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Sufficient conditions for non-zero entropy of closed relations

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Abstract. We introduce the notions of returns and well-aligned sets for closed relations on compact metric spaces and then use them to obtain non-trivial sufficient conditions for such a relation to have non-zero entropy. In addition, we give a characterization of finite relations with non-zero entropy in terms of Li–Yorke and DC2 chaos.

Key words: closed relations, Mahavier products, dynamical systems, entropy, Li-Yorke chaos, distributional chaos
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1. *Introduction*

In topological dynamics, the study of chaotic behaviour of a dynamical system is often based on some properties of continuous functions. One of the frequently studied properties of such functions in the theory of topological dynamical systems is the entropy of a continuous function $f : X \to X$ on a compact metric space X, which serves as a measure of the complexity of the dynamical system. This often leads to studying the entropy of the shift map σ on the inverse limit $\varprojlim(X, f)$. More precisely, suppose X is a compact metric space. If $f : X \to X$ is a continuous function, the inverse limit space generated by f is the subspace of $\prod_{i=0}^{\infty} X$ equipped with the usual product topology, given by

$$\varprojlim(X, f) := \left\{(x_0, x_1, x_2, x_3, \ldots) \in \prod_{i=0}^{\infty} X \mid \text{for each non-negative integer } i, x_i = f(x_{i+1})\right\},$$

also abbreviated as $\varprojlim f$. The map f on X induces a natural homeomorphism σ on $\varprojlim f$, called the shift map, defined by

$$\sigma(x_0, x_1, x_2, x_3, x_4, \ldots) = (x_1, x_2, x_3, x_4, \ldots)$$

for each $(x_0, x_1, x_2, x_3, x_4, \ldots)$ in $\varprojlim f$. Note that the shift map σ is the inverse function to the map defined by $(x_0, x_1, x_2, \ldots) \mapsto (f(x_0), x_0, x_1, x_2, \ldots)$.

To study such inverse limits $\varprojlim f$ and shift maps $\sigma : \varprojlim f \to \varprojlim f$, the study of backward orbits of points of dynamical systems (X, f) is also required; note that the inverse limit $\varprojlim f$ is the space of all backward orbits in (X, f). Such backward orbits of points are actually forward orbits of points in the dynamical system (X, f^{-1}) if f^{-1} is well defined. However, usually, f^{-1} is not a well-defined function; therefore, a more general tool is needed to study these properties. Note that for a continuous function $f : X \to X$, the set

$$\Gamma(f)^{-1} = \{(y, x) \in X \times X \mid y = f(x)\}$$

is a closed relation on X that describes best the dynamics of (X, f) in the backward direction when f^{-1} is not well defined. So, generalizing topological dynamical systems (X, f) to topological dynamical systems (X, F) with closed relations F on X by making the identification $(X, f) = (X, \Gamma(f))$ is only natural. So, what we are engaged in is the replacement of a map $f : X \to X$ by a closed relation F on X.

Recently, many such generalizations of dynamical systems were introduced and studied (see [1, 2, 7, 12, 14–19, 22], where more references may be found). However, there is not much known of such dynamical systems and, therefore, there are many properties of such set-valued dynamical systems that are yet to be studied. In [2], the notion of topological entropy $h(f)$ of continuous functions $f : X \to X$ on compact metric spaces X was generalized to the notion of topological entropy $\text{ent}(F)$ of closed relations F on compact metric spaces X. In this paper, we continue our research from [2]. We introduce the notions of returns and well-aligned sets for closed relations on compact metric spaces, and then use them to obtain non-trivial sufficient conditions for such relations to have non-zero entropy. In addition, we give a characterization of finite relations with non-zero entropy. We also show that, unlike topological entropy for closed relations on compact metric spaces in general, in the case of finite relations, positive entropy is equivalent to the shift map on the Mahavier product being Li–Yorke chaotic as well as equivalent to DC-2 distributional chaos for the shift map, as well as equivalent to F having a (k, ε)-return.

