Some spaces known to have CIP are n-cells [**15**], dendrites [**20**], convex subsets of Banach spaces [**22**], compact manifolds without boundary [**16**], and a class of polyhedra which includes all compact triangulable manifolds with or without boundary [**18**]. In [**22**, p. 553] it was asked if every locally connected continuum has CIP. This question was answered negatively in [**14**] where it was shown that, for each $n = 1, 2, \ldots,$ there exist $(n + 1)$-dimensional acyclic LC^{n-1} continua which do not have CIP.

The purpose of this paper is to give examples which show that the operations of taking products, cones, and wedges do not preserve CIP. Also, we show that retractions do not preserve CIP even for locally connected continua, and that strong deformation retracts of contractible continua with CIP need not have CIP. The motivation for considering each of the examples will be discussed as the paper progresses. We mention that some of our examples have stronger properties than is indicated above. For instance, when we show that CIP is not preserved by products, one of our factors is a 1-dimensional polyhedron.

2. Notation and lemmas. By a *compactum* we mean a compact metric space, and by a *continuum* we mean a connected compactum containing more than one point. A *subcontinuum* is a nonempty compact connected subset of a given continuum. The symbol S^n will be used to denote the boundary of the closed unit ball in Euclidean $(n + 1)$-dimensional space R^{n+1}.

Let Y and Z be metric spaces, and let $y \in Y$ and $x \in Z$. By the *wedge* $Y \bigvee_{y \sim z} Z$ *of* Y *and* Z we mean the space obtained by taking the disjoint union of Y and Z, and then identifying the point y with z.

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The symbols cl and $\times$ will denote the closure and (cartesian) product respectively. The symbol LC^n is as defined in [**5**, p. 30]. For sets A and B, $A - B$ denotes the complement of B in A. If ρ is a metric for Z, if $z \in Z$, and if A is a nonempty closed subset of Z, then we define

$$\rho[z, A] = \inf \{\rho(z, a) : a \in A\}.$$

The *cone over a compactum* M, denoted by Cone (M), will be regarded as the geometric cone over M as follows: Assume, without loss of generality, that $M \subset L$, L being some Euclidean space or Hilbert space, such that the closed linear span, $[M]$, of M is a proper subspace of L. Let $v \in L - [M]$. Then,

$$\text{Cone }(M) = \bigcup \{\overline{mv} : m \in M\}$$

where $\overline{mv}$ denotes the convex arc in L from m to v. By a *convex arc in* Cone (M) we mean a subarc of $\overline{mv}$ for some $m \in M$. Henceforth, the letter v will only be used as above, i.e., to denote the *vertex of* Cone (M). We will treat the points of Cone (M) as ordered pairs (m, t) where $m \in M$ and t is a real number. The points in $B = \text{Cone}(M) \cap [M]$, called the *base of* Cone(M), will be considered to have second coordinate $= 0$. The symbol π denotes the projection from Cone(M) onto $[0, \pi(v)]$ given by $\pi(m, t) = t$ for each $(m, t) \in \text{Cone}(M)$. For any $m \in M$, $(m, \pi(v))$ denotes, without confusion, the vertex v. The symbol P_B denotes the projection of Cone$(M) - \{v\}$ onto the base B. The following technical lemma will be used several times in Sections 3 and 4. Its proof is related to the proof of Theorem 1 of [**22**].

(2.1) LEMMA. *Let M be a compactum. Let K be a closed subset of* Cone(M) *such that $\pi^{-1}(s) \subset K \subset \pi^{-1}([0, s])$ for some $s \in [0, \pi(v)]$. Then: There is a mapping*

$$f : \pi^{-1}([0, s]) \to \pi^{-1}([0, s])$$

such that f has fixed point set equal to K and such that if

$$(m, r) \in [\pi^{-1}([0, s]) - K],$$

then $f(m, r) = (m, c)$ for some $c > r$.

Proof. Let $h: \pi^{-1}([0, s]) \times [0, 1] \to \pi^{-1}([0, s])$ be the homotopy given by

$$h((m, t), u) = (m, [1 - u] \cdot t + u \cdot s)$$

for each $(m, t) \in \pi^{-1}([0, s])$ and each $u \in [0, 1]$. Assuming that the metric ρ for Cone(M) has all its values $\leqq 1$, we define f by

$$f(z) = h(z, \rho[z, K])$$

for each $z \in \pi^{-1}([0, s])$. It is easy to verify that f has the desired properties. Therefore, we have proved (2.1).

The following lemma will be used in Sections 3–5. Its hypothesis and conclusion are stronger than those of Theorem 1 of [**22**].

(2.2) LEMMA. *Let M be a metric space such that there is a homotopy h: $M \times [0, 1] \to M$ satisfying: $h(m, 0) = m$ and $h(m, t) \neq m$ for each $m \in M$ and each $t > 0$. Then, for any metric space Z, $M \times Z$ has* CIP.

Proof. Assume that the metric ρ for $M \times Z$ has all its values $\leqq 1$. Let A be a nonempty closed subset of $M \times Z$. Define

$$f: M \times Z \to M \times Z$$

by $f(m, z) = (h(m, \rho[(m, z), A]), z)$ for each $(m, z) \in M \times Z$. Then f is a mapping having fixed point set equal to A.

