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C_2 BUILDING AND PROJECTIVE SPACE

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Abstract

We study the stability map from the rigid analytic space of semistable points in $\mathbb{P}^3$ to convex sets in the building of Sp_2 over a local field and construct a pure affinoid covering of the space of stable points.

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0. Introduction

Drinfeld introduced a p-adic symmetric space and used it to study the representations of GL(2) over a function field. Schneider and Stuhler use the map from the p-adic symmetric space to the building of GL(n) to study the cohomology of p-adic symmetric space. It is natural to ask for these results for any semisimple group. First we observe that the p-adic symmetric space of Drinfeld is the variety of points in the variety of Borel subgroups which are stable under the action of all maximal tori. The other point is that the map from the p-adic symmetric space to the building is just the interval of stability map. Everything make sense for any semisimple group except that in general the linearization used in the definition of stable points may result in the variety of stable points becomes smaller than the variety of semi-stable points. When this happens the interval of stability map will map a point in the p-adic space to a convex subset of the building. This phenomenon will almost always occur when the Borel subgroup is replaced by an arbitrary parabolic subgroup P of the semisimple group G. As a result it is not known how to prove in general even a result like Proposition 2.4 in Mumford [10]. Yet p-adic spaces constructed out of the flag varieties G/P could be interesting moduli space of periods (see Rapoport [13]). Also Moy [9] showed that the

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displacement function in a Bruhat-Tits building is convex and that convex sets in the Bruhat-Tits plays an important role in the representation theory of the group over the local field. It will be interesting if there is a relation between the representations of the group and the geometry of the variety of semi-stable points via the interval of stability map. In [5] Hsia Liang-Chung uses rigid geometry to study p-adic dynamic systems constructed out of a tree. It would be nice to have an analogue for a building. Thus we believe that it is worthwhile if only as experimental data to study the p-adic spaces when the variety of stable points is not the same as the variety of semi-stable points. The case we have chosen is the rank 2 group Sp_2 being the first case after $SL(2)$ and P is the maximal parabolic subgroup such that G/P is the projective 3-space. We point out that the calculations for other parabolics are also 'embedded' inside this case. We study in this paper those properties of the C_2 building which are related to the properties of the stable points in $\mathbb{P}^3$; in particular we shall use the $SL(2) \times SL(2)$ sub-building of the Sp_2 building to construct a pure affinoid covering of the p-adic space associated to G/P.

Let us describe the p-adic space we are studying. The maximal torus contained in the parabolic subgroup P acts on the flag variety G/P and we obtain the variety of stable points for this action as defined in Mumford [11]. Let $Y^s_{\mathscr{A}_0}$ denote the rigid analytic variety (see [1]) which has the same set of closed points as the variety of stable points above. The p-adic space we study here is $Y^s := \bigcap_{g \in G(F)} g \cdot Y^s_{\mathscr{A}_0}$. In the case when the stable points and the semistable points are the same these problems are studied by van der Put and Voskuil [12]. The case of quasi-split rank 1 group is studied by Voskuil [16]. This work started from a conversation with Voskuil in a cafe in Newtown. I would like to thank him for his generosity in sharing these ideas. Finally a raison d'être for Sp_2 is a response to Paul Sally's question: 'Do we know everything about Sp_2?' (Luminy Conference on Sp_2 1998) — I would like to thank him for his suggestion.

1. Buildings and flag varieties

In this section we give a summary of the general results on the p-adic spaces constructed out of the variety of stable points in flag varieties.

1.1. Let F be a p-adic field with ring of integers $\mathcal{O}$. Assume that p is odd. In this section we let G be an absolutely simple Chevalley group scheme over $\mathcal{O}$. Fix a maximal split torus T defined over F in G and choose a Borel subgroup B over F of G containing T. This fixes an ordering of the root system Φ of (G, T). Let $\mathscr{X}(T)$ (respectively $\mathscr{X}_*(T)$) be the lattice of characters (respectively one parameter subgroups) of T. Denote by $\langle \cdot, \cdot \rangle$ the perfect pairing between $\mathscr{X}(T)$ and $\mathscr{X}_*(T)$. Extend this to a pairing of $\mathscr{X}(T) \otimes \mathbb{R}$ and $\mathscr{X}_*(T) \otimes \mathbb{R}$.

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Fix a uniformiser π of F. Normalise the additive valuation v of F by $v(\pi) = 1$. Define the map $\nu : T \to \mathscr{X}_*(T) \otimes \mathbb{R}$ by $\langle \chi, \nu(t) \rangle = -v(\chi(t))$ for all $\chi \in \mathscr{X}(T)$. We regard $\mathscr{X}_*(T) \otimes \mathbb{R}$ as an affine space on which T acts by translation: $t \cdot z = z + \nu(t)$. The affine roots of (G, T) are the following affine functions on $\mathscr{X}_*(T) \otimes \mathbb{R}$: $(\alpha + n)(z) = \langle \alpha, z \rangle + n$ for $\alpha \in \Phi$. Denote the affine root system by Φ_{aff}. The affine root system gives a simplicial decomposition of $\mathscr{X}_*(T) \otimes \mathbb{R}$. The maximal simplices, called alcoves, are the closures of the connected components of the complement of the walls: $\beta(z) = 0$ for $\beta \in \Phi_{\text{aff}}$. The affine space $\mathscr{X}_*(T) \otimes \mathbb{R}$ endowed with this simplicial decomposition is called the apartment $\mathscr{A}$ attached to the torus T. The stabiliser G_σ in $G(F)$ of a simplex σ in the apartment $\mathscr{A}$ is a parabolic subgroup. All the maximal F-tori of G are conjugate. For $g \in G(F)$, the apartment attached to the torus gTg^{-1} is $g\mathscr{A}$ and the stabiliser of the simplex $g(\sigma)$ is $G_{g(\sigma)} = gG_\sigma g^{-1}$.

The Bruhat-Tits building $\mathscr{B}$ of G is defined to be $\bigcup_{g \in G(F)} g(\mathscr{A})/\sim$ where the equivalence relation $\sim$ is given by $\sigma_1 \sim \sigma_2$ if and only if $G_{\sigma_1} = G_{\sigma_2}$ [15, 2.1].

1.2. Fix a parabolic subgroup P defined over F of G containing the chosen Borel subgroup B. Write X for G/P. Let $\mathscr{L}$ be an ample line bundle of X. Choose a G-linearization of $\mathscr{L}$. This restricts to a T-linearization and we can define the variety of stable points $X^s(T, \mathscr{L})$ and the variety of semi-stable points $X^{ss}(T, \mathscr{L})$ with respect to this T-linearization of $\mathscr{L}$. In practice this is what we do. For a positive weight λ, there exists a G-module V_λ with highest weight λ. In V_λ there is a highest weight vector v_λ on which the maximal torus T acts with character λ. The G-orbit of the image of v_λ in the projective space $\mathbb{P}(V_\lambda)$ is isomorphic to the flag variety $X = G/P_\lambda$. The pullback of $\mathscr{O}(1)$ along the embedding $X \subset \mathbb{P}(V_\lambda)$ gives a line bundle $\mathscr{L}$ on X which has the G action induced by the the G action on V_λ. (See [6, 7].) This gives a G-linearization of $\mathscr{L}$. It induces a T-linearization of $\mathscr{L}$. Thus we can define the variety X^s (respectively X^{ss}) of stable (respectively semi-stable) points for the action of the torus T with respect to $\mathscr{L}$. Recall that a point x in X is said to be semi-stable with respect to $(T, \mathscr{L})$ if for some positive integer n there exists a T-invariant section f of $\mathscr{L}^{\otimes n}$ such that $f(x) \neq 0$ and the set of $y \in X$ such that $f(y) \neq 0$ is affine. A semi-stable point is said to be stable if moreover the set $y \in X$ such that $f(y) \neq 0$ is closed (see [11, Chapter 1.4]).

In our situation we have a simple criterion for stability. We can decompose

$$V_\lambda = \bigoplus_{\chi \in \mathscr{X}(T)} V_{\lambda,\chi}.$$

For x in V_λ, let us write x_χ for its component in $V_{\lambda,\chi}$. Let $\mu(x)$ denote the convex hull in $\mathscr{X}(T) \otimes \mathbb{R}$ of the set of χ such that $x_\chi \neq 0$. Then for any x in X the vertices of $\mu(x)$ is a subset of the W-orbit of λ and the edges of $\mu(x)$ are parallel to the roots (see [3]). The point x is semi-stable (respectively stable) if and only if 0 lies in $\mu(x)$

(respectively in the interior of $\mu(x)$). It is also known that $X^s = X^{ss}$ if and only if λ is not contained in a hyperplane through 0 spanned by roots [12, Theorem 1.1].

1.3. Let $\mathbb{C}$ denote a fixed completion of an algebraic closure of F. Write $\mathcal{O}_{\mathbb{C}}$ for the ring of integers of $\mathbb{C}$. Given an algebraic variety over F we can construct a rigid analytic variety which has the same set of closed points [1, 9.3.4]. We denote the analytification of $X^s(T, \mathcal{L}) \otimes \mathbb{C}$ and of $X^{ss}(T, \mathcal{L}) \otimes \mathbb{C}$ by $Y^s_{\mathcal{A}}$ and $Y^{ss}_{\mathcal{A}}$. We recall that $\mathcal{A}$ denotes the apartment attached to the torus T. Let

$$Y^{ss} := \bigcap_{g \in G(F)} g(Y^{ss}_{\mathcal{A}}) \quad \text{and} \quad Y^s := \bigcap_{g \in G(F)} g(Y^s_{\mathcal{A}}).$$

These are the rigid analytic flag varieties we study in this paper. For $G = SL(2)$ with the natural action on $X = \mathbb{P}^1$ the space Y^s is the Drinfeld upper half space.

We are interested in pure affinoid coverings of our rigid analytic spaces. Let Z be a rigid analytic space. A pure covering $\mathcal{U} = \{U_i\}$ of Z is an admissible covering by affinoid subspaces U_i satisfying the following conditions:

(1) For each i, U_i intersects a finite number of U_j.

(2) If $U_i \cap U_j \neq \emptyset$ then there exists a Zariski open affine set $V_{ij} \subset \overline{U}_j$ such that $U_i \cap U_j = R_i^{-1}(V_{ij})$ where $R_i : U_i \to \overline{U}_i$ is the reduction map [1, 7.1] and $U_i \cap U_j$ is an affinoid space having reduction $R_{ij} : U_i \cap U_j \to V_{ij}$.

To have a pure covering means that we can see that the reductions of the affinoids in the covering glue together nicely. There is a 1-1 correspondence between pure covering of Z and formal schemes over $\mathcal{O}$ whose generic fibre is Z and whose closed fibre is the reduction of Z with respect to the given pure covering (see [8]).

