An Analogue for Strong Summability of Abel's Summability Method

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1. *Introduction.* Given a series Σa_n, we define $A_n^{(k)}$, $k > -1$, by the relation

$$A_n^{(k)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(k)} a_\nu,$$

where $E_n^{(k)}$ is the binomial coefficient $\binom{k+n}{n}$. Let $c_n^{(k)} = A_n^{(k)}/E_n^{(k)}$. If $c_n^{(k)} \to s$ as $n \to \infty$, the series Σa_n is said to be summable $(C;\ k)$ to the sum s. If $k > 0$, $p \geqslant 1$ and if, as $n \to \infty$,

$$\sum_{\nu=0}^{n} |c_\nu^{(k-1)} - s|^p = o(n),$$

we say[1] that the series Σa_n is summable $[C;\ k, p]$ to the sum s, or that the series is strongly summable $(C;\ k)$ with index p to the sum s. If $a_n^{(k)}$ denotes the difference $c_n^{(k)} - c_{n-1}^{(k)}$, it is known[2] that necessary and sufficient conditions for summability $[C;\ k, p]$, $k > 0$, $p \geqslant 1$, to the sum s, are that Σa_n be summable $(C;\ k)$ to the sum s and that

$$\sum_{\nu=1}^{n} \{\nu |a_\nu^{(k)}|\}^p = o(n).$$

When $k = 0$, $p \geqslant 1$, we use this property to define summability $[C;\ 0, p]$.

The series Σa_n is said to be summable (A), or summable by Abel's method, to the sum s, if (i) the series

$$f(u) = \sum_{n=0}^{\infty} a_n e^{-n/u}$$

is convergent for every positive u and (ii) $f(u) \to s$ as $u \to \infty$ continuously. It is a natural analogue to say that Σa_n is strongly summable (A) with index p $(\geqslant 1)$ to the sum s, or that Σa_n is summable $[A;\ p]$, $p \geqslant 1$, to the sum s, if, in addition to (i) and (ii), we have, as $\omega \to \infty$,

$$\text{(iii)} \qquad \int_0^\omega |uf'(u)|^p\, du = o(\omega).$$

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[1] Hyslop, **2**. For the cases $k = 1$, $p = 1$ see respectively Kuttner, **4**, and Winn, **7**. For applications of strong summability to Fourier series see Paley, **6**, and Marcinkiewicz, **5**.

[2] Hyslop, **2**.

In this paper we show that summability $[A\,;\,p]$ implies summability $[A\,;\,q]$ for $p > q \geqslant 1$, and that summability $[C\,;\,k,\,p]$, $k \geqslant 0$, $p \geqslant 1$, implies summability $[A\,;\,p]$. The latter is of course the analogue for strong summability of the well-known result that summability $(C\,;\,k)$ implies summability (A).

2. *Preliminary Lemmas.* We state or derive here certain results which will be required in the proofs of the theorems.

LEMMA 1. *For $k > -1$, we have the formal identity*[1]

$$\sum_{n=0}^{\infty} n\,E_n^{(k)}\,a_n^{(k)}\,x^n = (1-x)^{-k}\sum_{n=0}^{\infty} na_n\,x^n.$$

LEMMA 2. *If*[2] $p > 1$, $f(x) \geqslant 0$, $K(x,\,y) \geqslant 0$ *and* $K(x,\,y)$ *is homogeneous of degree* -1, *and if*

$$\int_0^{\infty} K(x,\,1)\,x^{-1/p}\,dx = \lambda,$$

then
$$\int_0^{\infty} dy\left\{\int_0^{\infty} K(x,\,y)\,f(x)\,dx\right\}^p \leqslant \lambda^p \int_0^{\infty} \{f(x)\}^p\,dx.$$

LEMMA 3. *If $k > 0$, $p > 1$, $f(x) \geqslant 0$, $f(x) = 0$ for $x > k\omega$, then*

$$\int_1^{\omega} dy\left\{\int_1^{k\omega} x^k y^{-k-1} e^{-x/y} f(x)\,dx\right\}^p \leqslant \lambda^p \int_0^{k\omega} \{f(x)\}^p\,dx,$$

where $\lambda = \Gamma(k+1-p^{-1})$.