We mention that in cases where the reader is left to verify that a particular space has CIP, the following lemma may be useful.

(2.3) LEMMA. *A nonempty closed subset A of a space Z is a fixed point set of Z if A is contained in a retract B of Z such that B has* CIP.

3. Wedges. The wedge of two spaces was defined in Section 2. The fixed point property and the property of being an absolute retract or an absolute neighborhood retract are each preserved by taking wedges (see [**4**, p. 121] and [**5**, p. 90] respectively). The main purpose of this section is to give an example in (3.1) of a locally connected continuum having CIP whose wedge with itself at a specified point does not have CIP. This shows that a locally connected continuum can fail to have CIP and yet be the union of two locally connected continua having CIP. Thus, we have a different type of answer to the question in [**22**, p. 553] than the one mentioned in Section 1.

(3.1) *Example.* Let H be the *Hawaiian Earring*, i.e., $H = \bigcup_{j=1}^{\infty} S_j$ where

$$S_j = \{(x, y) \in R^2: (x - 2^{-j})^2 + y^2 = 2^{-2j}\}$$

for each $j = 1, 2, \ldots$. Let $C = H \times [0, 1]$; we call C the *Hawaiian Can.* Let $\theta = ((0, 0), 0) \in C$ and let $W = C \bigvee_{\theta \sim \theta} C$ (see Figure 1). We will prove:

(1) C has CIP;
(2) W does not have CIP.

To prove (1), let A be a nonempty closed subset of C. Let

$$L = \{(0, 0)\} \times [0, 1] \subset C.$$

First assume that $A \cap L = \emptyset$. Then there is a natural number J such that $A \cap (S_j \times [0, 1]) = \emptyset$ for each $j \geqq J$. Let

$$K = \bigcup_{j=1}^{J} (S_j \times [0, 1]).$$

By [**18**, 3.1], there is a mapping f from K into K with $A = A \cap K$ as its fixed point set. Note that there is a retraction r from C onto K. Clearly, $f \circ r$ is a

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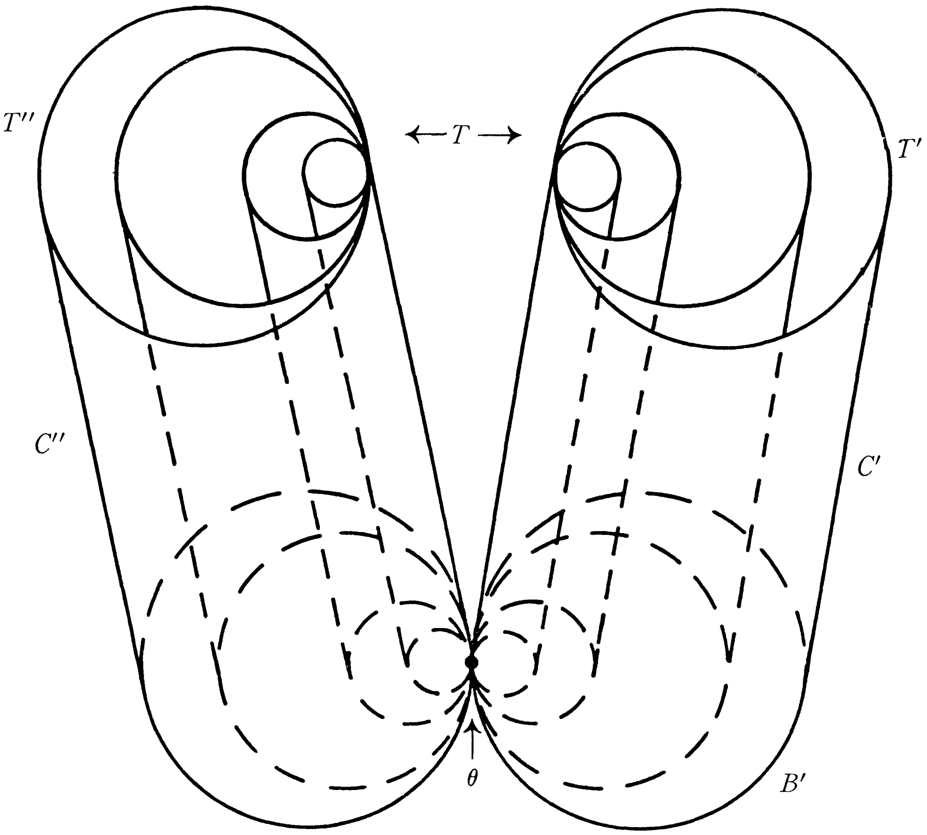


FIGURE 1

mapping from C into C having fixed point set equal to A. Next assume that $A \cap L \neq \emptyset$. Let

$$a = ((0,0), s) \in [A \cap L].$$

It is easy to see that there is a homotopy h: $H \times [0, 1] \rightarrow H$ such that $h(x, 0) = x$ for each $x \in H$ and $h(x, t) \neq x$ for each $x \in [H - \{(0, 0)\}]$ and $t > 0$. Let

$$k: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

be defined by $k(t_1, t_2) = [1 - t_2] \cdot t_1 + t_2 \cdot s$ for each $t_1, t_2 \in [0, 1]$. Let ρ denote the metric for C and assume, without loss of generality, that ρ has all its values $\leqq 1$. Define f: $C \rightarrow C$ by

$$f(x, t) = (h(x, \rho[(x, t), A]), k(t, \rho[(x, t), A]))$$

for each $(x, t) \in C$. It follows easily that f is continuous and that f has fixed point set equal to A. This completes the proof of (1).