1.4. The completion of $X^s(T, \mathcal{L}) \otimes \mathcal{O}_{\mathbb{C}}$ (respectively $X^{ss}(T, \mathcal{L}) \otimes \mathcal{O}_{\mathbb{C}}$) along the closed fibre will be denoted by $Y^s_{\mathcal{A},\mathcal{O}}$ (respectively $Y^{ss}_{\mathcal{A},\mathcal{O}}$). In particular, this means $Y^s_{\mathcal{A},\mathcal{O}}(\mathbb{C}) = X^s(T, \mathcal{L})(\mathcal{O}_{\mathbb{C}})$, $Y^{ss}_{\mathcal{A},\mathcal{O}}(\mathbb{C}) = X^{ss}(T, \mathcal{L})(\mathcal{O}_{\mathbb{C}})$.

Consider the maps $T(\mathbb{C}) \times Y^{ss}_{\mathcal{A},\mathcal{O}} \to Y^{ss}_{\mathcal{A}}$, and $T(\mathbb{C}) \times Y^s_{\mathcal{A},\mathcal{O}} \to Y^s_{\mathcal{A}}$ both defined by the action of the torus T on X. We shall construct an affinoid covering of Y^s by means of these maps and a natural affinoid covering of the analytic space $T \otimes F$ associated to the torus T. It is here the building of G enters the picture. The map ν extends uniquely to a map from $T(\mathbb{C})$ to $\mathcal{A}$. This defines the action of $T(\mathbb{C})$ on $\mathcal{A}$. For a simplex σ of the apartment $\mathcal{A}$, let T_σ denote the affinoid subspace of $T \otimes F$ given by the affinoid algebra [1, 6.1]: $F\langle \pi^n \chi \rangle$ where $\chi \in \mathcal{X}(T)$, $n \in \mathbb{Z}$, and $\chi + n \geq 0$ on σ. For the standard alcove σ_0 in $\mathcal{A}$ this affinoid algebra is $F\langle \alpha_1, \dots, \alpha_n, \pi\alpha_0^{-1} \rangle$, where $\alpha_1, \dots, \alpha_n$ is the basis of simple roots of $\Phi(G, T, B)$ and α_0 is the highest root. We see immediately that:

(1) $T_\sigma = \nu^{-1}(\sigma)$.

(2) The set of $\{T_\sigma\}$ for all simplices σ of A is an admissible covering of $T \otimes F$ by affinoids.

(3) If σ_1, σ_2 are two simplices then $T_{\sigma_1} \cap T_{\sigma_2}$ is empty if $\sigma_1 \cap \sigma_2 = \emptyset$ and is equal to $T_{\sigma_1 \cap \sigma_2}$ otherwise.

This leads us to introduce the analytic set $Y^s_{\mathscr{A},\sigma} := T_\sigma \cdot Y^s_{\mathscr{A},\mathscr{O}}$ and $Y^{ss}_{\mathscr{A},\sigma} := T_\sigma \cdot Y^{ss}_{\mathscr{A},\mathscr{O}}$ for $\sigma \in \mathscr{A}$. We shall study the covering by these sets [12, 3.3 and page 84].

1.5. We need the two maps r, I introduced by Voskuil.

The map r is used to compare the analytic sets coming from different apartments and different simplices. It is the ratio of the maximum absolute values of the torus invariants. Recall that $\mathscr{L}$ is the ample line bundle on $X = G/P$. Write $\Gamma(X, \mathscr{L}^{\otimes m})^T$ for the module of T-invariant sections. Pick an integer d such that the homogeneous T-invariants generate $\oplus_{n>0}\Gamma(X, \mathscr{L}^{\otimes dn})^T$ as a $\mathscr{O}$-algebra. Let $f_1, \dots, f_m$ be generators of $\Gamma(X, \mathscr{L}^{\otimes d})^T$. For two different apartments $\mathscr{A}_1$, $\mathscr{A}_2$, we define a function $r^{ss}_{\mathscr{A}_1,\mathscr{A}_2} : Y^{ss}_{\mathscr{A}_2} \to \mathbb{R}$ as follows: pick $g_1, g_2 \in G(F)$ so as to have $\mathscr{A}_1 = g_1(\mathscr{A})$, $\mathscr{A}_2 = g_2(\mathscr{A})$. Then as in [12, page 86] we put

$$r^{ss}_{\mathscr{A}_1,\mathscr{A}_2}(x) := \frac{\max_{1\le i\le m}\{|g_1^* f_i(x)|\}}{\max_{1\le i\le m}\{|g_2^* f_i(x)|\}}.$$

Here $g^*f(x)$ is $f(g^{-1}x)$. The value of $r^{ss}_{\mathscr{A}_1,\mathscr{A}_2}(x)$ only depends on the apartments $\mathscr{A}_1$, $\mathscr{A}_2$. The function r has the following properties:

(1) $r^{ss}_{g\mathscr{A}_1,g\mathscr{A}_2}(gx) = r^{ss}_{\mathscr{A}_1,\mathscr{A}_2}(x)$ for $g \in G(F)$ and $x \in Y^{ss}_{\mathscr{A}_2}$.

(2) $r^{ss}_{\mathscr{A}_1,\mathscr{A}_3} = r^{ss}_{\mathscr{A}_1,\mathscr{A}_2} r^{ss}_{\mathscr{A}_2,\mathscr{A}_3}$.

(3) If $\mathscr{A}_1$, $\mathscr{A}_2$ are apartments containing a simplex σ_1 and $x \in Y^{ss}_{\mathscr{A}_1,\sigma_1}$ then $r^{ss}_{\mathscr{A}_1,\mathscr{A}_2}(x) \ge 1$ [12, page 86 (c)].

(4) If $\sigma_0 \subset \mathscr{A}_0$, $\sigma_1 \subset \mathscr{A}_1$, $\mathscr{A}_2$ contains σ_0, σ_1 and $x \in Y^{ss}_{\mathscr{A}_0,\sigma_0} \cap Y^{ss}_{\mathscr{A}_1,\sigma_1}$ then $r^{ss}_{\mathscr{A}_2,\mathscr{A}_0}(x) \le r^{ss}_{\mathscr{A}_1,\mathscr{A}_0}(x)$ (from (2) and (3) above).

Now we can introduce

$$r^{ss}(x) = \begin{cases} \inf\{r^{ss}_{g\mathscr{A},\mathscr{A}}(x) : g \in G(F)\} & \text{if } x \in Y^{ss}_{\mathscr{A}}, \\ 0 & \text{if } x \notin Y^{ss}_{\mathscr{A}}. \end{cases}$$

1.6. The map I is a $G(F)$-invariant map from the variety Y^{ss} of semi-stable points to the set of convex subsets of the building $\mathscr{B}$ of G and $I(x)$ will be bounded if and only if x is stable. Recall the map $\nu : T(\mathbb{C}) \to \mathscr{A}$ which defines the action of $T(\mathbb{C})$ on $\mathscr{A}$. Let $0 \in \mathscr{A}_0$ be the vertex where the affine roots $\alpha_1, \dots, \alpha_n$ take the value 0. For $x \in Y^{ss}_{\mathscr{A}}$, the interval of T-stability $I_{\mathscr{A}}(x)$ is defined as the closure of the set $\{t \cdot 0 \in \mathscr{A} : x \in t \cdot Y^{ss}_{\mathscr{A},O}\}$, where t runs through points of T in the algebraic closure of F. (See [16, 2.3]) We put $I_{g\mathscr{A}}(x) = g(I_{\mathscr{A}}(g^{-1}x))$. For $x \in Y^{ss}$, we define the interval of G-stability $I(x)$ to be the set of all $z \in \mathscr{B}$ such that for any apartment $\mathscr{A}'$

containing z we have $z \in I_{\mathscr{A}'}(x)$ [16, 4.7]. We recall the following properties of I. Assume that $x \in Y^{ss}$.

(1) $I(x) = \cup\{I_{\mathscr{A}'}(x) : r^{ss}_{\mathscr{A}',\mathscr{A}}(x) = r^{ss}(x)\}$.

(2) $I(x)$ is convex; it is bounded if and only if $x \in Y^s$ and $I(x) = \{t \cdot 0\}$ if and only if $x \in t \cdot Y^s_{\mathscr{A}',\mathscr{O}}$ for all apartments $\mathscr{A}'$ containing $t \cdot 0$.

When $Y^{ss} = Y^s$ the interval of stability $\mathscr{I}$ defines a map from the analytic space Y^s to the building $\mathscr{B}$ and this is the map used by Drinfeld and Schneider-Stuhler (see [4, 14, Section 1]).

2. Action of Sp_2 on $\mathbb{P}^3$

Let G be the symplectic group over $\mathscr{O}$ defined by the form

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The group $G(F)$ of F rational points consists of 4×4 matrices g with coefficients in F such that ${}^t g J g = J$, where ${}^t g$ denotes the transposed matrix of g. We choose a maximal torus T_0 over $\mathscr{O}$ so that $T_0(F)$ consists of matrices

$$t = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}$$

for $t_1, t_2 \in F^\times$. We choose the Borel subgroup B of G to be the upper triangular matrices in G. This fixes a basis $\{\alpha_1, \alpha_2\}$ of the root system of (G, T_0) which is of type C_2. In standard notation the root system C_2 is $\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2$. It has simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = 2e_2$. The highest root is $\alpha_0 = 2\alpha_1 + \alpha_2$.

The fundamental weights are $\omega_1 = e_1, \omega_2 = e_1 + e_2$. Consider positive weights of the form $\lambda = n_1\omega_1 + n_2\omega_2$ with positive integers n_1, n_2. Let W be the Weyl group of this root system and W_λ be the stabilizer of the weight λ. Associated to λ is the parabolic subgroup $P_\lambda = B W_\lambda B$ of G. Let us write P for P_{λ_1}. Then

$$P(F) = \left\{ \begin{pmatrix} a & * & * \\ 0 & g & * \\ 0 & 0 & a^{-1} \end{pmatrix} : a \in F^\times, g \in SL(2, F) \right\}.$$

Let v_1, v_2, v_3, v_4 be the standard basis of F^4. Write $[u, v, \ldots]$ for the subspace spanne by the vectors $u, v, \ldots$. The group P is the stabilizer of the isotropic line $[v_1]$, so P i

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the intersection with G of the stabilizer in $SL(4)$ of the flag $[v_1] \subset [v_1]^\perp = [v_1, v_2, v_3]$. We see that G/P is isomorphic to the set of isotropic lines in the 4-dimensional affine space. But every line is isotropic. So $G/P \cong \mathbb{P}^3$, the projective 3-space.