When constructing a closed relation F on a compact metric space X, a standard topological dynamical system (X_F^+, σ), where X_F^+ is the Mahavier product of the relation F and σ is the shift map on X_F^+, is constructed at the same time. This is a new way of constructing topological dynamical systems with interesting properties. Therefore, it is only natural to study the properties of the relation F that imply interesting topological or dynamical properties of the dynamical system (X_F^+, σ). Here, we study the properties of the closed relation F on X (such as returns and well-aligned sets) that imply that the shift map σ on X_F^+ has positive entropy. Then, for example, one can apply these results to obtain mappings on compact metric spaces with non-zero entropy. Oprocha constructed in [21] a transitive zero-entropy homeomorphism on the Lelek fan. He asked then if there existed

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also a transitive non-zero entropy homeomorphism on the Lelek fan [**21**, Question 3.11]. In [**3**], Banič, Erceg and Kennedy constructed a transitive homeomorphism on the Lelek fan. The construction was based on producing a closed relation F on $X = [0, 1]$ in such a way that the Mahavier product X_F^+ is a Lelek fan and the shift map σ on it is transitive. In [**2**], the authors showed that the entropy of the relation F from [**3**] has positive entropy. In Example 4.12, we use the introduced theory of returns to shorten and simplify that proof from [**2**] and then we use Theorem 3.16 to show that this implies that the shift map σ on X_F^+ from [**3**] has non-zero entropy. This gives an answer to Oprocha's question [**21**, Question 3.11] in the affirmative. The same technique of closed relations on compact metric spaces was used later in [**4**] to produce a transitive homeomorphism on the Cantor fan. The notions of returns can be used to give simple proofs that a transitive homeomorphism on the Cantor fan from [**4**] also has positive entropy. See Examples 4.10 and 4.11.

We proceed as follows. In §2, basic definitions and notation that are needed later in the paper are given and presented. In §3, the topological entropy for closed relations is defined and in addition, basic results from [**2**] are presented. In §4, our first main result as well as illustrative examples and corollaries are given and proved. In our last section, §5, we restrict ourselves to finite relations on compact metric spaces. Here, our second main result, a characterization of finite relations with non-zero entropy, is presented and proved.

2. *Definitions and notation*

First, we define some properties from the continuum theory and the theory of inverse limits that are used later in the paper.

Definition 2.1. Let X be a non-empty compact metric space. We always use ρ to denote the metric on X.

Definition 2.2. Suppose X is a non-empty compact metric space. If for each non-negative integer n, $f_n : X \to X$ is a continuous function, *the inverse limit space generated by* (f_n) is the subspace of $\prod_{i=0}^{\infty} X$ equipped with the usual product topology, given by

$$\varprojlim(X, f_n) := \left\{(x_0, x_1, x_2, x_3, \ldots) \in \prod_{i=0}^{\infty} X \mid \text{for each non-negative integer } i,\, x_i = f_i(x_{i+1})\right\}.$$

Definition 2.3. *A continuum* is a non-empty connected compact metric space. A continuum is *degenerate* if it consists of only a single point. Otherwise it is *non-degenerate*. *A subcontinuum* is a subspace of a continuum which itself is a continuum.

Next, we define chainable continua (using inverse limits); see [**20**, §XII] for more details.

Definition 2.4. A continuum X is *chainable* if there is a sequence (f_n) of continuous surjections $f_n : [0, 1] \to [0, 1]$ such that X is homeomorphic to $\varprojlim([0, 1], f_n)$.

Definition 2.5. A continuum X is *decomposable* if there are proper subcontinua A and B of X $(A, B \neq X)$ such that $X = A \cup B$. A continuum is *indecomposable* if it is not decomposable. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable.

Definition 2.6. *A pseudoarc* is any non-degenerate hereditarily indecomposable chainable continuum.

Bing showed in [5] that any two pseudoarcs are homeomorphic. Next, we present basic definitions and well-known results about closed relations and Mahavier products.

Definition 2.7. We use:

(1) $\mathbb{N}$ to denote the set of positive integers;
(2) $\mathbb{N}_m$ to denote the set $\{1, 2, 3, \ldots, m\}$ for each positive integer m;
(3) $\mathbb{Z}_+$ to denote the set of non-negative integers; and
(4) $\mathbb{Z}$ to denote the set of integers.

Let m be a positive integer. For any set X, we use:

(1) X^m to denote the Cartesian product $\prod_{i=0}^{m-1} X$;
(2) $X^{\mathbb{Z}_+}$ to denote the Cartesian product $\prod_{i=0}^{\infty} X$.