To prove (2), let $C' \subset W$ and $C'' \subset W$ denote the two copies of C which have been wedged at θ to produce $W = C' \cup C''$ (see Figure 1). Let T' denote the top of C', $T' = \{(x, 1) \in C'\}$, and let T'' denote the top of C'', $T'' = \{(x, 1) \in C''\}$. Let $T = T' \cup T''$. We will prove that T is not the fixed point set of any mapping from W into W. To prove this, suppose that there is a mapping $f: W \to W$ such that f has fixed point set equal to T. Since $f(\theta) \neq \theta$, we assume without loss of generality that $f(\theta) \in [C'' - C']$. Then, by the continuity of f, there exists an open subset V of W such that $\theta \in V$ and $f[V] \subset C'' - C'$. Let B' denote the bottom of the Can C', $B' = \{(x, 0) \in C'\}$. Then, there exists a circle $S' \subset [V \cap B']$. Note that $f[S'] \subset C'' - C'$. Let

$$Y' = \{(y, t) \in C' : y \in S' \text{ and } 0 \leqq t \leqq 1\}.$$

Geometrically, Y' is the cylinder in C' above S'. Let Q' denote the top of the cylinder Y', $Q' = \{(y, 1) \in Y'\}$. Let $\psi: Q' \times [0, 1] \to Y'$ be a homotopy such that $\psi(q', 0) = q'$ for each $q' \in Q'$ and such that $\psi[Q' \times \{1\}] = S'$. Then, since $Q' \subset T$ and T is the fixed point set of f, $f \circ \psi: Q' \times [0, 1] \to W$ is a homotopy such that

(a) $f \circ \psi(q', 0) = q'$ for each $q' \in Q'$

and, since $\psi[Q' \times \{1\}] = S'$ and $f[S'] \subset C'' - C'$,

(b) $f \circ \psi[Q' \times \{1\}] \subset C'' - C'$.

It is easy to see that there is a retraction r from W onto Y' such that $r[C''] = \{\theta\}$. By using (a) and (b) above, it follows easily that $r \circ f \circ \psi$: $Q' \times [0, 1] \to Y'$ is a homotopy contracting Q' to θ in Y'. This is not possible. Therefore, T is not the fixed point set of any mapping from W into W and, thus, we have proved (2).

In the following example we modify (3.1) so as to obtain examples which possess higher orders of local connectedness than the example in (3.1).

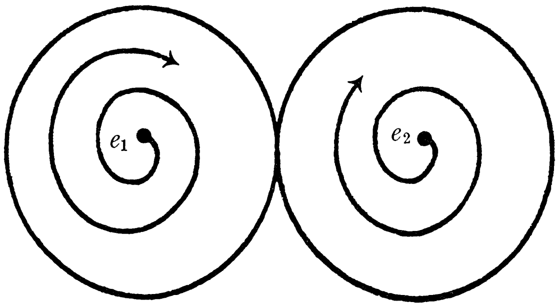
(3.2) *Example*. Let $H_n = \bigcup_{j=1}^{\infty} S_j^n$ where

$$S_j^n = \{(x_1, x_2, \ldots, x_{n+1}) \in R^{n+1}: (x_1 - 2^{-j})^2 + x_2^2 + \ldots + x_{n+1}^2 = 2^{-2j}\}$$

for each $j = 1, 2, \ldots$. Let $C_n = H_n \times [0, 1]$ and let θ_n denote the origin in R^{n+2}. Then: C_n is an $(n + 1)$-dimensional LC^{n-1} continuum having CIP, while $W_n = C_n \bigvee_{\theta_n \sim \theta_n} C_n$ does not have CIP. The proof is analogous to the one in (3.1).

Next we give an example to show that wedging does not preserve CIP for 1-dimensional unicoherent continua. Our example is not locally connected, as must be the case since 1-dimensional unicoherent locally connected continua are dendrites; thus, their wedge has CIP by [**20**, 3.1].

(3.3) *Example.* Let X be a circle with a spiral and let Y denote the wedge of X with itself as indicated in Figure 2. Note that X is a unicoherent 1-dimensional continuum. It follows using (2.2) that X has CIP. However, the reader may readily check that the set $\{e_1, e_2\}$ consisting of the two end points of the spirals is not a fixed point set of Y.



Y

FIGURE 2

In comparing the examples in (3.1) through (3.3), the following questions come to mind.

(3.4) *Questions.* Are there two 1-dimensional locally connected continua having CIP whose wedge does not have CIP? In fact, does every 1-dimensional locally connected continuum have CIP? Note that these two questions have affirmative answers for 1-dimensional absolute retracts (i.e., dendrites) by using [**20**, 3.1].