3. C_2 buildings

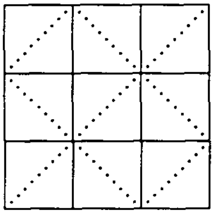
In the Bruhat-Tits building $\mathscr{B}$ of the group $G = Sp_2$ associated to a maximal torus T is an apartment $\mathscr{A}$ in $\mathscr{B}$. For each simplex $\sigma \in \mathscr{A}$ we have defined the affinoid subspace $T_\sigma \subset T$.

For the standard alcove σ_0 defined by $\alpha_1, \alpha_2, 1 - \alpha_0$, we have

$$T_{\sigma_0} = Sp\left(k\langle \alpha_1, \alpha_2, \pi\alpha_0^{-1}\rangle\right) = Sp\left(k\langle t_1 t_2^{-1}, t_2^2, \pi t_1^{-2}\rangle\right).$$

For each maximal F-torus T in G, we can find a subgroup H of G such that T lies in H and H is defined over F and is isomorphic to $SL_2 \times SL_2$ over F. For $i = 1, 2, 3, 4$, let us write $U_i(F)$ for the subgroup consisting of transformations taking x_j to x_j if $j \neq i$ and $x_i \to x_i + u_i x_{5-i}$ with $u_i \in F$. For T_0, the subgroup H is generated by the groups $U_i(F)$ for $i = 1, 2, 3, 4$.

Let $\mathscr{I}$ denote the building of the group $H(F) \cong SL_2(F) \times SL_2(F)$. The inclusions of groups $T(F) \subset H(F) \subset G(F)$ gives rise to inclusions of simplicial complexes $\mathscr{A} \subset \mathscr{I} \subset \mathscr{B}$. To make these inclusions simplicial one has to split each $SL_2 \times SL_2$ chamber in two $Sp(4)$ chambers (see picture below). We will always assume that the simplicial structures of the $SL_2 \times SL_2$ buildings are arranged in this way.



In the picture the dotted lines indicate walls occurring only in the $Sp(4)$ building and solid lines indicate walls in the $SL_2 \times SL_2$ building.

We give a description of the stabilizer G_{σ_0} in $G(F)$ of the standard alcove σ_0. Let U_α be the root subgroup of G with respect to the maximal torus T_0 corresponding to the root α. Then $U_\alpha(F) \cong F$ and $U_{n+\alpha}(F) = \{x \in U_\alpha(F) \cong F : v(x) \geq n\}$. It is known that G_{σ_0} is generated by $U_{n+\alpha}$ for those $n + \alpha \geq 0$ on σ_0 (see [15, 3.1.1]). It follows that for $g \in G(F)$ with matrix (g_{ij}), we have $g \in G_{\sigma_0}$ if and only if $v(g_{ij}) \geq v(m_{ij})$, where

$$(m_{ij}) = \begin{pmatrix} 1 & \pi & \pi & \pi \\ 1 & 1 & \pi & \pi \\ 1 & 1 & 1 & \pi \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Thus G_{σ_0} stabilizes the following $\mathcal{O}$-submodules of F^4: $M_0 = \langle v_1, v_2, v_3, v_4\rangle$, $M_1 = \langle \pi v_1, \pi_2, v_3, v_4\rangle$, $M_2 = \langle \pi v_1, v_2, v_3, v_4\rangle$ and $M_2^\vee = \langle v_1, v_2, v_3, \pi^{-1} v_4\rangle$.

4. Stable points

For ease of reading we introduce here all the analytic sets we shall use.

4.1. Stable points related to an apartment Write $\mathscr{A}_0$ for the apartment attached to the torus T_0. We write the coordinates of $\mathbb{P}^3$ as x_1, x_2, x_3, x_4. The T_0-invariants are generated by x_1x_4 and x_2x_3. It follows that

$$\begin{aligned} Y^{ss}_{\mathscr{A}_0} &= \{x \in \mathbb{P}^3 : x_1x_4 \neq 0 \quad \text{or} \quad x_2x_3 \neq 0\} \\ Y^{s}_{\mathscr{A}_0} &= \{x \in \mathbb{P}^3 : x_i \neq 0, i = 1 \cdots 4\}. \end{aligned}$$

For any simplex σ in an apartment $\mathscr{A}$ of the building $\mathscr{B}$ of the group G, we can find an element $g \in G(F)$ such that $\sigma = g(\sigma_0)$ and $\mathscr{A} = g(\mathscr{A}_0)$. We put $Y^s_{\mathscr{A}} = gY^s_{\mathscr{A}_0}$. This definition does not depend on the choice of the element g. As in general we put $Y^{ss} = \bigcap_{g\in G(F)} g \cdot Y^{ss}_{\mathscr{A}_0}$ and $Y^s = \bigcap_{g\in G(F)} g \cdot Y^s_{\mathscr{A}_0}$.

We define the following analytical subspace of $Y^s_{\mathscr{A}_0}$:

$$Y^s_{\mathscr{A}_0,\mathcal{O}} := \left\{x \in \mathbb{P}^3 : \left|\frac{x_1}{x_4}(x)\right| = 1, \left|\frac{x_2}{x_3}(x)\right| = 1\right\}.$$

The space $Y^s_{\mathscr{A}_0,\mathcal{O}}$ is not affinoid. Let $Y^s_{\mathscr{A}_0,\mathcal{O},n}$ be the set of $x \in Y^s_{\mathscr{A}_0,\mathcal{O}}$ such that $|\pi^{n+1}| \le |x_1x_4/x_2x_3| \le |\pi^n|$.

Let $0 \in \mathscr{A}_0$ be the vertex where the affine roots α_1 and α_2 take the value 0. As $G(F)$ acts on both the flag variety and the building we can put $Y^s_{\mathscr{A}_0,t\cdot 0} = t \cdot Y^s_{\mathscr{A}_0,\mathcal{O}}$.

The map $T \times Y^s_{\mathscr{A}_0,\mathcal{O}} \to Y^s_{\mathscr{A}_0}$ given by $(t, x) \mapsto t \cdot x$ is surjective. So the analytic sets $Y^s_{\mathscr{A}_0,\sigma} := T_\sigma \cdot Y^s_{\mathscr{A}_0,\mathcal{O}}$ cover $Y^s_{\mathscr{A}_0}$ for $\sigma \in \mathscr{A}_0$. We have for $\sigma = \sigma_0$

$$\begin{aligned} Y^s_{\mathscr{A}_0,\sigma_0} &= \{t \cdot x : t \in T_{\sigma_0}, x \in Y^s_{\mathscr{A}_0,\mathcal{O}}\} \\ &= \left\{x \in \mathbb{P}^3 : |\pi| \le \left|\frac{x_1}{x_4}(x)\right| \le \left|\frac{x_2}{x_3}(x)\right| \le 1\right\}. \end{aligned}$$

We put $Y^s_{\mathscr{A}_0,\sigma,n} := T_\sigma \cdot Y^s_{\mathscr{A}_0,\sigma,n}$. For any $\sigma \in \mathscr{A} \subset \mathscr{B}$, we can find an element $g \in G(F)$ such that $\sigma = g(\sigma_0)$ and $g\mathscr{A} = g(\mathscr{A}_0)$. We now define: $Y^s_{\mathscr{A},\sigma,n} := g(Y^s_{\mathscr{A}_0,\sigma_0,n})$. This definition does not depend on the choice of the element g.

We put $Y^*_{\mathscr{A}_0,\sigma_0}$ for the set of $x \in Y^s_{\mathscr{A}_0,\sigma_0}$ such that $|g^*x_i/x_i| = 1$, $|g^*x_{5-i}/x_{5-i}| = 1$, $\forall g \in G_{\sigma_0}$, and $i = 1$ if $1 \le |x_1x_4/x_2x_3|$, $i = 2$ if $|x_1x_4/x_2x_3| \le 1$. We write $Y^*_{\mathscr{A}_0,\sigma,n}$ for the set of $x \in Y^s_{\mathscr{A}_0,\sigma,n}$ such that $|g^*x_i/x_i| = 1$, $|g^*x_{5-i}/x_{5-i}| = 1$, $\forall g \in G_\sigma$, $i = 1$ if $n \le -1$ and $i = 2$ if $n \ge 0$.

4.2. Stable points related to a sub-building Suppose H_0 is the subgroup of G containing the torus T_0 and is isomorphic to $SL_2 \times SL_2$ over F, let $\mathscr{I}_0$ denote the building of the group $H_0(F)$ and we put $Y^s_{\mathscr{I}_0} = \bigcap_{h \in H_0(F)} hY^s_{\mathscr{A}_0}$. This is the set of x in $\mathbb{P}^3$ which is stable for each maximal F torus in H_0. If $\mathscr{I} = g(\mathscr{I}_0)$ we put $Y^s_{\mathscr{I}} = gY^s_{\mathscr{I}_0}$. If $\sigma \in \mathscr{A} \subset \mathscr{I}$ then we take: $Y^s_{\mathscr{I},\sigma,n} := \bigcap_{h \in H_\sigma} hY^s_{\mathscr{A},\sigma,n} = \bigcap_{\substack{\mathscr{A}' \ni \sigma \\ \mathscr{A}' \subset \mathscr{I}}} Y^s_{\mathscr{A}',\sigma,n}$. Here $H = gH_0g^{-1}$ and H_σ is the stabilizer of the simplex σ in $H(F)$ and $\mathscr{I}$ is the building belonging to H. We have

$$Y^s_{\mathscr{I}_0,\sigma,n} = \left\{x \in Y^s_{\mathscr{A}_0,\sigma,n} : |g^*x_i/x_i| = 1,\ \forall g \in H_\sigma,\ i = 1, \dots, 4\right\}.$$

We define the affinoid subspace $Y^*_{\mathscr{I}_0,\sigma_0,n} \subset Y^s_{\mathscr{I}_0,\sigma_0,n}$ as follows:

$$Y^*_{\mathscr{I}_0,\sigma_0,n} := \begin{cases} \left\{x \in Y^s_{\mathscr{I}_0,\sigma_0,n} : |g^*x_1/x_1| = |g^*x_4/x_4| = 1, \forall g \in P_{\sigma_0}\right\} & n \le -1; \\ \left\{x \in Y^s_{\mathscr{I}_0,\sigma_0,n} : |g^*x_2/x_2| = |g^*x_3/x_3| = 1, \forall g \in P_{\sigma_0}\right\} & n \ge 0. \end{cases}$$

Furthermore, $Y^*_{\mathscr{I},\sigma,n} \subset Y^s_{\mathscr{I},\sigma,n}$ is defined as being $g(Y^n_{\sigma_0,H_0})$ where $g \in G(F)$ is such that $g(\mathscr{I}_0) = \mathscr{I}$ and $g(\sigma_0) = \sigma$. This definition does not depend on the choice of g.