For each non-negative integer i and for each $\mathbf{x} = (x_0, x_1, x_2, \ldots) \in X^{\mathbb{Z}_+}$ (or $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{m-1}) \in X^m$), we use $\pi_i(\mathbf{x})$ or $\mathbf{x}(i)$ or $\mathbf{x}_i$ to denote the ith coordinate x_i of the point $\mathbf{x}$. For all non-negative integers k and ℓ such that $k \leq \ell$, we use $[k, \ell]$ to denote the set $\{k, k+1, k+2, \ldots, \ell\}$ and $\pi_{[k,\ell]} : \prod_{i=0}^{\infty} X \to \prod_{i=k}^{\ell} X$ to denote the projection that is defined by $\pi_{[k,\ell]}(x_0, x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_\ell, x_{\ell+1}, \ldots) = (x_k, x_{k+1}, x_{k+2}, \ldots, x_\ell)$. For $k = \ell$, we use π_k to denote the projection $\pi_{[k,k]}$. For a non-empty compact metric space X, we use $p_1 : X \times X \to X$ and $p_2 : X \times X \to X$ to denote *the standard projections* defined by $p_1(s, t) = s$ and $p_2(s, t) = t$ for all $(s, t) \in X \times X$.

Definition 2.8. Let X and Y be metric spaces, and let $f : X \to Y$ be a function. We use $\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$ to denote *the graph of the function f*.

Definition 2.9. Let X be a non-empty compact metric space and let $F \subseteq X \times X$ be a relation on X. If F is closed in $X \times X$, then we say that F is *a closed relation on X*.

Definition 2.10. Let X be a set and let F be a relation on X. Then we define $F^{-1} = \{(y, x) \in X \times X \mid (x, y) \in F\}$ to be *the inverse relation of the relation F on X*.

Definition 2.11. Let X be a non-empty compact metric space and let F be a closed relation on X. Then we call

$$X_F^m = \left\{(x_0, x_1, x_2, x_3, \ldots, x_m) \in \prod_{i=0}^{m} X \mid \text{for each } i \in \{0, 1, 2, \ldots, m-1\}, (x_i, x_{i+1}) \in F\right\}$$

for each positive integer m, *the mth Mahavier product of F*, and we call

$$X_F^+ = \left\{(x_0, x_1, x_2, x_3, \ldots) \in \prod_{i=0}^{\infty} X \mid \text{for each non-negative integer } i, (x_i, x_{i+1}) \in F\right\}$$

the infinite Mahavier product of F.

In our previous papers, we have also used $\star_{i=0}^{m-1} F$ to denote the mth Mahavier product and $\star_{i=0}^{\infty} F$ to denote the infinite Mahavier product of F.

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Definition 2.12. Let X be a non-empty compact metric space and let F be a closed relation on X. The function $\sigma : X_F^+ \to X_F^+$, defined by

$$\sigma(x_0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)$$

for each $(x_0, x_1, x_2, x_3, \ldots) \in X_F^+$, is called *the shift map on* X_F^+.

Definition 2.13. We use Σ_2 to denote the set

$$\Sigma_2 = \{\mathbf{s} : \mathbb{Z}_+ \to \{0, 1\}\}$$

and for each non-negative integer j, we use Σ_2^j to denote the set

$$\Sigma_2^j = \{\mathbf{s} : [0, j] \to \{0, 1\}\}.$$

Finally, we define the operation $\star$ on Mahavier products.

Definition 2.14. Let X be a non-empty compact metric space and let F be a closed relation on X. Also, let $m, n \in \mathbb{Z}_+$ and let $(x_0, x_1, x_2, \ldots, x_n) \in X_F^n$ and $(y_0, y_1, y_2, \ldots, y_m) \in X_F^m$ be such that $x_n = y_0$. Then we define $(x_0, x_1, x_2, \ldots, x_n) \star (y_0, y_1, y_2, \ldots, y_m)$ by

$$(x_0, x_1, x_2, \ldots, x_n) \star (y_0, y_1, y_2, \ldots, y_m) = (x_0, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m).$$

Observation 2.1. Let X be a non-empty compact metric space and let F be a closed relation on X. Also, let $m, n \in \mathbb{Z}_+$, and let $(x_0, x_1, x_2, \ldots, x_n) \in X_F^n$ and $(y_0, y_1, y_2, \ldots, y_m) \in X_F^m$ be such that $x_n = y_0$. Then $(x_0, x_1, x_2, \ldots, x_n) \star (y_0, y_1, y_2, \ldots, y_m) \in X_F^{n+m}$.