(3.5) *Questions.* Are there two acyclic continua having CIP whose wedge does not have CIP? What if the two continua are also 1-dimensional (note that the continuum X of (3.3) is unicoherent but not acyclic)? Are there two contractible continua having CIP whose wedge does not have CIP?

The following result about wedges will be used in the proof of (4.3).

(3.6) PROPOSITION. *Let M be any given compactum and let N be an arcwise connected continuum having* CIP *such that N does not have the fixed point property. Then, for any point $q \in N$,* Cone(M) $\bigvee_{v\sim q} N$ *has* CIP.

Proof. Let $W = \text{Cone}(M) \bigvee_{v\sim q} N$ and let p denote the wedge point of W. To prove that W has CIP, let A be a nonempty closed subset of W. Let $A_1 = A \cap \text{Cone}(M)$ and let $A_2 = A \cap N$. We take two cases.

Case 1. $A_1 = \emptyset$. Then $A = A_2 \subset N$. Thus, since N has CIP, there is a mapping $f\colon N \to N$ with fixed point set equal to A. Extend f to a mapping

$\bar{f}: W \to W$ by letting

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in N \\ f(p), & \text{if } x \in \text{Cone}(M). \end{cases}$$

Clearly, $\bar{f}$ has fixed point set equal to A.

Case 2. $A_1 \neq \emptyset$. Whether $A_2 \neq \emptyset$ or $A_2 = \emptyset$, the hypotheses on N imply there exists a mapping $f: N \to N$ whose fixed point set is equal to A_2. By (2.1) with $s = \pi(v)$, we have that there is a mapping

$$g: \text{Cone}(M) \to \text{Cone}(M)$$

with fixed point set equal to $A_1 \cup \{p\} = A_1 \cup \{v\}$ such that if $z \in [\text{Cone}(M) - (A_1 \cup \{p\})]$, then

(#) z moves under g towards p on the convex arc in Cone(M) from z to p.

If $p \in A$, then $k_1: W \to W$ defined by

$$k_1(x) = \begin{cases} f(x), & \text{if } x \in N \\ g(x), & \text{if } x \in \text{Cone}(M) \end{cases}$$

is a mapping whose fixed point set is equal to A. For the rest of the proof, assume $p \notin A$. Then there exists $t_0 < \pi(v)$ such that (#) occurs for each $z = (m, s) \in \text{Cone}(M)$ such that $t_0 \leqq s \leqq \pi(v)$. Since $f(p) \neq p$ and since N is arcwise connected, there is an arc γ in N from p to $f(p)$. By using γ and the convex arcs in $\pi^{-1}([t_0, \pi(v)])$, it is easy to see how to modify g so as to obtain a mapping

$$g': \text{Cone}(M) \to \text{Cone}(M) \cup \gamma$$

such that $g'(p) = f(p)$ and g' has fixed point set equal to A_1. Then $k_2: W \to W$ defined by

$$k_2(x) = \begin{cases} f(x), & \text{if } x \in N \\ g(x), & \text{if } x \in \text{Cone}(M) \end{cases}$$

is a mapping whose fixed point set is equal to A.

By Cases 1 and 2, we have proved that W has CIP. This completes the proof of (3.6).

4. Retracts. In [**14**] it was asked if every compact absolute retract has CIP. Since the Hilbert cube I_∞ has CIP [**22**, 1.1, p. 554], this question would have an affirmative answer if CIP were a retract invariant for I_∞. The result in (4.1) shows that CIP is not in general a retract invariant. Though we do not know if CIP is a retract invariant for I_∞, the proof of (4.1) will be used in (4.2) to show that CIP is not a retract invariant even for spaces with nice local properties.

(4.1) THEOREM. *Any metric space can be embedded as a retract in a metric space having* CIP.

Proof. Let Z be a given metric space. Then Z is embedded in $S^1 \times Z$ as a retract and $S^1 \times Z$ has CIP by (2.2).

(4.2) *Examples.* Let W_n be as in (3.2). Then $S^1 \times W_n$ is an $(n + 2)$-dimensional LC^{n-1} continuum. Hence (see the proof of (4.1)), CIP is not a retract invariant for the class of $(n + 2)$-dimensional LC^{n-1} continua. By taking X_n to be as in [**14**] and using the proof of (4.1), we obtain the acyclic continuum X_n, which does not have CIP [**14**], as a retract of the $(n + 2)$-dimensional LC^{n-1} continuum $S^1 \times X_n$ which has CIP.

We have seen that CIP is not preserved by retractions. Next we give an example of a contractible continuum X with CIP which has, as a strong deformation retract, a subcontinuum not having CIP.