4.3. Remarks We have a $SL_2 \times SL_2$-equivariant map $\psi : Y^s_{\mathscr{A}_0} \to \mathbb{P}^1 \times \mathbb{P}^1$ given by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_4) \times (x_2, x_3)$. It is clear that $Y^s_{\mathscr{I}_0} := \bigcap_{g \in SL_2(F) \times SL_2(F)} gY^s_{\mathscr{A}_0} = \psi^{-1}(\Omega_1 \times \Omega_1)$. Here $\Omega_1 \subset \mathbb{P}^1$ is Drinfeld symmetric space. Let Φ denote the closure of the graph of ψ in $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore we take $\Phi_{\mathscr{I}_0} := \Phi \cap (\mathbb{P}^3 \times \Omega_1 \times \Omega_1)$. The group $SL_2(F) \times SL_2(F)$ acts on $\Phi_{\mathscr{I}_0}$. We take on $\Phi \subset \mathbb{P}^3 \times \mathbb{P}^1$ the coordinates $(x_1, x_2, x_3, x_4) \times (z_1, z_4) \times (z_2, z_3)$ with $z_1x_4 = z_4x_1$ and $z_2x_3 = z_3x_4$. A pure affined covering of $\Omega_1 \times \Omega_1$ is found by taking the $SL_2(F) \times SL_2(F)$-images of the following affinoid subspace $\mathbb{F}$ given as the set of $z \in \mathbb{P}^1 \times \mathbb{P}^1$ satisfying the following conditions: $|\pi| \le |z_1/z_4| \le 1$, $|\pi| \le |z_2/z_3| \le 1$, $|z_1/z_4 - c| = |z_2/z_3 - c| = 1$, $|\pi z_4/z_1 - c| = |\pi z_3/z_2 - c| = 1$ for all units c in F (see [2, page 172]). The space $\Phi \cap (\mathbb{P}^3 \times \mathbb{F})$ is not affinoid, but it can be covered by the following two affinoid spaces:

$$\mathbb{F}^+ := \left\{x \in \Phi \cap (\mathbb{P}^3 \times \mathbb{F}) : \left|\frac{x_1x_4}{x_2x_3}\right| \le 1\right\},$$

$$\mathbb{F}^- := \left\{x \in \Phi \cap (\mathbb{P}^3 \times \mathbb{F}) : \left|\frac{x_1x_4}{x_2x_3}\right| \ge 1\right\}.$$

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The covering $\mathbb{F} := \{g(\mathbb{F}^+), g(\mathbb{F}^-) : g \in SL_2(F) \times SL_2(F)\}$ is pure and covers all of $\Phi_{\mathscr{I}_0}$. Since the components of the reduction of $\Omega_1 \times \Omega_1$ with respect to the covering are all proper, the reduction of $\Phi_{\mathscr{I}_0}/\Gamma$ of $\Phi_{\mathscr{I}_0}$ by a discrete co-compact subgroup $\Gamma \subset SL_2(F) \times SL_2(F)$ is a proper analytical variety. It is in fact algebraic. One can embed $Y^s_{\mathscr{I}_0} \hookrightarrow \Phi_{\mathscr{I}_0}$. So one can view $\Phi_{\mathscr{I}_0}/\Gamma$ as a compactification of $Y^s_{\mathscr{I}_0}/\Gamma$. See Voskuil [16]. Note that using the embedding above one has

$$(\mathbb{F}^+ \cup \mathbb{F}^-) \cap Y^s_{\mathscr{I}_0} = \bigcup_{n \in \mathbb{Z}} (Y^s_{\mathscr{I}_{0,\sigma_0,n}} \cup Y^s_{\mathscr{I}_{0,\sigma_1,n}}).$$

Here σ_0 is the standard alcove defined by the affine roots α_1, α_2 and $1 - \alpha_0$. The alcove σ_1 is determined by the affine roots $-\alpha_1$, $1 - \alpha_2$ and α_0. Note that $\sigma_1 \cup \sigma_0$ forms an $SL_2 \times SL_2$-chamber in the building of $SL_2(F) \times SL_2(F)$. One has $T_{\sigma_1} = Sp(F\langle t_2 t_1^{-1}, \pi t_2^{-2}, t_1^2\rangle)$ and $T_{(\sigma_1 \cup \sigma_2)} := T_{\sigma_1} \cup T_{\sigma_2} = Sp(F\langle \pi t_1^{-2}, t_1^2, \pi t_2^{-2}, t_2^2\rangle)$.

5. Convex sets in C_2 buildings

To each semi-stable point we assign a convex subset of the building, namely its interval of stability. We use the explicit torus invariants to give an explicit description of the intervals of stability.

5.1. Let us recall that the interval of stability is defined for x in $Y^{ss}_{\mathscr{A}_0}$ as $I_{\mathscr{A}_0}(x) = \{t^{-1} \cdot 0 : t \cdot x \in Y^{ss}_{\mathscr{A}_0, \mathscr{O}}\}$. For the $SL_2 \times SL_2$ sub-building $\mathscr{I}_0$ introduced in Section 4.2, we define for $x \in Y^s_{\mathscr{I}_0,\sigma}$ the set $I_{\mathscr{I}_0}(x)$ to be the union of $I_{\mathscr{A}}(x)$ over those $\mathscr{A}$ in $\mathscr{I}_0$ such that $\mathscr{A} \ni \sigma$. Then $I_{\mathscr{I}_0}(x) = \bigcup_{g \in H_\sigma} g \cdot I_{\mathscr{A}}(x)$.

5.2. We introduce the centre of the interval of stability. Let $0 \in \mathscr{A}_0$ be the vertex where the affine roots α_1 and α_2 take the value 0. We define a T_0 equivalent map $v_{\mathscr{A}_0} : Y^s_{\mathscr{A}_0} \to \mathscr{A}_0$ by

$$\begin{cases} v_{\mathscr{A}_0}(x) = 0 & \text{if } x \in Y^s_{\mathscr{A}_0,\mathscr{O}}; \\ v_{\mathscr{A}_0}(t \cdot x) = t \cdot 0 & \text{otherwise.} \end{cases}$$

Note that $(t^* x_1 / t^* x_4)(x) = t_1^{-2}(x_1/x_4)(x) = -(2\alpha_1 + \alpha_2)(t) \cdot (x_1/x_4)(x)$ and that $(t^* x_2 / t^* x_3)(x) = t_2^{-2}(x_2/x_3)(x) = -\alpha_2(t)(x_2/x_3)(x)$. Let v be the additive valuation of F normalised such that $v(\pi) = 1$, where π is a uniformiser. For $x \in Y^s_{\mathscr{A}_0,\mathscr{O}}$, one has $v((x_1/x_4)(x)) = 0 = (-(2\alpha_1 + \alpha_2))(0)$ and $v((x_2/x_3)(0)) = 0 = -\alpha_2(0)$. From this and the description of the action of the torus on the apartment given in [15, 1.1], one easily gets the following description of $v_{\mathscr{A}_0}$:

$$\begin{cases} (2\alpha_1 + \alpha_2)(v_{\mathscr{A}_0}(x)) = -v((x_1/x_4)(x)); \\ \alpha_2(v_{\mathscr{A}_0}(x)) = -v((x_2/x_3)(x)). \end{cases}$$

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We like to remark that $Y^s_{\mathscr{A}_0,\sigma} = v^{-1}_{\mathscr{A}_0}(\sigma)$ for $\sigma \in \mathscr{A}_0$.

5.3. We describe the interval of stability in terms of the action of the torus on the 'origin' 0. We have

$$\begin{aligned} Y^{ss}_{\mathscr{A}_0,\mathscr{O}} = &\{|x_1/x_4| = 1, |x_2x_3/x_1x_4| \le 1, |x_2/x_4| \le 1, |x_3/x_4| \le 1\} \\ &\cup \{|x_2/x_3| = 1, |x_1x_4/x_2x_3| \le 1, |x_1/x_3| \le 1, |x_4/x_3| \le 1\}. \end{aligned}$$

Take x in $Y^{ss}_{\mathscr{A}_0}$ and let t be the diagonal matrix with entries t_1, t_2, t_2^{-1}, t_1^{-1}. Since $\alpha_1 = e_1 - e_2$ and $\alpha_2 = 2e_2$, we have $\alpha_1(t) = t_1t_2^{-1}$ and $\alpha_2(t) = t_2^2$. Note that $(t^*x_1/t^*x_4)(x) = t_1^{-2}(x_1/x_4)(x) = -(2\alpha_1+\alpha_2)(t)(x_1/x_4)(x)$ and that $(t^*x_2/t^*x_3)(x) = t_2^{-2}(x_2/x_3)(x) = -\alpha_2(t)(x_2/x_3)(x)$.

Suppose x and tx are in the part of $Y^{ss}_{0,\mathscr{A}_0}$ given by

$$\left\{\left|\frac{x_2}{x_3}\right| = 1, \left|\frac{x_1x_4}{x_2x_3}\right| \le 1, \left|\frac{x_1}{x_3}\right| \le 1, \left|\frac{x_4}{x_3}\right| \le 1\right\}.$$

Then

$$1 = \left|\frac{(tx)_2}{(tx)_3}\right| = \left|\frac{t_2x_2}{t_2^{-1}x_3}\right| = |t_2|^2.$$

From the definition of ν: $\langle\alpha, \nu(t)\rangle = -v(\alpha(t))$ we get from $|t_2|^2 = 1$ that $\nu(t) = -v(t_1)e_1$. The action of T on the apartment $\mathscr{A}$ is then $t^{-1}v = v + v(t_1)e_1$. Suppose x_1 and x_4 are not 0. Then from $|t_1x_1/t_2^{-1}x_3| = |(tx)_1/(tx)_3| \le 1$ we get $|t_1| \le |x_3/x_1|$ and from $|t_1^{-1}x_4/t_1^{-1}x_3| = |(tx)_4/(tx)_3| \le 1$ we get $|x_4/x_3| \le |t_1|$. Recall that

$$I_{\mathscr{A}_0}(x) = \left\{t^{-1}\cdot 0 : tx \in Y^{ss}_{\mathscr{A}_0,\mathscr{O}}\right\}.$$

We see that in this case $x_1, x_4 \ne 0$.

$$I_{\mathscr{A}_0}(x) = \left\{0 + ce_1 : \left|\frac{x_4}{x_3}\right| \le |c| \le \left|\frac{x_3}{x_1}\right|\right\},$$

and so $I_{\mathscr{A}_0}(x)$ is an interval. If $x_4 \ne 0$ and $x_1 = 0$, we see that $I_{\mathscr{A}_0}(x)$ is a half-line $0 + ce_1$ with $|x_4/x_3| \le |c|$ and if $x_1 = 0$ and $x_4 = 0$ then $I_{\mathscr{A}_0}(x)$ is the full line $0 + ce_1$ with $-\infty \le |c| \le \infty$.