3. *Topological entropy of closed relations on compact metric spaces*

In this section, we summarize the generalization of topological entropy to closed relations on a compact metric space introduced in [2], where the entropy of a closed relation F on X is defined using Mahavier products $X_{F^{-1}}^m$. To simplify the notion of entropy of closed relations and to avoid any possible confusion when dealing with F and F^{-1} at the same time, we define here the entropy of F using Mahavier products X_F^m. This new definition is equivalent to the definition of entropy from [2].

Definition 3.1. Let X be a non-empty compact metric space and let $\mathcal{S}$ be a family of subsets of X. We use $|\mathcal{S}|$ to denote the cardinality of $\mathcal{S}$.

Definition 3.2. Let X be a non-empty compact metric space and let $\mathcal{S}$ be a family of subsets of X. For each positive integer n, we use $\mathcal{S}^n$ to denote the family

$$\mathcal{S}^n = \{S_0 \times S_1 \times S_2 \times \cdots \times S_{n-1} \mid S_0, S_1, S_2, \ldots, S_{n-1} \in \mathcal{S}\}.$$

We call the elements $S_0 \times S_1 \times S_2 \times \cdots \times S_{n-1}$ of $\mathcal{S}$ *the n-boxes* (*generated by the family* $\mathcal{S}$).

Definition 3.3. Let X be a non-empty compact metric space and let $\mathcal{U}$ be an open cover for X. We use $N(\mathcal{U})$ to denote

$$N(\mathcal{U}) = \min\{|\mathcal{V}| \mid \mathcal{V} \text{ is a finite subset of } \mathcal{U} \text{ that covers } X\}.$$

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Definition 3.4. Let X be a non-empty compact metric space, let n be a positive integer, let K be a closed subset of the product X^n and let $\mathcal{U}$ be a non-empty family of open subsets of X^n such that $K \subseteq \bigcup \mathcal{U}$. We use $N(K, \mathcal{U})$ to denote

$$N(K, \mathcal{U}) = \min \left\{ |\mathcal{V}| \mid \mathcal{V} \text{ is a non-empty subfamily of } \mathcal{U} \text{ such that } K \subseteq \bigcup \mathcal{V} \right\}.$$

Observation 3.1. Let X be a non-empty compact metric space, let K be a closed subset of the product X^n and let $\mathcal{U}$ be a non-empty family of open subsets of X^n such that $K \subseteq \bigcup \mathcal{U}$. Note that even in the case when $K = \emptyset$, $N(K, \mathcal{U})$ is a positive integer since $|\mathcal{V}| \geq 1$ for each non-empty subfamily $\mathcal{V}$ of $\mathcal{U}$. Also, note that for each positive integer m and for each open cover α for X,

$$1 \leq N(X_F^m, \alpha^{m+1}) \leq (N(\alpha))^{m+1},$$

and that for some positive integer m, X_F^m may be empty. Note that even in this case,

$$1 \leq N(X_F^m, \alpha^{m+1}) \leq (N(\alpha))^{m+1}.$$

Observation 3.2. Let X be a non-empty compact metric space, let F be a closed relation on X such that $p_2(F) \subseteq p_1(F)$ and let α be an open cover for X. Note that for each positive integer m,

$$N(X_F^m, \alpha^{m+1}) \leq N(X_F^{m+1}, \alpha^{m+2}).$$

LEMMA 3.3. *Let X be a non-empty compact metric space, let F be a closed relation on X and let α be an open cover for X. Then*

$$N(X_F^{m+n}, \alpha^{m+n+1}) \leq N(X_F^m, \alpha^{m+1}) \cdot N(X_F^n, \alpha^{n+1})$$

for all positive integers m and n.

Proof. Let m and n be positive integers, and let:

(1) $N(X_F^m, \alpha^{m+1}) = k_m$ and let $\gamma = \{\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_{k_m-1}\}$ be a subfamily of α^{m+1} such that $X_F^m \subseteq \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{k_m-1}$; and

(2) $N(X_F^n, \alpha^{n+1}) = k_n$ and let $\delta = \{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_{k_n-1}\}$ be a subfamily of α^{n+1} such that $X_F^n \subseteq \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_{k_n-1}$.