(4.3) *Example.* Let $Y = S^1 \cup \mathscr{S}$ be the circle S^1 with the spiral $\mathscr{S}$ given by

$$\mathscr{S} = \{[1 + (1/t)] \cdot e^{it}: t \geqq +1\}.$$

Let $N = \text{Cone}(Y)$. Let M denote the Cantor set. Let

$$X = \text{Cone}(M) \bigvee_{v_1 \sim v_2} N$$

where v_1 denotes the vertex of Cone(M) and v_2 denotes the vertex of N. By [**22**, p. 556], Cone(M) does not have CIP. Clearly, Cone(M) is a strong deformation retract of X. It remains to show that X has CIP. Note that N does not have the fixed point property ([**4**, pp. 129–130] or [**13**]). Hence, once we show that N has CIP, we will know from (3.6) that X has CIP. To prove that N has CIP, let A be a nonempty closed subset of N. By (2.1), we may assume $v_2 \notin N$. Then, letting

$$m = \max \{\pi(a): a \in A\},$$

there exist r, s, and t such that

$$m < r < s < t < \pi(v_2).$$

Let $B = A \cup \pi^{-1}(s)$. Let $f: \pi^{-1}([0, s]) \to \pi^{-1}([0, s])$ be as in (2.1) with fixed point set equal to B. Let $T = \pi^{-1}([t, \pi(v_2)])$. Since T is homeomorphic to N, there is a fixed point free mapping $g: T \to T$ such that $g[\pi^{-1}(t)] = \{v_2\}$ [**4**, pp. 129–130]. For any $y \in Y$, let $\alpha(y)$ denote the convex arc from (y, r) to (y, t). By (2.1), $f(y, r) = (y, c)$ for some c such that $r < c < s$. Define k_y on $\alpha(y)$ by: For any λ such that $0 \leqq \lambda \leqq 1$,

$$k_y(y, [1 - \lambda]r + \lambda t) = (y, [1 - \lambda]c + \lambda\pi(v_2)).$$

Observe that $k_y(y, r) = (y, c) = f(y, r)$ and that $k_y(y, t) = v_2 = g(y, t)$. It follows that $j: N \to N$ defined by

$$j(y, u) = \begin{cases} f(y, u), & \text{if } 0 \leqq u \leqq r \\ k_y(y, u), & \text{if } r \leqq u \leqq t \\ g(y, u), & \text{if } t \leqq u \leqq \pi(v_2) \end{cases}$$

is a continuous function with fixed point set equal to A. Therefore, we have proven that N has CIP. This completes the verifications for (4.3).

5. Products. By a *polyhedron* we mean a compact connected polyhedron. By using the proof of Corollary 3 in [**6**, p. 145], one can apply [**18**, 3.1] to show that: (1) if X is a polyhedron, then $X \times [0, 1]$ has CIP and (2) the product of any two 1-dimensional polyhedra has CIP. The main purpose of this section is to show in (5.1) that CIP is not preserved by products of 1-dimensional continua even when one of the factors is a 1-dimensional polyhedron. We also obtain some results about topological groups and indecomposable continua.

(5.1) *Example.* Let X be the 1-dimensional planar continuum drawn in Figure 3. In the n^{th} row we have a null sequence of circles $C_1^n, C_2^n, \ldots$ converging to the point p_n such that, for each i, C_i^n is "connected to" C_{i+1}^n by a line, half of which spirals in on C_i^n and the other half of which spirals in on C_{i+1}^n. As $n \to \infty$, the rows converge to the point p. The rows are connected by the arc from p to q. It is not difficult to verify that X has CIP (we leave the proof to the reader, who will find (2.3) helpful). We will use the following fact about X:

(1) If L is a locally connected subcontinuum of X such that $p_n \in L$, then $L = \{p_n\}$.

Let $Y = Y_1 \cup Y_2 \cup Y_3$, drawn in Figure 4, where $Y_1 = S^1$ and

$$Y_2 = \{(x, y) \in R^2: x^2 + (y-2)^2 = 1\},$$
$$Y_3 = \{(0, y) \in R^2: 3 \leqq y \leqq 4\}.$$

It is easy to verify that Y has CIP (we omit the proof). Let $e = (0, 4) \in Y$. We will use the following fact (whose proof we will briefly indicate):

(2) There exists $\epsilon > 0$ such that if $f: Y \to Y$ is within ϵ of the identity map on Y, then f does not have fixed point set equal to $\{e\}$.

Sketch of proof of (2). Take $\epsilon = 1$. Suppose that $f: Y \to Y$ is within ϵ of the identity map on Y and that f has fixed point set equal to $\{e\}$. If $f(0, 3) \in Y_3$, then it follows that f is not single-valued at $(0, 1)$. Hence $f(0, 3) \in Y_2$. Then, by first considering the images of points near $(0, 3)$, it follows that $f(0, 1) \in Y_2$. This implies that f has a fixed point in Y_1. This completes our proof of (2).