Similarly suppose x and tx are in the part of $Y^{ss}_{0,\mathscr{A}_0}$ given by

$$\left\{\left|\frac{x_1}{x_4}\right| = 1, \left|\frac{x_2x_3}{x_1x_4}\right| \le 1, \left|\frac{x_2}{x_4}\right| \le 1, \left|\frac{x_3}{x_4}\right| \le 1\right\}.$$

Then in case $(x_1, x_4 \ne 0)$

$$I_{\mathscr{A}_0}(x) = \left\{0 + ce_2 : \left|\frac{x_2}{x_1}\right| \le |c| \le \left|\frac{x_4}{x_2}\right|\right\}.$$

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It is now clear that $I_{\mathscr{A}_0}(x)$ is the set of q in $\mathscr{A}_0$ such that:

$$\left|\alpha_2(q) + v\left(\frac{x_2}{x_3}(x)\right)\right| \le v(\epsilon), \quad (2\alpha_1+\alpha_2)(q) = -v\left(\frac{x_1}{x_4}(x)\right)$$

if $|\epsilon| \le 1$, and such that

$$\left|(2\alpha_1+\alpha_2)(q) + v\left(\frac{x_1}{x_4}(x)\right)\right| \le -v(\epsilon), \quad \alpha_2(q) = -v\left(\frac{x_2}{x_3}(x)\right),$$

if $|\epsilon| \ge 1$. Here $\epsilon \in \overline{F}$ is such that $|(x_2x_3/x_1x_4)(x)| = |\epsilon|$.

6. Some estimates on coordinates

In order that we can use the r maps to study the simplicial decomposition of the stable points we need some estimates on the absolute values of the coordinates given by the torus invariants of the stable points under the action of the group. This will be done in the next few lemmas.

LEMMA 6.1. *Suppose* $x \in Y^s_{\mathscr{A}_0,\sigma_0}$ *and that* $g \in G_{\sigma_0}$.

(a) *If* $|(x_1x_4/x_2x_3)(x)| = 1$, *then* $|(g^*x_i/x_i)(x)| \le 1$ *for* $i = 1, \dots, 4$.

(b) *If* $|(x_1x_4/x_2x_3)(x)| < 1$, *then* $|(g^*x_2/x_2)(x)| \le 1$, $|(g^*x_3/x_3)(x)| \le 1$ *and* $|(g^*x_1g^*x_4/x_2x_3)(x)| \le 1$.

(c) *If* $|(g^*x_i/x_i)(x)| \ge 1$, *for* $i = 1, \dots, 4$, *and that* $|(x_1x_4/x_2x_3)(x)| = |\epsilon^2|$ *with* $\epsilon \in \overline{F}$, *then*

$$|(g^*x_2/g^*x_3)(x)| = |(x_2/x_3)(x)|,$$
$$|(x_1/x_4)(x)|\,|\epsilon| \le |(g^*x_1/g^*x_4)(x)| \le |(x_1/x_4)(x)|\,|\epsilon^{-1}| \quad \text{if } |\epsilon| \le 1,$$
$$|(g^*x_1/g^*x_4)(x)| = |(x_1/x_4)(x)|,$$
$$|(x_3/x_3)(x)|\,|\epsilon^{-1}| \le |(g^*x_2/g^*x_3)(x)| \le |(x_2/x_3)(x)|\,|\epsilon| \quad \text{if } |\epsilon| \ge 1.$$

PROOF. For $g^{-1} \in G_{\sigma_0}$, write $x_i(g^{-1}(x)) = \sum g_{ij}x_j(x)$. One has

$$\left|\frac{g^*x_i}{x_i}(x)\right| = \left|\frac{\sum g_{ij}x_j(x)}{x_i(x)}\right| \le \max_j \left|\frac{g_{ij}x_j(x)}{x_i(x)}\right|.$$

It follows from the explicit description of G_{σ_0} that $\max_j |g_{ij}x_j(x)/x_i(x)| = 1$. From this part (a) follows. The proofs of the rest of this lemma and the next lemma are similar, we omit them. □

LEMMA 6.2. *Let* $x \in Y^*_{\mathscr{A},\sigma}$ *and let* $H \cong SL_2(F) \times SL_2(F)$ *be determined by* $\mathscr{A}$, *that is the building* $\mathscr{I}$ *of* H *contains* $\mathscr{A}$. *Then for all* $g \in H_\sigma$ *we have* $|g^*x_i\, x_i(x)| \le 1$, *for* $1 \le i \le 4$.

LEMMA 6.3. *Suppose* $x \in Y^*_{\mathscr{I}_{0,\sigma_0}}$ *and* $g \in G_{\sigma_0}$ *satisfies* $|g^*x_1g^*x_4/g^*x_2g^*x_3(x)|=1$. *Then* $x \in Y^s_{g\mathscr{I}_{0,\sigma_0}}$. *Furthermore* $|x_1/x_4(x)| = 1$, *if* $|x_2x_3/x_1x_4(x)|<1$ *and* $|x_2/x_3(x)| = |\pi|$, *if* $|x_2x_3/x_1x_4(x)| > 1$.

PROOF. Let $\epsilon \in \overline{F}$ be such that $|\epsilon^2| = |x_1x_4/x_2x_3(x)|$. Take $y = (x_1, \epsilon x_2, \epsilon x_3, x_4)$. Then $|x_1x_4/x_2x_3(y)| = 1$ and $y \in Y^s_{\mathscr{A}_{0,\sigma_0}}$. From the explicit description of G_{σ_0} we see that for all $g \in G_{\sigma_0}$ we have $\max_j\{|(g_{ij}x_j/x_i)(y)|\} = 1$, where $g^*x_i = \sum_j g_{ij}x_j$.

First we look at the case $|\epsilon| < 1$. Since $x \in Y^*_{\mathscr{I}_{0,\sigma_0}}$, we have $|(g^*x_2/x_2)(x)| = |(g^*x_3/x_3)(x)| = 1$. Write $g^*x_1 = \sum a_jx_j$ and $g^*x_4 = \sum b_jx_j$. We get

$$\max|a_jx_j/x_1(y)| = \max|b_jx_j/x_4(y)| = 1.$$

This means that the product of

$$\max\left\{|a_1\epsilon^{-1}x_1(x)|, |a_2x_2(x)|, |a_3x_3(x)|, |a_4\epsilon^{-1}x_4(x)|\right\}$$

and

$$\max\left\{|b_1\epsilon^{-1}x_1(x)|, |b_2x_2(x)|, |b_3x_3(x)|, |b_4\epsilon^{-1}x_4|\right\}$$

is equal to $|x_2x_3(x)|$.

Now take $g \in P_{\sigma_0}$ as in the assumption. Then $|g^*x_1g^*x_4/x_2x_3(x)| = 1$. Just as before we obtain $\max\{|a_ix_i(x)|\}\max\{|b_ix|\} = |x_2x_3(x)|$. Comparing these two values of $|x_2x_3(x)|$ and using $|\epsilon| < 1$ we see that

$$\begin{cases}\max\left\{|a_1\epsilon^{-1}x_1(x)|, |a_4\epsilon^{-1}x_4(x)|\right\} = \max\{|a_2x_2(x)|, |a_3x_3(x)|\}\\ \max\left\{|b_1\epsilon^{-1}x_1(x)|, |b_4\epsilon^{-1}x_4(x)|\right\} = \max\{|b_2x_2(x)|, |b_3x_3(x)|\}.\end{cases}$$

The description of G_{σ_0} shows that $|a_i| = 1$ and $|b_2| = |b_3| = |\pi|$. Hence

$$|x_2x_3(x)| = \max\{|x_2|, |x_3|\}\max\{|\pi x_2|, |\pi x_3|\}.$$

Since $x \in Y^*_{\mathscr{I}_{0,\sigma_0}}$ we have $|(x_2/x_3)(x)| \le 1$. Hence $|x_2x_3| = |x_3||\pi x_3|$, that is, $|x_2/x_3| = |\pi|$. Furthermore

$$|(g^*x_1/g^*x_4)(x)| = |(g^*x_2/g^*x_3)(x)| = |(x_3/x_2)(x)| = |\pi|,$$

hence $x \in Y^s_{g\mathscr{I}_{0,\sigma_0}}$. The proofs of other cases are similar. □

REMARK. The lemma and Figure 1 show that one has $|(x_1x_4/x_2x_3)(x)| \neq 1$ and $|(g^*x_1g^*x_4/g^*x_2g^*x_3)(x)| = 1$ for some $g \in G_{\sigma_0}$ only in the following cases: $(1, \epsilon a, \epsilon b, c)$ and $(\epsilon\pi, \pi a, b, \epsilon c)$ with $|\epsilon| < 1$, $|a| - |b| = |c| = 1$.

LEMMA 6.4. *Let* $\mathscr{A} = g\mathscr{A}_0$, $x \in gY^s_{\mathscr{A}_0}$, *and* $h \in G_{I_{\mathscr{A}}(x)}$. *Then* $|(h^*g^*x_i/g^*x_i)(x)| \le 1$.

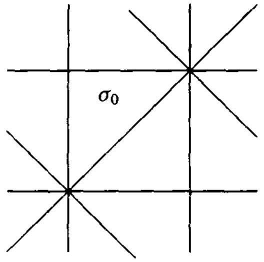

FIGURE 1.

PROOF. It is sufficient to proof this for $\mathscr{A} = \mathscr{A}_0$. We know that $G_{I_{\mathscr{A}_0}(x)}$ is spanned by $U_{n+\alpha}$ for those roots α such that $\alpha(q) \geq n$ holds for all $q \in I_{\mathscr{A}_0}(x)$. We only do the case $\alpha = \alpha_2$. The others are similar. We take $h \in U_{n+\alpha_2}$. We can write $h^*(x) = (x_1, x_2, x_3 - ax_2, x_4)$. It follows from the explicit description of $I_{\mathscr{A}_0}(x)$ that $v(a) \geq -v(x_2/x_3(x))$ and so $|(h^*x_3/x_3)(x)| \leq 1$. □

7. Applications of the r map

In this section we shall prove a number of lemmas explaining how we can use the r maps to study the partitioning of the stable points by the action of the torus. For example we shall show that $x \in Y^{ss}$ if and only if $r^{ss}(x) > 0$.