In Lemma 2 take $K(x,\,y)$ to be $x^k y^{-k-1} e^{-x/y}$. Then

$$\int_0^{\infty} K(x,\,1)\,x^{-1/p}\,dx = \int_0^{\infty} x^{k-1/p}\,e^{-x}\,dx = \Gamma(k+1-p^{-1}) = \lambda.$$

Also

$$\int_1^{\omega} dy\left\{\int_1^{k\omega} x^k y^{-k-1} e^{-x/y} f(x)\,dx\right\}^p \leqslant \int_0^{\infty} dy\left\{\int_0^{\infty} x^k y^{-k-1} e^{-x/y} f(x)\,dx\right\}^p$$

$$\leqslant \lambda^p \int_0^{\infty} \{f(x)\}^p\,dx = \lambda^p \int_0^{k\omega} \{f(x)\}^p\,dx.$$

LEMMA 4. *If $p > 1$, $f(x) \geqslant 0$, $f(x) = 0$ for $x > \omega$, then*

$$\int_1^{\omega} dy\left\{\int_0^{\omega} y^{-1} e^{-x/y} f(x)\,dx\right\}^p \leqslant \lambda^p \int_0^{\omega} \{f(x)\}^p\,dx,$$

where $\lambda = \Gamma(1-p^{-1})$.

The proof is almost the same as that of Lemma 3.

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[1] Kogbetliantz, 3.

[2] Hardy, Littlewood and Pólya, 1, 229.

LEMMA 5. *If* $k > 0$, $p > 1$, $\phi_\nu \geqslant 0$ *for* $\nu = 0, 1, 2, \ldots,$ *then*

(i) $$\int_1^\omega du \left\{ \sum_{\nu=1}^{m-1} \nu^k u^{-k-1} e^{-\nu/u} \phi_\nu \right\}^p \leqslant \lambda^p \sum_{\nu=0}^{n} \phi_\nu{}^p,$$

(ii) $$\int_1^\omega du \left\{ \sum_{\nu=m+2}^{n} \nu^k u^{-k-1} e^{-\nu/u} \phi_\nu \right\}^p \leqslant \lambda^p \sum_{\nu=0}^{n} \phi_\nu{}^p,$$

where $m = [ku]$, $n = [k\omega]$ *and* $\lambda = \Gamma(k+1-p^{-1})$.

Clearly the function $x^k e^{-x/u}$ increases for $0 < x < ku$ and decreases for $x > ku$. In Lemma 3 take

$$\begin{aligned} f(x) &= \phi_\nu, \quad \nu \leqslant x < \nu+1, \quad \nu = 0, 1, \ldots, n-1. \\ &= 0, \quad x \geqslant n. \end{aligned}$$

Then

$$\begin{aligned} \int_1^\omega du \left\{ \sum_{\nu=1}^{m-1} \nu^k u^{-k-1} e^{-\nu/u} \phi_\nu \right\}^p &\leqslant \int_1^\omega du \left\{ \sum_{\nu=1}^{m-1} \int_\nu^{\nu+1} x^k u^{-k-1} e^{-x/u} f(x)\, dx \right\}^p \\ &\leqslant \int_1^\omega du \left\{ \int_1^{k\omega} x^k u^{-k-1} e^{-x/u} f(x)\, dx \right\}^p \leqslant \lambda^p \int_0^{k\omega} \{f(x)\}^p\, dx \\ &= \lambda^p \int_0^n \{f(x)\}^p\, dx = \lambda^p \sum_{\nu=0}^{n-1} \int_\nu^{\nu+1} \phi_\nu{}^p\, dx \leqslant \lambda^p \sum_{\nu=0}^{n} \phi_\nu{}^p, \end{aligned}$$

which proves (i). The proof of (ii) is similar, but in this case it is convenient to define $f(x)$ as follows:

$$\begin{aligned} f(x) &= \phi_\nu, \quad \nu-1 \leqslant x < \nu, \quad \nu = 1, 2, \ldots, n, \\ &= 0, \quad x \geqslant n. \end{aligned}$$

LEMMA 6. *If* $p > 1$, $r = [\omega]$ *and* $\phi_\nu \geqslant 0$ *for* $\nu = 0, 1, 2, \ldots,$ *then*

$$\int_1^\omega du \left\{ \sum_{\nu=0}^{r} u^{-1} e^{-\nu/u} \phi_\nu \right\}^p \leqslant \lambda^p \sum_{\nu=0}^{r} \phi_\nu{}^p,$$

where $\lambda = \Gamma(1-p^{-1})$.

This follows from Lemma 4 as Lemma 5 follows from Lemma 3.

LEMMA 7. *If* $b_\nu \geqslant 0$, $\nu = 1, 2, \ldots, s$, *and* $p \geqslant 1$, *then*

$$\left(\sum_{\nu=1}^{s} b_\nu \right)^p \leqslant s^p \sum_{\nu=1}^{s} b_\nu{}^p.$$

The proof of this inequality is immediate.