Observe that

$$X_F^m \times X_F^n \subseteq \bigcup_{i=0}^{k_m-1} \bigcup_{j=0}^{k_n-1} \Gamma_i \times \Delta_j.$$

Let $\pi : X_F^m \times X_F^n \to X_F^m \times X_F^{n-1}$ be the projection which omits the first coordinate in X_F^n:

$$\pi((x_0, x_1, x_2, \ldots, x_m), (y_0, y_1, y_2, \ldots, y_n)) = ((x_0, x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_n))$$

for each $((x_0, x_1, x_2, \ldots, x_m), (y_0, y_1, y_2, \ldots, y_n)) \in X_F^m \times X_F^n$. Also, let

$$Z = \{(\mathbf{x}, \mathbf{y}) \in X_F^m \times X_F^n \mid \mathbf{x}_m = \mathbf{y}_0\}.$$

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Then $\pi(Z)$ is homeomorphic to X_F^{m+n}. Furthermore, note that for each $i \in \{0, 1, 2, \ldots, k_m - 1\}$ and for each $j \in \{0, 1, 2, \ldots, k_n - 1\}$,

$$\pi(\Gamma_i \times \Delta_j) \subseteq \alpha^{m+n+1}$$

and the result follows. □

We use the following lemma in the proof of Theorem 3.5.

LEMMA 3.4. *Let (a_m) be a sequence in $\mathbb{R}$ such that:*

(1) *for each positive integer m, $a_m \geq 0$; and*

(2) *for all positive integers m and n, $a_{m+n} \leq a_m + a_n$.*

Then the limit $\lim_{m\to\infty} a_m/m$ exists and

$$\lim_{m\to\infty} \frac{a_m}{m} = \inf\left\{\frac{a_m}{m} \mid m \text{ is a positive integer}\right\}.$$

Proof. The proof can be found in [**24**, Theorem 4.9, p. 87]. □

THEOREM 3.5. *Let X be a non-empty compact metric space, let F be a closed relation on X and let α be an open cover for X. Then the limit*

$$\lim_{m\to\infty} \frac{\log N(X_F^m, \alpha^{m+1})}{m}$$

exists.

Proof. For each positive integer m, let

$$a_m = \log N(X_F^m, \alpha^{m+1}).$$

By Observation 3.1, for each positive integer m, $N(X_F^m, \alpha^{m+1})$ is a positive integer. It follows that for each positive integer m, $a_m \geq 0$. Next, we prove that the above limit exists. By Lemma 3.4, it suffices to show that for all positive integers m and n, $a_{m+n} \leq a_m + a_n$. It follows (using Lemma 3.3) that for all positive integers m and n,

$$\begin{aligned} a_{m+n} &= \log N(X_F^{m+n}, \alpha^{m+n+1}) \leq \log(N(X_F^m, \alpha^{m+1}) \cdot N(X_F^n, \alpha^{n+1})) \\ &= \log N(X_F^m, \alpha^{m+1}) + \log N(X_F^n, \alpha^{n+1}) = a_m + a_n. \end{aligned}$$

□

Definition 3.5. Let X be a non-empty compact metric space, let F be a closed relation on X and let α be an open cover for X. We define *the entropy of F with respect to the open cover α* by

$$\text{ent}(F, \alpha) = \lim_{m\to\infty} \frac{\log N(X_F^m, \alpha^{m+1})}{m}.$$

Observation 3.6. Let X be a non-empty compact metric space, let F be a closed relation on X and let α be an open cover for X. Since X is non-empty, it follows that α is non-empty. Therefore, $\log N(X_F^m, \alpha^{m+1})$ is well defined and is a non-negative number (even in the case when $X_F^m = \emptyset$ for some positive integer m), since $N(X_F^m, \alpha^{m+1})$ is

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defined in Definition 3.4 as

$$N(X_F^m, \alpha^{m+1}) = \min\left\{|\mathcal{V}| \mid \mathcal{V} \textit{ is a non-empty subfamily of } \alpha^{m+1} \text{ such that } X_F^m \subseteq \bigcup \mathcal{V}\right\}.$$