7.1. Using the torus invariants x_1x_4, x_2x_3 we define for $x \in Y^{ss}_{\mathscr{A}_0}$ and $g \in G(F)$

$$r^{ss}_{g\mathscr{A}_0, \mathscr{A}_0}(x) = \frac{\max\{|g^*(x_1x_4)(x)|, |g^*(x_2x_3)(x)|\}}{\max\{|(x_1x_4)(x)|, |(x_2x_3)(x)|\}}.$$

Recall we also have introduced

$$r^{ss}(x) = \begin{cases} \inf\left\{r^{ss}_{g\mathscr{A}_0, \mathscr{A}_0}(x) : g \in G(F)\right\} & \text{if } x \in Y^{ss}_{\mathscr{A}_0}; \\ 0 & \text{if } x \notin Y^{ss}_{\mathscr{A}_0}. \end{cases}$$

We also introduce here another function. Take $x \in Y^{ss}$. Let $g(\mathscr{A}_0)$ be an apartment such that $r^{ss}_{g\mathscr{A}_0, \mathscr{A}_0}(x) = r^{ss}(x)$. Suppose $|g^*(x_jx_{5-j})(x)| \geq |g^*(x_ix_{5-i})(x)|$, $j \neq i, 5-i$. Then we define: $r^s_{g\mathscr{A}_0}(x) = |g^*(x_ix_{5-i}/x_jx_{5-j})(x)|$. For $x \in Y^{ss}$, we define $r^s(x) = \inf\{r^s_{\mathscr{A}}(x) : r^{ss}_{\mathscr{A}, \mathscr{A}_0}(x) = r^{ss}(x)\}$.

7.2. Let α be the root of gTg^{-1} associated with g^*x_j/g^*x_{5-j} and assume $r^s_A(x) < 1$. For x in Y^{ss}, let L_x be the set of $q \in \mathscr{A}$ such that $\alpha(q) = \alpha(v_{\mathscr{A}}(x)) = -v((g^*x_j/g^*x_{5-j})(x))$. Let $\sigma \in \mathscr{A}$ be an alcove such that $v_{g\mathscr{A}_0}(x) \in \sigma$. Then we define: $T(x)_{\mathscr{A}} := \bigcup_{g\in G_\sigma} g(L_x)$. By definition $T(x)_{\mathscr{A}} \subset \mathscr{B}$ is a sub tree of the building. Furthermore $I_{\mathscr{A}}(x) \subset T(x)_{\mathscr{A}}$. The next lemma shows that $T(x)_{\mathscr{A}}$ does not depend on the choice of $\mathscr{A}$. So we take $T(x) := T(x)_{\mathscr{A}}$.

LEMMA 7.1. *Let $x \in Y^{ss}$ and $\mathscr{A}_j, j = 1,2$ be such that $r^{ss}_{\mathscr{A}_j,\mathscr{A}_0}(x) = r^{ss}(x)$ and $r^s_{\mathscr{A}_j}(x) < 1$. Let $v_{\mathscr{A}_j}(x) \in \sigma_j \in \mathscr{A}_j$. Then either $\sigma_1 \cap \sigma_2 \neq \emptyset$ or $v_{\mathscr{A}_j}(x) \in T(x)_{\mathscr{A}_1} = T(x)_{\mathscr{A}_2}$.*

PROOF. Let $g_j \in G(F)$ be such that $g_j(\sigma_0) = \sigma_j$, $g_j(\mathscr{A}_0) = \mathscr{A}_j$. Let $\mathscr{A}$ be such that $\sigma_1, \sigma_2 \in \mathscr{A}$. Let $f_j \in G_{\sigma_j}$ be such that $f_j g_j(\mathscr{A}_0) = \mathscr{A}$.

Since $r^{ss}_{\mathscr{A}_j,\mathscr{A}_0}(x) = r^{ss}(x)$ we have $r^{ss}_{\mathscr{A},\mathscr{A}_0}(x) = r^{ss}(x)$. For each j let i_j be such that $|g_j^*(x_{i_j}x_{5-i_j})(x)| = \max\{|g_j^*(x_i x_{5-i})(x)| : i = 1,2\}$. Since $r^{ss}_{\mathscr{A}_j\mathscr{A}_0}(x) = r^{ss}(x)$ and $r^s_{\mathscr{A}_j}(x) < 1$ we have $|f_j^* g_j^*(x_{i_j}x_{5-i_j})(x)| = \max\{|f_j^* g_j^*(x_i x_{5-i})(x)| : i = 1,2\}$.

If $f_1^* g_1^*(x_{i_1}x_{5-i_1})(x) = f_2^* g_2^*(x_{i_2}x_{5-i_2})(x)$ then one clearly has $T(x)_{\mathscr{A}_1} = T(x)_{\mathscr{A}_2}$. If $f_1^* g_1^*(x_{i_1}x_{5-i_1})(x) \neq f_2^* g_2^*(x_{i_2}x_{5-i_2})(x)$ then $r^s_{\mathscr{A}}(x) = 1$. Now by Lemma 6.3 we have $v_{\mathscr{A}}(x) \in \sigma_1$ and $v_{\mathscr{A}}(x) \in \sigma_2$. Hence $\sigma_1 \cap \sigma_2 \neq \emptyset$. □

LEMMA 7.2. *There exists an apartment $\mathscr{A}$ satisfying $r^{ss}_{\mathscr{A},\mathscr{A}_0}(x) = r^{ss}(x)$ and $r^s_{\mathscr{A}}(x) = 0$ if and only if $r^s(x) = 0$.*

PROOF. Let g_i be such that $r^{ss}_{g_i\mathscr{A}_0,\mathscr{A}_0}(x) = r^{ss}(x)$ and $r^s_{g_i\mathscr{A}_0}(x) \to 0$. Let $\sigma \in g_1\mathscr{A}_0$ be an alcove such that $x \in Y^{ss}_{g_1\mathscr{A}_0,\sigma}$. Since $r^{ss}_{g_1\mathscr{A}_0,\mathscr{A}_0}(x) = r^{ss}(x)$, for all $h \in G_\sigma$ we have $r^{ss}_{hg_1\mathscr{A}_0,\mathscr{A}_0}(x) = r^{ss}(x)$. We will construct a sequence $h_i \in G_\sigma$ such that $r^s_{h_i g_1\mathscr{A}_0}(x) \to 0$. The compactness of G_σ implies $h_i \to h$. Then $r^s_{hg_1\mathscr{A}_0}(x) = 0$ and the lemma follows. □

We know that $v_{g_i\mathscr{A}_0}(x) \in T(x)$. Find a line $L_i \subset T(x)$ containing $v_{g_i\mathscr{A}_0}(x)$ and $v_{g_1\mathscr{A}_0}(x)$ such that L_i contains half of the interval of stability $I_{g_i\mathscr{A}_0}(x)$. Next choose an apartment $\mathscr{A}_i$ containing L_i. Since L_i contains $v_{g_1\mathscr{A}_0}(x)$, one has $\mathscr{A}_i = h_i g_1\mathscr{A}_0$ for some $h_i \in \mathscr{G}_\sigma$. The fact that $\mathscr{A}_i$ contains half of $I_{g_i\mathscr{A}_0}(x)$ implies that $r^s_{\mathscr{A}_i}(x) \le r^s_{g_i\mathscr{A}_0}(x)/2$. Hence $r^s_{h_i g_1\mathscr{A}_0}(x) \to 0$, we are done.

LEMMA 7.3. *If $x \in Y^*_{\mathscr{A},\sigma}$ then for all $\mathscr{A}'$ containing $I_{\mathscr{A}}(x)$ one has $r^s_{\mathscr{A}'}(x) \le r^s_{\mathscr{A}}(x)$.*

PROOF. Suppose $I_{\mathscr{A}}(x) \subset \mathscr{A}'$. Then we have an $h \in G_{I_{\mathscr{A}}(x)}$ such that $\mathscr{A}' = h(\mathscr{A})$. It is sufficient to prove the lemma in the case $\mathscr{A} = \mathscr{A}_0$. Furthermore, we only treat the case $x \in Y^*_{\mathscr{A}_0,\sigma}$ with $|(x_1x_4/x_2x_3)(x)| \ge 1$. The other case is similar. In this case we have $|(h^*x_1/x_1)(x)| = |(h^*x_4/x_4)(x)| = 1$ since $h \in G_{I_{\mathscr{A}}(x)} \subset G_\sigma$. Furthermore we have $|(h^*x_2/x_2)(x)| \le 1$, $|(h^*x_3/x_3)(x)| \le 1$. Hence

$$r^s_{\mathscr{A}'}(x) = |(h^*x_2 h * x_3/h^*x_4 h^*x_1)(x)| \le |(x_2x_3/x_4x_1)(x)| = r^s_{\mathscr{A}_0}(x).$$

This proves the lemma. □

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LEMMA 7.4. *If* $x \in Y^{ss}_{\mathscr{A}_0}$, *then there exists an* $h \in G(F)$ *such that* $x \in h \cdot Y^s_{\mathscr{A}_0}$ *and* $r^{ss}_{h\mathscr{A}_0, \mathscr{A}_0}(x) = 1$.

PROOF. If $x \in Y^s_{\mathscr{A}_0}$ we can take $h = \mathrm{id}$. So let us assume that $x \notin Y^s_{\mathscr{A}_0}$. Then either $x_1x_4 = 0$ or $x_2x_3 = 0$, and not both since $x \in Y^{ss}_{\mathscr{A}_0}$. It is sufficient to treat only the case $x_1x_4 \neq 0$ and $x_2x_3 = 0$. Take $h_1(x) = (x_1, x_2 + ax_1, x_3, x_4 - ax_3)$ and $h_2(x) = (x_1 - bx_2, x_2, x_3 + bx_4, x_4)$. Then choose $a, b \in F$ such that $h_2, h_1(x) \in Y^s_{\mathscr{A}_0}$. By taking $|a|$, $|b|$ sufficiently small we can get that $r^{ss}_{\mathscr{A}_0, h_2h_1\mathscr{A}_0}(h_2h_1(x)) = 1$. Therefore $h = (h_2h_1)^{-1}$ satisfies the lemma. ☐

LEMMA 7.5. *If* $\sigma \subset \mathscr{A} \subset \mathscr{I}$ *and* $x \in Y^*_{\mathscr{I},\sigma}$ *then for all* $g \in G_\sigma$ *we have* $r^{ss}_{g\mathscr{A},\mathscr{A}}(x) = 1$.