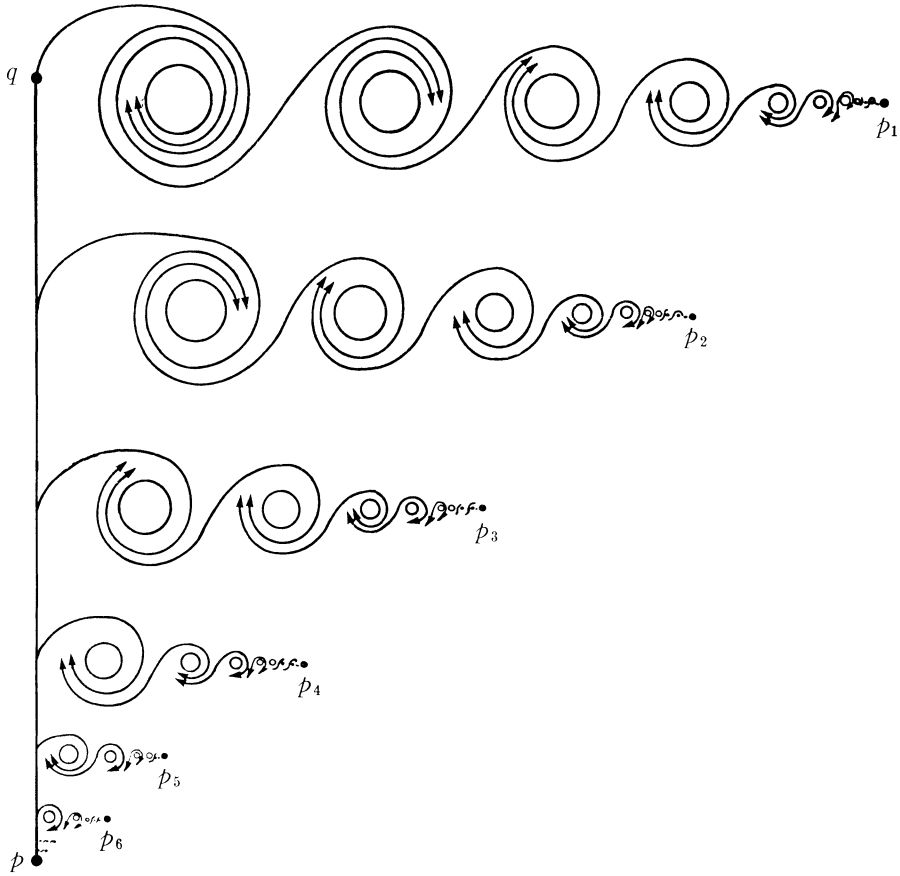


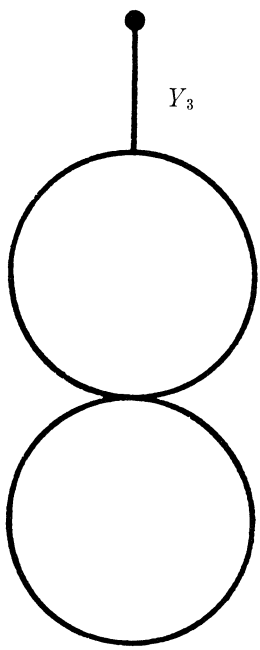
FIGURE 3

We now show, using (1) and (2), that $X \times Y$ does not have CIP. The points p_n, p, and e are as in Figures 3 and 4. Let

$$A = \{(p_n, e) \in X \times Y: n = 1, 2, \ldots\} \cup \{(p, y) \in X \times Y: y \in Y\}.$$

Suppose that there is a mapping $g: X \times Y \to X \times Y$ such that g has fixed point set equal to A. Let π_X denote the projection of $X \times Y$ onto X given by $\pi_X(x, y) = x$ for each $(x, y) \in X \times Y$. Since Y is a locally connected continuum and since $g(p_n, e) = (p_n, e)$ for each $n = 1, 2, \ldots,$ it follows easily using (1) that $\pi_X(g[\{p_n\} \times Y]) = \{p_n\}$ for each $n = 1, 2, \ldots$. Hence,

(3) $g[\{p_n\} \times Y] \subset \{p_n\}] \times Y$ for each $n = 1, 2, \ldots$.



Y

FIGURE 4

For each $n = 1, 2, \ldots$, define

$$j_n: \{p\} \times Y \to \{p_n\} \times Y$$

by $j_n(p, y) = (p_n, y)$ for each $(p, y) \in \{p\} \times Y$. Then, by (3), $j_n^{-1} \circ g \circ j_n = k_n$ is a mapping from $\{p\} \times Y$ into $\{p\} \times Y$ for each $n = 1, 2, \ldots$. Also, for each $n = 1, 2, \ldots, k_n$ has fixed point set equal to $\{(p, e)\}$. Finally, since the sequence $\{p_n\}_{n=1}^{\infty}$ converges to p and since g leaves each point of $\{p\} \times Y$ fixed, it follows easily that the sequence $\{k_n\}_{n=1}^{\infty}$ converges to the identity map on $\{p\} \times Y$. Thus, we have a contradiction to (2). Therefore, there is no mapping from $X \times Y$ into $X \times Y$ with fixed point set equal to A. This completes our presentation of (5.1).

(5.2) *Questions*. Is there a continuum X having CIP such that $X \times [0, 1]$ does not have CIP? Are there two locally connected continua X and Y, each of which has CIP, such that $X \times Y$ does not have CIP?

In (2.2) we gave a sufficient condition in order that a product have CIP. We gave an application of (2.2) in (4.1). We now give an application of (2.2) to topological groups.

(5.3) THEOREM. *Let* $(G, \cdot)$ *be a metrizable topological group which contains an arc. Then, for any metric space* Z, $G \times Z$ *has* CIP.