Therefore, the entropy $\mathrm{ent}(F, \alpha)$ with respect to the open cover α is always well defined – also in the case when $X_F^m = \emptyset$ for some positive integer m. Note that if for some positive integer m_0, $X_F^{m_0} = \emptyset$, then for each $m \geq m_0$, also $X_F^m = \emptyset$. It follows that in this case, $N(X_F^m, \alpha^{m+1}) = 1$ and, therefore,

$$\mathrm{ent}(F, \alpha) = \lim_{m\to\infty} \frac{\log N(X_F^m, \alpha^{m+1})}{m} = \lim_{m\to\infty} \frac{\log 1}{m} = 0.$$

Definition 3.6. Let X be a metric space and let $\mathcal{S}$ and $\mathcal{T}$ be families of subsets of X. We say that the family $\mathcal{S}$ *refines* the family $\mathcal{T}$ if for each $S \in \mathcal{S}$, there is $T \in \mathcal{T}$ such that $S \subseteq T$. The notation $\mathcal{T} \leq \mathcal{S}$ means that the family $\mathcal{S}$ refines the family $\mathcal{T}$.

PROPOSITION 3.7. [2, Proposition 1] *Let X be a non-empty compact metric space and let F be a closed relation on X. For all non-empty open covers α and β,*

$$\alpha \leq \beta \Longrightarrow \mathrm{ent}(F, \alpha) \leq \mathrm{ent}(F, \beta).$$

Proof. Let α and β be any open covers for X such that $\alpha \leq \beta$. Then $\alpha^n \leq \beta^n$ for each positive integer n. Let m be a positive integer, let $k = N(X_F^m, \beta^{m+1})$ and let $\{B_0, B_1, B_2, \ldots, B_{k-1}\} \subseteq \beta^{m+1}$ be such that $X_F^m \subseteq B_0 \cup B_1 \cup B_2 \cup \cdots \cup B_{k-1}$. For each $i \in \{0, 1, 2, \ldots, k-1\}$, let $A_i \in \alpha$ such that $B_i \subseteq A_i$. Therefore, $X_F^m \subseteq A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_{k-1}$ and $N(X_F^m, \alpha^{m+1}) \leq N(X_F^m, \beta^{m+1})$ follows. Therefore, $\mathrm{ent}(F, \alpha) \leq \mathrm{ent}(F, \beta)$. □

PROPOSITION 3.8. [2, Proposition 2] *Let X be a non-empty compact metric space and let α be a non-empty open cover for X. For all closed relations F and G on X,*

$$F \subseteq G \Longrightarrow \mathrm{ent}(F, \alpha) \leq \mathrm{ent}(G, \alpha).$$

Proof. The proposition follows from the fact that $N(X_F^m, \alpha^{m+1}) \leq N(X_G^m, \alpha^{m+1})$ for each positive integer m. □

Definition 3.7. Let X be a non-empty compact metric space, let F be a closed relation on X and let $E = \{\mathrm{ent}(F, \alpha) \mid \alpha \text{ is an open cover for } X\}$. We define *the entropy of F* by

$$\mathrm{ent}(F) = \begin{cases} \sup(E); & E \text{ is bounded in } \mathbb{R}, \\ \infty; & E \text{ is not bounded in } \mathbb{R}. \end{cases}$$

Observation 3.9. Let X be a non-empty compact metric space and let F be a closed relation on X. Note that if for some positive integer m_0, $X_F^{m_0} = \emptyset$, then, as seen in Observation 3.6, for each open cover α for X, $\mathrm{ent}(F, \alpha) = 0$. Therefore, in this case, $E = \{\mathrm{ent}(F, \alpha) \mid \alpha \text{ is an open cover for } X\} = \{0\}$ and since E is bounded in $\mathbb{R}$, it follows that $\mathrm{ent}(F) = \sup(E) = \sup\{0\} = 0$.

The following theorems summarize the basic properties of $\mathrm{ent}(F)$.