PROOF. We may assume that $\sigma = \sigma_0$ and $\mathscr{I} = \mathscr{I}_0$ after replacing x by $g^{-1}(x)$. It is sufficient to treat the case $|x_1x_4/x_2x_3(x)| < 1$. Then

$$r^{ss}_{g\mathscr{A}_0, \mathscr{A}_0}(x) = \max\{|g^*(x_1x_4)/x_2x_3(x)|, |g^*(x_2x_3)/x_2x_3(x)|\}.$$

Since $Y^*_{\mathscr{I}_0,\sigma_0} \subset Y^s_{\mathscr{A}_0,\sigma_0}$ it follows from Lemma 6.1 (b) that for all $g \in G_{\sigma_0}$ we have $r^{ss}_{gA_0,A_0}(x) = 1$. So for all $\mathscr{A}$ with $\sigma_0 \in \mathscr{A}$ we have $r^{ss}_{\mathscr{A},\mathscr{A}_0} = 1$. ☐

LEMMA 7.6. *If* $x \notin Y^{ss}_{\mathscr{A}_0}$ *then for all* $\mathscr{I} \subset \mathscr{B}$, *and for all* $\sigma \in \mathscr{I}$ *we have* $x \notin Y^*_{\mathscr{I},\sigma}$.

PROOF. Let us assume $x \notin Y^{ss}_{\mathscr{A}_0}$. Then $x_1x_4 = x_2x_3 = 0$. It is sufficient to treat the case where $x_1 = x_2 = 0$.

First we assume that $x_3x_4 \neq 0$. Take $a, b \in F^\times$ and $h \in G(F)$ given by $h(y_1, y_2, y_3, y_4) = (y_1 + ay_4, y_2 + by_3, y_3, y_4)$. Now $h(x) = (ax_4, bx_3, x_3, x_4) \in Y^s_{\mathscr{A}_0}$. Hence $x \in Y^s_{h^{-1}\mathscr{A}_0}$. Suppose $x \in Y^*_{\mathscr{I},\sigma}$ and that σ lies in $\mathscr{A} \subset \mathscr{I}$. By taking $|a|$, $|b|$ sufficiently small we can make sure that $r^{ss}_{h^{-1}\mathscr{A}_0,\mathscr{A}}(x) < 1$. Now we can find a $\sigma_1 \in h^{-1}\mathscr{A}_0$ such that $x \in Y^s_{h^{-1}\mathscr{A}_0,\sigma_1}$. Next we can find $\widetilde{\mathscr{A}}$ containing σ and σ_1. Now $r^{ss}_{\widetilde{\mathscr{A}},\mathscr{A}}(x) \le r^{ss}_{h^{-1}\mathscr{A}_0,\mathscr{A}}(x) < 1$. Since $\widetilde{A} = fA$ for some $f \in P_\sigma$ this contradicts the fact that $x \in Y^*_{\mathscr{I},\sigma}$.

Now suppose that also $x_3x_4 = 0$. It is enough to treat the case $x_3 = 0$. So $x = (0, 0, 0, 1)$. Now we use $\widetilde{h} = h_2h_1h$ where $h_1(x_1, x_2, x_3, x_4) = (x_1 + bx_3, x_2 + bx_4, x_3, x_4)$ and $h_2(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3 + cx_2, x_4)$ with $b, c \in F^\times$. Now $h_2h_1h(x) = (a, b, bc, 1)$. Again $\widetilde{h}(x) \in Y^s_{\mathscr{A}_0}$. Now apply the same proof as above to finish this case. ☐

LEMMA 7.7. *If* $\sigma \subset \mathscr{A} \subset \mathscr{I}$ *and* $x \in Y^*_{\mathscr{I},\sigma}$ *then* $r^{ss}(x) > 0$.

PROOF. We may assume that $\sigma = \sigma_0$ and $\mathscr{I} = \mathscr{I}_0$ after replacing x by $g^{-1}(x)$. By Lemma 7.5 for all $\mathscr{A}$ with $\sigma_0 \in \mathscr{A}$ we have $r^{ss}_{\mathscr{A},\mathscr{A}_0} = 1$. Take an apartment $h\mathscr{A}_0$.

If $r^{ss}_{h\mathscr{A}_0,\mathscr{A}_0}(x) = 0$ then $x \notin hY^{ss}_{\mathscr{A}_0}$. From Lemma 7.6 it follows that $x \notin Y^*_{\mathscr{I}_0,\sigma_0}$. Therefore this cannot occur. Now we assume that $r^{ss}_{hA_0,A_0}(x) \notin 0$. We have $x \in h \cdot Y^{ss}_{\mathscr{A}_0}$. Using Lemma 7.4 we find an apartment $g(h(\mathscr{A}_0))$ such that $x \in gh(Y^s_{\mathscr{A}_0})$ and $r^{ss}_{gh\mathscr{A}_0,h\mathscr{A}_0}(x) = 1$. Thus $r^{ss}_{h\mathscr{A}_0,\mathscr{A}_0}(x) = r^{ss}_{gh\mathscr{A}_0,\mathscr{A}_0}(x)$. Since $x \in gh(Y^s_{\mathscr{A}_0})$ there exists a $\sigma_1 \in gh(\mathscr{A}_0)$ such that $x \in Y^s_{gh\mathscr{A}_0,\sigma_1}$. Take an apartment $\mathscr{A}''$ containing both σ_0 and σ_1. By general properties of the r function we have $r^{ss}_{\mathscr{A}'',\mathscr{A}_0}(x) \leq r^{ss}_{gh\mathscr{A}_0,\mathscr{A}_0}(x)$. Therefore, $r^{ss}_{h\mathscr{A}_0,\mathscr{A}_0}(x) \geq 1$. So we have proved the lemma. □

LEMMA 7.8. *We have $x \in Y^{ss}$ if and only if $r^{ss}(x) > 0$.*

PROOF. If $x \notin Y^{ss}$ then there exists an element $h \in G(F)$ such that $x \notin hY^{ss}_{\mathscr{A}_0}$. Hence $r^{ss}(x) = 0$. Conversely assume $x \in Y^{ss}$ and that $r^{ss}(x) = 0$. Take a sequence $g_i \in G(F)$ such that $r^{ss}_{g_i\mathscr{A}_0,\mathscr{A}_0}(x) \to 0$. Since $x \in gX^{ss}$ we can use Lemma 7.4 to find an $h_i \in G(F)$ such that $x \in h_iY^s_{\mathscr{A}_0}$ and $r^{ss}_{h_i\mathscr{A}_0,\mathscr{A}_0}(x) = r^{ss}_{g_i\mathscr{A}_0\mathscr{A}_0}(x)$. So let us replace the sequence g_i by the sequence h_i. Since $x \in h_iY^s_{\mathscr{A}_0}$ there exist $\sigma_i \in h_i\mathscr{A}_0$ such that $x \in Y^s_{h_i\mathscr{A}_0,\sigma_i}$. We can assume that $h_1\mathscr{A}_0 = A_0$ and $\sigma_1 = \sigma_0$. Find an apartment $\mathscr{A}_i$ containing both σ_0 and σ_i. Then there exist $\widetilde{f}_i \in G_{\sigma_i}$ and $f_i \in G_{\sigma_0}$ such that $\mathscr{A}_i = f_i(\mathscr{A}_0) = \widetilde{f}_ih_i(\mathscr{A}_0)$. Using Lemma 6.1 we find that $r^{ss}_{\mathscr{A}_i,\mathscr{A}_0}(x) = r^{ss}_{f_i\mathscr{A}_0,\mathscr{A}_0}(x) \leq r^{ss}_{h_i\mathscr{A}_0,\mathscr{A}_0}(x)$. So we have constructed a sequence $f_i \in G_{\sigma_0}$ such that $r^{ss}_{f_i,\mathscr{A}_0,\mathscr{A}_0}(x) \to 0$. Since $G_{\sigma 0}$ is compact, $f_i \to f$. Clearly $r^{ss}_{f\mathscr{A}_0,\mathscr{A}_0}(x) = 0$. Hence $x \notin f\, Y^{ss}_{\mathscr{A}_0}$ and so $x \notin Y^{ss}$. This gives the required contradiction. □

LEMMA 7.9. $Y^{ss} \subset \bigcup_{\mathscr{I},\sigma} Y^*_{\mathscr{I},\sigma}$.

PROOF. Since $x \in Y^{ss}$ one has $r^{ss}(x) > 0$. We can find $g \in G(F)$ such that $r^{ss}_{g\mathscr{A}_0,\mathscr{A}_0}(x) = r^{ss}(x)$. If $x \in Y^s_{g\mathscr{I}_0}$ then $x \in Y^s_{g\mathscr{I}_0,\sigma}$ for some $\sigma \subset \mathscr{I}$. Since $r^{ss}_{g\mathscr{A}_0,\mathscr{A}_0}(x) = r^{ss}(x)$ we must have $x \in Y^*_{g\mathscr{I}_0,\sigma}$. So let us assume that $x \notin Y^s_{gI_0}$. We can assume that $g = \mathrm{id}$. There exists an $\mathscr{A} \subset \mathscr{I}_0$ such that $x \in Y^s_{\mathscr{A}_0}$. Again we can assume $\mathscr{A} = \mathscr{A}_0$. We only treat the case $x_1x_4 \neq 0$, $x_2x_3 = 0$. The other case is similar. Since $x \in Y^{ss}$ there cannot be a relation $x_1 + ax_4 = 0$ for some $a \in F$. From this it follows that $x \in Y^s_{h\mathscr{I}_0}$ where h is as in the proof of Lemma 7.4. Since we have $r^{ss}_{h\mathscr{A}_0,\mathscr{A}_0}(x) = 1$, the lemma follows. □

8. Affinoid coverings

In this section we shall give a construction of a pure affinoid covering of Y_s which yields a reduction consisting of proper components.

THEOREM 8.1. *When we take all the $SL_2(F) \times SL_2(F)$ sub-buildings $\mathscr{I}$ of the C_2 building $\mathscr{B}$, all simplices $\sigma \in \mathscr{I}$ and all integers n the affinoids $Y^*_{\mathscr{I},\sigma,n}$ cover Y^{ss}.*

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This follows from Lemma 7.6 and Lemma 7.9.

The analytic space $Y^s_{\mathscr{A}_0,\mathscr{O}}$ is not affinoid, but we can cover it by affinoids $Y^s_{\mathscr{A}_0,\mathscr{O},n}$. And $\{Y^s_{\mathscr{A}_0,\mathscr{O},n}$ for $n \in Z\}$ is a pure affinoid covering of $Y^s_{\mathscr{A}_0,\mathscr{O}}$. While $\{Y^s_{\mathscr{A}_0,\sigma,n} \mid \sigma \in A_0, n \in Z\}$ gives a pure affinoid covering of $Y^s_{\mathscr{A}_0}$.