3. *Strong Abel Summability.*

The first of our two theorems follows almost immediately from Hölder's inequality.

THEOREM 1. *If* Σa_n *is summable* $[A; p]$ *it is also summable* $[A; q]$ *for* $p > q \geqslant 1$.

It is sufficient to show that, when $p > q \geqslant 1$,

$$\int_1^\omega |uf'(u)|^p\, du = o(\omega)$$

implies that $$\int_1^\omega |uf'(u)|^q \, du = o(\omega).$$

Denote $|uf'(u)|$ by $g(u)$. Then, by Hölder's inequality,

$$\int_1^\omega g^q \, du \leqslant \left\{ \int_1^\omega (g^q)^{p/q} du \right\}^{q/p} \left\{ \int_1^\omega 1 du \right\}^{1-q/p}$$
$$= o(\omega^{q/p}) \, O(\omega^{1-q/p}) = o(\omega).$$

4. *Strong Cesàro and Abel Summability.* The following is the main theorem:

THEOREM 2. *If* Σa_n *is summable* $[C; k, p]$, $k \geqslant 0$, $p \geqslant 1$, *then it is summable* $[A; p]$ *to the same sum.*

We suppose throughout, as we may without loss of generality, that the sum of the series in the Cesàro sense is zero. Thus $c_n^{(k)} = o(1)$ as $n \to \infty$.

Clearly conditions (i) and (ii) in the definition of summability $[A; p]$ are satisfied. It is only necessary therefore to show that the hypothesis implies (iii). We consider four cases separately.

Case (i), $k > 0$, $p > 1$. We have, by Lemma 1,

$$\int_1^\omega |uf'(u)|^p \, du = \int_1^\omega u^{-p} \left| \sum_{\nu=1}^{\infty} \nu a_\nu e^{-\nu/u} \right|^p du$$
$$= \int_1^\omega u^{-p} (1-e^{-1/u})^{kp} \left| \sum_{\nu=1}^{\infty} \nu E_\nu^{(k)} a_\nu^{(k)} e^{-\nu/u} \right|^p du$$
$$= O\left[\int_1^\omega \left\{ \left| u^{-k-1} \sum_{\nu=1}^{n} \nu E_\nu^{(k)} a_\nu^{(k)} e^{-\nu/u} \right| + \left| u^{-k-1} \sum_{\nu=n+1}^{\infty} \nu E_\nu^{(k)} a_\nu^{(k)} e^{-\nu/u} \right| \right\}^p du \right]$$
$$= O\left\{ \int_1^\omega \left| u^{-k-1} \sum_{\nu=1}^{n} \nu E_\nu^{(k)} a_\nu^{(k)} e^{-\nu/u} \right|^p du \right\}$$
$$+ O\left\{ \int_1^\omega \left| u^{-k-1} \sum_{\nu=n+1}^{\infty} \nu E_\nu^{(k)} a_\nu^{(k)} e^{-\nu/u} \right|^p du \right\},$$

by Lemma 7. Denote these integrals respectively by $I_1(\omega)$ and $I_2(\omega)$. Then, writing ϕ_ν for $|\nu a_\nu^{(k)}|$, we have

$$I_1(\omega) = O\left[\int_1^\omega \left\{ \sum_{\nu=1}^{n} \nu^k u^{-k-1} e^{-\nu/u} \phi_\nu \right\}^p du \right]$$
$$= O[I_{1,1}(\omega) + I_{1,2}(\omega) + I_{1,3}(\omega)],$$

where

$$I_{1,1}(\omega) = \int_1^\omega \left\{ \sum_{\nu=1}^{m-1} \nu^k u^{-k-1} e^{-\nu/u} \phi_\nu \right\}^p du,$$

$$I_{1,2}(\omega) = \int_1^\omega u^{-p(k+1)} \{ m^k e^{-m/u} \phi_m + (m+1)^k e^{-(m+1)/u} \phi_{m+1} \}^p \, du,$$

$$I_{1,3}(\omega) = \int_1^\omega \left\{ \sum_{\nu=m+2}^{n} \nu^k u^{-k-1} e^{-\nu/u} \phi_\nu \right\}^p du.$$