Proof. Let e denote the identity element of G. Since G contains an arc, we have by translation that e belongs to an arc in G. Let γ be an arc in G having e as one of its end points. Let $\alpha: [0, 1] \to \gamma$ be a homeomorphism onto γ such that $\alpha(0) = e$. Define $h: G \times [0, 1] \to G$ by $h(x, t) = \alpha(t) \cdot x$ for each $(x, t) \in G \times [0, 1]$. Note that $h(x, 0) = x$ for each $x \in G$ and that $h(x, t) \neq x$ for all $x \in G$ and $t > 0$. Hence, by (2.2), $G \times Z$ has CIP.

(5.4) THEOREM. *Any locally compact metrizable topological group* $(G, \cdot)$ *has* CIP.

Proof. If G is not totally-disconnected, then G contains an arc [**8**, p. 663] and hence, by (5.3), G has CIP. Next, assume G is totally-disconnected. Then, since G is locally compact, there is an open compact subgroup H of G by [**24**, 29E5, p. 215]. Note that $\Lambda = \{x \cdot H: x \in G\}$ is a cover of G by mutually disjoint open compact subsets of G. Let A be a nonempty closed subset of G and let $a_0 \in A$. We define a retraction r from G onto A as follows: Let $L \in \Lambda$. If $L \cap A = \emptyset$, let $r(z) = a_0$ for each $z \in L$. If $L \cap A \neq \emptyset$ then, since L is compact and totally-disconnected, we may let r on L be a retraction from L onto $L \cap A$ by using [**12**, 0(*c*), p. 165]. Defining r in the above manner for each $L \in \Lambda$, we see from the properties of Λ that r is a continuous function on G. Therefore, since r has fixed point set equal to A, we have proved that G has CIP.

We now use (5.4) to give an example of an indecomposable continuum which has CIP. A continuum is said to be *indecomposable* [**10**, p. 139] provided that it is not the union of any two proper subcontinua.

(5.5) *Example.* Let D be the dyadic solenoid, i.e., D is the inverse limit of the inverse sequence $\{D_n, f_n\}_{n=1}^{\infty}$ where, for each $n = 1, 2, \ldots, D_n = S^1$ and $f_n(z) = z^2$ for each $z \in S^1$. Since D is a continuum which admits the structure of a topological group [**10**, p. 145], D has CIP by (5.4). Also, D is indecomposable [**10**, p. 145].

In relation to (5.3) and (5.4), we ask the following question.

(5.6) *Question.* Does every metrizable topological group have CIP?

In (5.5) we gave an example of an indecomposable continuum having CIP. A continuum is said to be *hereditarily indecomposable* provided that each of its subcontinua is indecomposable.

(5.7) *Question.* Is there an hereditarily indecomposable continuum having CIP [comp., (6.2)]? We do not know if the pseudo-arc has CIP. It follows from [**7**] that each subcontinuum of the pseudo-arc is a fixed point set of the pseudo-arc. Also, it can be shown using [**7**], [**9**] to obtain a shift, and [**3**, Theorem 14] that any two-point subset of the pseudo-arc is a fixed point set of the pseudo-arc.

6. Cones. In [**22**, p. 556] it was shown that the cone over the Cantor set does not have CIP. Note that the Cantor set has CIP since every nonempty closed subset of the Cantor set is a retract of the Cantor set [**12**, p. 165]. Hence, CIP is not preserved by taking cones over compacta. In the following example we show that CIP is not preserved by taking cones over continua.

(6.1) *Example.* Let X be any continuum which contains no arc (= *arcless*) and let $Y = X \times S^1$. By (2.2), Y has CIP. We show that Cone(Y) does not have CIP. Let B denote the base of Cone(Y). Suppose that there is a mapping f: Cone$(Y) \to$ Cone(Y) such that f has fixed point set equal to B. Since $f(v) \neq v$, there exists a unique simple closed curve $S \subset B$ such that $f(v) \in$ Cone(S), i.e., $f(v) = ((x, s), t)$ for some $((x, s), t) \neq v$ and, thus, $S = \{x\} \times S^1 \subset B$. For future use, let us note that since X is arcless,

(1) S is an arc component of B.

Let $E = f^{-1}(v) \cap$ Cone(S). We prove that

(2) E separates v from S in Cone(S).

Suppose (2) is false. Then, since Cone(S) is a locally connected continuum (in fact, a 2-cell), there exists an arc $\alpha \subset$ [Cone$(S) - E$] from a point $p \in S$ to v. Hence, using the compactness of $f^{-1}(v)$, there exists an arc $\beta \subset$ [Cone$(Y) - f^{-1}(v)$] from a point $q \in [B - S]$ to v. Let $\gamma = \alpha \cup \beta$. Since $f(p) = p$ and $f(q) = q$, we have that $p, q \in f[\gamma]$. Also, note that $v \notin f[\gamma]$ and that $f[\gamma]$ is a locally connected continuum. Thus, $P_B(f[\gamma])$ is a locally connected subcontinuum of B such that $p, q \in P_B(f[\gamma])$. This gives a contradiction to (1) since $p \in S$ and $q \in [B - S]$. Hence, we have proved (2). It follows from (2) and well-known properties of 2-cells (see, for example, [**21**, 3.2]) that some subcontinuum K of E separates v from S in Cone(S). Choose and fix a number t_0 such that

$$\sup (\pi[K]) < t_0 < \pi(v).$$

Then, since f is continuous and $f[K] = \{v\}$, there is a simple closed curve C,

(3) $C \subset [\text{Cone}(S) \cap \pi^{-1}((0, t_0))]$,

such that C separates v from S in Cone(S) and

(4) $f[C] \subset \pi^{-1}((t_0, \pi(v)])$.