PROPOSITION 8.2. *The following are pure affinoid coverings*:

(a) $Y^s_{\mathscr{A}_0,\mathscr{O}} = \bigcup_{n\in\mathbb{Z}} Y^s_{\mathscr{A}_0,\mathscr{O},n}$.
(b) $Y^s_{\mathscr{A}} = \bigcup_{\sigma\in A, n\in\mathbb{Z}} Y^s_{\mathscr{A},\sigma,n}$.
(c) $Y^s_{\mathscr{I}} = \bigcup_{n\in\mathbb{Z},\sigma\in\mathscr{I}} Y^s_{\mathscr{I},\sigma,n}$.

PROOF. We prove part (c). First we remark that $Y^s_{\mathscr{I}_0,\sigma,n}$ is the set of x in $Y^s_{\mathscr{A}_0,\sigma,n}$ such that $|g * x_i/x_i(x)| = 1$ for all $g \in H_\sigma$ and $i = 1, \dots, 4$. Furthermore, for $x \in Y^s_{\mathscr{A}_0,\sigma,n}$ one has $|g * x_i/x_i(x)| \le 1$. So $Y^s_{\mathscr{I}_0,\sigma,n} \subset Y^s_{\mathscr{A}_0,\sigma,n}$ is an open affinoid subspace of the form: $Y^s_{\mathscr{I}_0,\sigma,n} = Y^s_{\mathscr{A}_0,\sigma,n} - R^{-1}(V_{\mathscr{A}_0,\sigma,n})$. Here R denotes the canonical reduction map of $Y^s_{\mathscr{A}_0,\sigma,n}$ and $V_{\mathscr{A}_0,\sigma,n}$ is a closed subvariety. To see that the covering is pure consider the intersection $Y^s_{\mathscr{I},\sigma,n} \cap Y^s_{\mathscr{I},\sigma',m}$. Take an apartment $\mathscr{A} \subset \mathscr{I}$ containing σ and σ'. It follows from part (a) of the proposition that $Y^s_{\mathscr{A},\sigma,n} \cap Y^s_{\mathscr{A},\sigma',m}$ is pure . Since $Y^s_{\mathscr{I},\sigma,n} = Y^s_{\mathscr{A},\sigma,n} - R^{-1}(V_{\mathscr{A},\sigma,n})$ and $Y^s_{\mathscr{I},\sigma',m} = Y^s_{\mathscr{A},\sigma',m} - R^{-1}(V_{\mathscr{A},\sigma',m})$, the intersection $Y^s_{\mathscr{I},\sigma,n} \cap Y^s_{\mathscr{I},\sigma',m}$ is also pure. This completes the proof. □

8.1. A construction We give the construction of the affinoids which will be used to cover the space of stable points. Suppose $\mathscr{A}$ is $g \cdot \mathscr{A}_0$. For $\sigma_1, \sigma_2 \in \mathscr{A}$ let $\Delta(\sigma_1, \sigma_2)$ denote the convex hull of (σ_1, σ_2). For $\sigma \in \mathscr{A}$ let $Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)'$ be the set of x in $Y^*_{\mathscr{A},\sigma}$ such that $v_{\mathscr{A}}(x) \in \sigma$, and that $\Delta(\sigma_1, \sigma_2)$ contains $I_{\mathscr{A}}(x)$, $\sigma_1 \cap I_{\mathscr{A}}(x) \neq \emptyset$, $\sigma_2 \cap I_{\mathscr{A}}(x) \neq \emptyset$, and that for all $h \in G_{\Delta(\sigma_1,\sigma_2)}$ we have $|h^* g^* x_i/g^* x_i(x)| = 1$. We will always assume σ_1, σ_2 chosen in such a way $Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)'$ is nonempty. If $x \in Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)'$ and $\mathscr{A}' \supset \Delta(\sigma_1, \sigma_2)$ then $I_{\mathscr{A}'}(x) = I_{\mathscr{A}}(x)$. Next we introduce an open subaffinoid $Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)$ of $Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)'$ such that for x in this subaffinoid and for $A' \supset I_{\mathscr{A}}(x)$ we get $I_{\mathscr{A}'}(x) = I_{\mathscr{A}}(x)$. To do this we take certain functions f in the affinoid algebra of $Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)'$ demand that $|f(x)| = 1$ to get our subaffinoid. Suppose that the end points of the interval $I_{\mathscr{A}}(x)$ are P_1, P_2 with $P_i \in \sigma_i$ and that the center $v_{\mathscr{A}}(x)$ of $I_{\mathscr{A}}(x)$ lies in σ. Suppose that the wall L of σ_1 corresponds a certain root and the line L is defined by $|x_i/x_j(x)| = |\pi^n|$ say. Then we know for x in $Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)'$ we have $|x_i/x_j(x)| \le |\pi^n|$. Now choose $|a| = 1$ then

(1) P_1 is in the interior of σ_1 if and only if $|x_i/x_j(x)| < |\pi^n|$, which in turn implies $|x_i + \pi^n a x_j/x_j(x)| = |\pi^n a x_j/x_j(x)| = |\pi^n|$;
(2) P_1 is in $\sigma_1 \cap L$ if and only if $|x_i/x_j(x)| = |\pi^{-n}|$, which in turn implies $|x_i + \pi^n a x_j/x_j(x)| \le |\pi^n|$.

So we can take f as being $\pi^{-n}(x_i + \pi^n a x_j/x_j)$. From the construction we now have: If $x \in Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)$ and $I_{\mathscr{A}}(x) \subset \mathscr{A}'$ then $I_{\mathscr{A}}(x) = I_{\mathscr{A}'}(x)$. Finally, if $\mathscr{A}$ is in the $SL_2 \times SL_2$ building $\mathscr{I}$ and H is the $SL_2 \times SL_2$ acting on $\mathscr{I}$ and $\mathscr{A} = g \cdot \mathscr{A}_0$ we put $Y(\mathscr{I}, \sigma, \sigma_1, \sigma_2)$ to be the set of x in $Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)$ such that $|h^*g^*x_i/g * x_i(x)| = 1$ for all $h \in H_\sigma$.

LEMMA 8.3. *Suppose that* $\sigma \in \mathscr{A} \subset \mathscr{I}$ *and* $x \in Y(\mathscr{I}, \sigma, \sigma_1, \sigma_2)$.

(a) *If* $h(I_{\mathscr{A}}(x)) \subset \mathscr{A}'$ *for some* $h \in H$ *then* $I_{\mathscr{A}'}(x) = h(I_{\mathscr{A}}(x))$.

(b) *If* $h \in G_\sigma$ *then* $I_{h\mathscr{A}}(x) = h(I_{\mathscr{A}}(x))$ *if and only if* $h(I_{\mathscr{A}}(x)) \subset \mathscr{I}$.

(c) *If* $h \in G_\sigma$ *then* $I_{h\mathscr{A}}(x) \neq h(I_{\mathscr{A}}(x))$ *if and only if* $r^s_{\mathscr{A}}(x) \leq r^s_{h\mathscr{A}}(x)$.

This follows immediately from the definitions.

LEMMA 8.4. (a) *If* $r^s(x) \neq 0$ *then* x *is in one of the* $Y(\mathscr{I}, \sigma, \sigma_1, \sigma_2)$.

(b) $r^s(x) \neq 0$ *if and only if* $x \in Y^s$.

PROOF. (a) It suffices to take $\mathscr{A}$ such that $r^s_{\mathscr{A}}(x) = r^s(x)$.

(b) Find an $\mathscr{A}$ such that $x \in Y(\mathscr{A}, \sigma, \sigma_1, \sigma_2)$ for some $\sigma, \sigma_1, \sigma_2$. Now observe that there exists an $h \in P_\sigma$ such that $r^s_{h\mathscr{A}}(x) = 0$ if and only if $x \notin Y^s$. □

LEMMA 8.5. *Suppose* $x \in Y(\mathscr{I}_1, \sigma, \sigma_1, \sigma_2) \cap Y(\mathscr{I}_2, \tau, \tau_1, \tau_2)$. *Let* $\sigma \in \mathscr{A}_1 \subset \mathscr{I}_1$ *and* $\tau \in \mathscr{A}_2 \subset \mathscr{I}_2$. *Then there exists* $h \in H_{1,\sigma}$ *such that* $h(I_{\mathscr{A}_1}(x)) = I_{A_2}(x)$ *and* $I_{\mathscr{I}_1}(x) = I_{\mathscr{I}_2}(x)$.

PROOF. Take an apartment $\widetilde{A}$ containing σ and τ. Now $r^s_{\widetilde{A}}(x)$ depends on the intersection of $S_{x,H_1} \cap \widetilde{A}$ and also on the intersection $S_{x,H_2} \cap \widetilde{A}$. If $S_{s,H_1} \neq s_{x,H_2}$ then we can change $\widetilde{A}$ a little such that $S_{x,H_1} \cap \widetilde{A}$ changes and $S_{x,H_2} \cap \widetilde{A}$ remains the same. The change of $S_{x,H_1} \cap \widetilde{A}$ means that $r^s_{\widetilde{A}}(x)$ changes whereas since $S_{x,h_2} \cap \widetilde{A}$ does not change $r^s_{\widetilde{A}}$ does not change $r^s_{\widetilde{A}}(x)$ also does not change. This is absurd and one must have $S_{x,H_2} \cap \widetilde{A} = S_{x,H_1} \cap \widetilde{A}$. Since both $\sigma, \tau \in \widetilde{A}$ it easily follows that one must have $I_{\mathscr{I}_1}(x) = I_{\mathscr{I}_2}(x)$. □

THEOREM 8.6. *The family of sets* $Y(\mathscr{I}, \sigma, \sigma_1, \sigma_2)$ *obtained by taking all* $SL_2 \times SL_2$ *buildings* $\mathscr{I} \subset \mathscr{B}$, $\sigma, \sigma_1, \sigma_2$ *in all apartments* $\mathscr{A}$ *in* $\mathscr{I}$ *gives a pure covering of* Y^s. *Furthermore the reduction with respect to this covering consists of proper components.*

PROOF. Let $x \in Y^s$. Then we can find $\mathscr{A} \subset \mathscr{B}$ such that $r^s_A(x) = r^s(x)$. The proposition above shows that $I_x := I_{\mathscr{I}}(x)$ is uniquely determined. Here $\mathscr{I}$ is the $SL_2 \times SL_2$ building containing $\mathscr{A}$. This is clear, since we have a unique S_x for each $x \in Y^s$. From this one easily concludes that the covering is pure. That the reduction consists of proper components is proved using the same method as in [12, Theorem 3.6, part 5]. □

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