By Lemma 5, $I_{1,1}(\omega)$ and $I_{1,3}(\omega)$ are each equal to $O\left\{\sum_{\nu=0}^{n} \phi_\nu{}^p\right\}$ which, by hypothesis, is $o(n)$, or $o(\omega)$. Also

$$I_{1,2}(\omega) \leqslant \int_1^\omega \{(ku)^k e^{-k} u^{-k-1}\}^p \, (\phi_m+\phi_{m+1})^p \, du.$$

Clearly by hypothesis $\phi_m{}^p$, ϕ_{m+1}^p are each $o(\omega)$. Hence

$$I_{1,2}(\omega) = o\left\{\omega \int_1^\omega u^{-p}\, du\right\} = o\left\{\omega \int_1^\infty u^{-p}\, du\right\} = o(\omega).$$

Returning now to $I_2(\omega)$ we have

$$\begin{aligned}\left|\sum_{\nu=n+1}^{\infty} \nu E_\nu^{(k)} a_\nu^{(k)} e^{-\nu/u}\right| &= \left|\sum_{\nu=n+1}^{\infty} \nu E_\nu^{(k)} \{c_\nu^{(k)} - c_{\nu-1}^{(k)}\} e^{-\nu/u}\right| \\ &= \left|\sum_{\nu=n+1}^{\infty} \nu E_\nu^{(k)} c_\nu^{(k)} e^{-\nu/u} - \sum_{\nu=n+1}^{\infty} (\nu-1) E_{\nu-1}^{(k)} c_{\nu-1}^{(k)} e^{-\nu/u}\right. \\ &\qquad \left. - \sum_{\nu=n+1}^{\infty} (k+1) E_{\nu-1}^{(k)} c_{\nu-1}^{(k)} e^{-\nu/u}\right| \\ &\leqslant \left|(1-e^{-1/u}) \sum_{\nu=n+1}^{\infty} \nu E_\nu^{(k)} c_\nu^{(k)} e^{-\nu/u}\right| + |n E_n^{(k)} c_n^{(k)} e^{-n/u}| \\ &\qquad + \left|\sum_{\nu=n+1}^{\infty} (k+1) E_{\nu-1}^{(k)} c_{\nu-1}^{(k)} e^{-\nu/u}\right|,\end{aligned}$$

and we must now show that the three integrals

$$I_{2,1}(\omega) = \int_1^\omega \left| u^{-k-1} (1-e^{-1/u}) \sum_{\nu=n+1}^{\infty} \nu E_\nu^{(k)} c_\nu^{(k)} e^{-\nu/u}\right|^p du,$$

$$I_{2,2}(\omega) = \int_1^\omega |u^{-k-1} n E_n^{(k)} c_n^{(k)} e^{-n/u}|^p \, du,$$

$$I_{2,3}(\omega) = \int_1^\omega \left| u^{-k-1} \sum_{\nu=n+1}^{\infty} E_{\nu-1}^{(k)} c_{\nu-1}^{(k)} e^{-\nu/u}\right|^p du,$$

are each $o(\omega)$.

We use the fact that $c_\nu^{(k)} = o(1)$ as $\nu \to \infty$.

Dealing first with $I_{2,1}(\omega)$, we have

$$\begin{aligned} I_{2,1}(\omega) &= o\left[\int_1^\omega u^{-p(k+2)}\, du \left\{\sum_{\nu=n+1}^{\infty} \nu^{k+1} e^{-\nu/u}\right\}^p\right] \\ &= o\left[\int_1^\omega u^{-p(k+2)}\, du \left\{\int_n^\infty x^{k+1} e^{-x/u}\, dx + O(n^{k+1} e^{-n/u})\right\}^p\right] \\ &= o\left[\int_1^\omega du \left\{\int_{n/u}^\infty y^{k+1} e^{-y}\, dy\right\}^p\right] + o\left[\omega^{p(k+1)} \int_1^\omega u^{-p(k+2)} e^{-pn/u}\, du\right].\end{aligned}$$

In the first of these expressions, we may replace the lower limit of the inner integral by zero, and the expression is clearly $o(\omega)$. By means of the substitution $y = pn/u$ it is easy to see that the second expression is $o(\omega^{1-p}) = o(\omega)$.

Also
$$I_{2,2}(\omega) = o\left\{\omega^{p(k+1)} \int_1^\omega u^{-p(k+1)} e^{-pn/u}\, du\right\},$$

and the substitution $y = pn/u$ shows that this is also $o(\omega)$.