Since C is a simple closed curve in the manifold interior of the 2-cell Cone(S), it follows from the Jordan Curve Theorem [**23**, p. 104] that Cone$(S) - C$ has exactly two components V and W, with $v \in V$ and $S \subset W$, and that C is the boundary in Cone(S) of each of V and W. Furthermore, it is easy to see using basic facts about the topology of the plane that $D = V \cup C$ is a 2-cell and $Q = W \cup C$ is an annulus with manifold boundary equal to $C \cup S$. Hence, since C is a retract of Q, there is a retraction r_1 from Cone(S) onto D such that $r_1[Q] = C$. Define r from

$$M = \text{Cone}(S) \cup f[D]$$

onto D by

$$r(z) = \begin{cases} r_1(z), & \text{if } z \in \text{Cone}(S) \\ v, & \text{if } z \in [M - \text{Cone}(S)]. \end{cases}$$

We show that r is continuous. Since

$$M - \text{Cone}(S) = f[D] - \text{Cone}(S),$$

we have that $\text{cl}[M - \text{Cone}(S)] \subset f[D]$. Hence, since $f[D]$ is a locally connected continuum and since each arc from M − Cone(S) to Cone(S) must go through v (by (1)), it follows easily that

(5) $\text{cl}[M - \text{Cone}(S)] \cap \text{Cone}(S) \subset \{v\}$.

By (5), r is continuous. Thus, r is retraction from M onto D. Let g denote the restriction of f to D. Since $r \circ g$ maps D into D and D is a 2-cell, there exists $p \in D$ such that $r \circ g(p) = p$ [**23**, 3.3, p. 243], i.e., $r(f(p)) = p$. First suppose that $f(p) \in D$. Then, since r is the identity on D, $r(f(p)) = f(p)$. Thus, since $r(f(p)) = p$, we have that $f(p) = p$. Hence, $p \in B$. However, $p \in D$ and $D \cap B = \emptyset$. Thus, we have a contradiction. Therefore,

(6) $f(p) \notin D$.

Next, suppose that $f(p) \in Q$. Then, since $r_1[Q] = C$, $r_1(f(p)) \in C$. Thus, since $p = r(f(p)) = r_1(f(p))$, we have that $p \in C$. Hence, by (4), $\pi(f(p)) > t_0$. Therefore, since $Q \subset \pi^{-1}([0, t_0))$ by (3), it follows that

(7) $f(p) \notin Q$.

Since Cone$(S) = D \cup Q$, we have by (6) and (7) that $f(p) \notin$ Cone(S). Since $p \in D$, $f(p) \in M$. Hence, $f(p) \in [M - \text{Cone}(S)]$ which implies that $r(f(p)) = v$. Thus, since $r(f(p)) = p$, $p = v$. Hence, $f(p) = f(v) \in$ Cone(S) by definition of S. Thus, we have a contradiction. Therefore, there is no mapping f: Cone$(Y) \to$ Cone(Y) such that f has fixed point set equal to B. Therefore, Cone(Y) does not have CIP.

In connection with (6.1), let us note the following comments and questions.

The argument in [**22**, p. 556], which shows that the cone over the Cantor set does not have CIP, can be applied to show that the cone over any arcless continuum X does not have CIP. We used $Y = Y \times S^1$ in (6.1) in order to assure that the continuum Y, over which we took the cone, has CIP. This leads to the following question which was asked for a more restrictive class of continua in (5.7).

(6.2) *Question.* Is there an arcless continuum Z such that Z has CIP? If the answer is "yes", then Y in (6.1) can be replaced by such a Z and the proof in (6.1) can be replaced by the one in [**22**, p. 556].

With respect to the next question, note that Y in (6.1) is of dimension $\geqq 2$.

(6.3) *Question.* Is there a 1-dimensional continuum having CIP whose cone does not have CIP?

It may be possible to answer (6.2) affirmatively with a rational Z (for constructions of rational arcless continua, see [**1**], [**2**], or [**11**]). This would give an affirmative answer to (6.3) since rational continua are 1-dimensional [**23**, p. 99].

The most basic question related to this section would seem to be the following one:

(6.4) *Question.* Is there a locally connected continuum having CIP whose cone does not have CIP?

Added in proof. Questions (3.4) and (3.5) have been answered. In a paper to appear in the Canadian Mathematical Bulletin entitled *A note on fixed point sets and wedges*, the authors have shown that a wedge of two 1-dimensional contractible planar continua having CIP need not have CIP. In a paper to appear in the Pacific Journal of Mathematics entitled *Fixed point sets of 1-dimensional Peano continua*, the first author and E. D. Tymchatyn have shown that every 1-dimensional continuum has CIP.

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University of Saskatchewan,
Saskatoon, Saskatchewan