THEOREM 3.10. *Let X be a non-empty compact metric space. For all closed relations F and G on X,*

$$F \subseteq G \Longrightarrow \mathrm{ent}(F) \leq \mathrm{ent}(G).$$

Proof. The theorem follows directly from Proposition 3.8. □

THEOREM 3.11. *Let X be a non-empty compact metric space and let F be a closed relation on X. Then*

$$\mathrm{ent}(F^{-1}) = \mathrm{ent}(F).$$

Proof. Let T be the homeomorphism on X^{m+1} which reverses the order of the coordinates. Then $T(X_F^m) = X_{F^{-1}}^m$ and $T(X_{F^{-1}}^m) = X_F^m$. Note that for any subset U of α^{m+1} which covers X_F^m, the set $T(U)$ is a subset of α^{m+1} which covers $X_{F^{-1}}^m$, and that for any subset U of α^{m+1} which covers $X_{F^{-1}}^m$, the set $T(U)$ is a subset of α^{m+1} which covers X_F^m. The result follows. □

In Theorem 3.12, we show that the entropy of closed relations F on a non-empty compact metric space X is a generalization of the well-known topological entropy of continuous functions $f : X \to X$. Before stating and proving the theorem, we give the following definitions.

Definition 3.8. Let X be a set, let $f : X \to X$ be a function and let $\mathcal{S}$ be a family of subsets of X. Then we define $f^{-1}(\mathcal{S}) = \{f^{-1}(S) \mid S \in \mathcal{S}\}$.

Definition 3.9. Let X be a set and let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m-1}$ be families of subsets of X. Then we define

$$\vee_{i=0}^{m-1} \mathcal{A}_i = \{A_0 \cap A_1 \cap A_2 \cap \cdots \cap A_{m-1} \mid \text{for each } i \in \{0, 1, 2, \ldots, m-1\}, A_i \in \mathcal{A}_i\}.$$

Definition 3.10. Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. For any open cover α for X, we define

$$h(f, \alpha) = \lim_{m \to \infty} \frac{\log(N(\vee_{i=0}^{m} f^{-i}(\alpha)))}{m}.$$

Definition 3.11. Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. Also, let $E = \{h(f, \alpha) \mid \alpha \text{ is an open cover for } X\}$. Then we define

$$h(f) = \begin{cases} \sup(E); & E \text{ is bounded in } \mathbb{R}, \\ \infty; & E \text{ is not bounded in } \mathbb{R}, \end{cases}$$

to be *the entropy of the function f*.

The following theorem shows that the topological entropy of a continuous function $f : X \to X$ on compact metric spaces X is just a special case of the entropy of closed relations F on X. Therefore, the concept $\mathrm{ent}(F)$ is a generalization of the concept of $h(f)$.

THEOREM 3.12. *Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. Then*

$$\mathrm{ent}(\Gamma(f)) = h(f).$$

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Proof. Observe that for each $x \in X$ and for all $A_0, A_1, A_2, \ldots, A_n \in \alpha$, $(x, f(x), f^2(x) \ldots, f^n(x)) \in A_0 \times A_1 \times A_2 \times \cdots \times A_n$ if and only if $x \in \bigcap_{i=0}^{n} f^{-i}(A_i)$. So, a collection of elements of α^{n+1} covers $X^n_{\Gamma(f)}$ if and only if the corresponding elements of $\vee_{i=0}^{n} f^{-i}(\alpha)$ cover X. Therefore, for any open cover α for X and for any positive integer m,

$$N(X^m_{\Gamma(f)}, \alpha^{m+1}) = N(\vee_{i=0}^{m} f^{-i}(\alpha)).$$

Finally, we prove that $\operatorname{ent}(\Gamma(f)) = h(f)$:

$$\begin{aligned}
\operatorname{ent}(\Gamma(f)) &= \sup\{\operatorname{ent}(\Gamma(f), \alpha) \mid \alpha \text{ is an open cover for } X\} \\
&= \sup\left\{\lim_{m\to\infty} \frac{\log N(X^m_{\Gamma(f)}, \alpha^{m+1})}{m} \mid \alpha \text{ is an open cover for } X\right\} \\
&= \sup\left\{\lim_{m\to\infty} \frac{\log N(\vee_{i=0}^{m} f^{-i}(\alpha))}{m} \mid \alpha \text{ is an open cover for } X\right\} \\
&= \sup\{h(f, \alpha) \mid \alpha \text{ is an open cover for } X\} \\
&= h(f).
\end{aligned}$$

This completes the proof. □

For a more thorough discussion of topological entropy $h(f)$, see [24]. In Theorem 3.16, we present a relationship between $\operatorname{ent}(F)$ and $h(\sigma)$. We use the following notation in its proof.