Finally
$$\begin{aligned} I_{2,3}(\omega) &= o\left[\int_1^\omega u^{-p(k+1)} du \left\{\sum_{\nu=n+1}^{\infty} \nu^k e^{-\nu/u}\right\}^p\right] \\ &= o\left[\int_1^\omega u^{-p(k+1)} du \left\{\int_{n+1}^{\infty} x^k e^{-x/u}\, dx\right\}^p\right] \\ &= o(\omega). \end{aligned}$$

The theorem is therefore proved for $k > 0$, $p > 1$.

Case (ii), $k = 0$, $p > 1$. We proceed as in Case (i) but replace n by r, where $r = [\omega]$. The proof that

$$I_2(\omega) = \int_1^\omega \left| u^{-1} \sum_{\nu=r+1}^{\infty} \nu a_\nu e^{-\nu/u} \right|^p du = o(\omega)$$

is unaltered, except that $c_\nu^{(k)}$ is replaced by A_ν, where $A_\nu = \sum_{\mu=0}^{\nu} a_\mu$. Also, by Lemma 6,

$$\begin{aligned} I_1(\omega) &= O\left[\int_1^\omega \left\{\sum_{\nu=1}^{r} u^{-1} e^{-\nu/u} \nu |a_\nu|\right\}^p du\right] \\ &= O\left\{\sum_{\nu=1}^{r} (\nu |a_\nu|)^p\right\} = o(r) = o(\omega), \end{aligned}$$

by hypothesis.

An independent proof of this case is not strictly necessary since summability $[C;\ 0, p]$ implies[1] summability $[C;\ k, p]$, $k > 0$, $p \geqslant 1$.

Case (iii), $k > 0$, $p = 1$. In this case we have

$$\begin{aligned} I_1(\omega) &= O\left(\int_1^\omega du \sum_{\nu=1}^{n} u^{-k-1} \nu^k e^{-\nu/u} \nu |a_\nu^{(k)}|\right) \\ &= O\left(\sum_{\nu=1}^{n} \nu^{k+1} |a_\nu^{(k)}| \int_1^\omega u^{-k-1} e^{-\nu/u}\, du\right) \\ &= O\left(\sum_{\nu=1}^{n} \nu |a_\nu^{(k)}| \int_{\nu/\omega}^{\nu} y^{k-1} e^{-y}\, dy\right) \\ &= o(n) = o(\omega). \end{aligned}$$

For $I_2(\omega)$ we merely replace p by unity throughout the argument in Case (i).

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[1] Hyslop, 2.

Case (iv), $k=0$, $p=1$. The truth of the theorem in this case may be inferred from Case (iii) and the consistency theorem for strong Cesàro summability which has been quoted above. For the sake of completeness, however, and because the preceding arguments require modification in this case, we think it desirable to insert a short independent proof.

The proof that $I_2(\omega)=o(\omega)$ presents no difficulty. Also, if $r=[\omega]$ and $0<\delta<1$, we have

$$\begin{aligned} I_1(\omega) &= O\left(\int_1^\omega u^{-1}\,du \sum_{\nu=1}^{r} \nu|a_\nu| e^{-\nu/u}\right) \\ &= O\left(\omega^\delta \sum_{\nu=1}^{r} \nu|a_\nu| \int_1^\omega u^{-1-\delta} e^{-\nu/u}\,du\right) \\ &= O\left(\omega^\delta \sum_{\nu=1}^{r} \nu^{1-\delta}|a_\nu| \int_0^\infty y^{\delta-1} e^{-y}\,dy\right) \\ &= O\left(\omega^\delta \sum_{\nu=1}^{r} \nu^{1-\delta}|a_\nu|\right). \end{aligned}$$

Denoting[1] $\sum_{\mu=0}^{\nu} \mu|a_\mu|$ by B_ν and noting that $B_\nu=o(\nu)$ by hypothesis, we have, on summing by parts,

$$\begin{aligned} \omega^\delta \sum_{\nu=1}^{r} \nu^{1-\delta}|a_\nu| &= \omega^\delta \sum_{\nu=1}^{r-1} B_\nu\{\nu^{-\delta}-(\nu+1)^{-\delta}\}+B_r(\omega/r) \\ &= O\left(\omega^\delta \sum_{\nu=1}^{r-1} \nu^{-\delta-1} B_\nu\right)+o(\omega) \\ &= o\left(\omega^\delta \sum_{\nu=1}^{r-1} \nu^{-\delta}\right)+o(\omega) \\ &= o(\omega^\delta r^{1-\delta})+o(\omega)=o(\omega). \end{aligned}$$

The theorem is therefore completely proved.

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[1] The subsequent argument is substantially due to Winn. See Winn, **7**.

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