Definition 3.12. Let X be a non-empty compact metric space, let F be a closed relation on X and let $\mathcal{U}$ be an open cover for $X^{\mathbb{Z}_+}$. Then we use $\mathcal{U} \cap X_F^+$ to denote

$$\mathcal{U} \cap X_F^+ = \{U \cap X_F^+ \mid U \in \mathcal{U}\}.$$

Next, let m be a positive integer and let $\mathcal{V}$ be an open cover for X^m. Then we use $\mathcal{V} \times X^{\mathbb{Z}_+}$ to denote

$$\mathcal{V} \times X^{\mathbb{Z}_+} = \{V \times X^{\mathbb{Z}_+} \mid V \in \mathcal{V}\},$$

which is an open cover for $X^{\mathbb{Z}_+}$.

LEMMA 3.13. *Let X be a non-empty compact metric space, let F be a closed relation on X such that $p_2(F) \subseteq p_1(F)$ and let σ be the shift map on X_F^+. Then*

$$\operatorname{ent}(F, \alpha) = h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+)$$

for any open cover α for X and for any positive integer m.

Proof. First, we define $\sigma_X : X^{\mathbb{Z}_+} \to X^{\mathbb{Z}_+}$ to be the function defined by

$$\sigma_X(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots)$$

for each $(x_0, x_1, x_2, \ldots) \in X^{\mathbb{Z}_+}$. Note that $(\sigma_X)|_{X_F^+} = \sigma$. Let α be an open cover for X and let m be a positive integer. First, note that for each positive integer k and for any $V \in \alpha^{m+1}$,

$$\sigma_X^{-k}(V \times X^{\mathbb{Z}_+}) = X^k \times V \times X^{\mathbb{Z}_+}.$$

It follows that $\sigma^{-k}(\alpha^{m+1} \times X^{\mathbb{Z}_+}) = X^k \times \alpha^{m+1} \times X^{\mathbb{Z}_+}$ and, therefore,

$$N(X_F^+, \vee_{i=0}^k \sigma_X^{-i}(\alpha^{m+1} \times X^{\mathbb{Z}_+})) = N(X_F^+, \alpha^{m+k+1} \times X^{\mathbb{Z}_+}).$$

Let $\pi : X_F^+ \to X_F^{m+k}$ be the projection to the first $m+k+1$ coordinates. Note that π is surjective since $p_2(F) \subseteq p_1(F)$. Then

$$N(X_F^+, \alpha^{m+k+1} \times X^{\mathbb{Z}_+}) = N(X_F^{m+k}, \alpha^{m+k+1})$$

follows since for each $\mathbf{x} \in X_F^+$ and for each $V \in \alpha^{m+k+1}$,

$$\mathbf{x} \in V \times X^{\mathbb{Z}_+} = \pi^{-1}(V) \Longleftrightarrow \pi(\mathbf{x}) \in V.$$

Note also that for all positive integers m and k,

$$\begin{aligned}
N(\vee_{i=0}^k \sigma^{-i}((\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+)) &= N(\vee_{i=0}^k (\sigma_X^{-i}(\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap \sigma^{-i}(X_F^+))) \\
&= N(\vee_{i=0}^k (\sigma_X^{-i}(\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+)) = N(X_F^+, \vee_{i=0}^k (\sigma_X^{-i}(\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+)) \\
&= N(X_F^+, \vee_{i=0}^k \sigma_X^{-i}(\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) = N(X_F^+, \vee_{i=0}^k \sigma_X^{-i}(\alpha^{m+1} \times X^{\mathbb{Z}_+})).
\end{aligned}$$

It follows that for all positive integers m and k,

$$N(X_F^{m+k}, \alpha^{m+k+1}) = N(\vee_{i=0}^k \sigma^{-i}((\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+)).$$

Next, we prove that

$$\lim_{k\to\infty} \frac{\log N(X_F^{m+k}, \alpha^{m+k+1})}{k} = \lim_{k\to\infty} \frac{\log N(X_F^k, \alpha^{k+1})}{k}.$$

For each positive integer n, let $a_n = \log N(X_F^n, \alpha^{n+1})$. Then $a_k = \log N(X_F^k, \alpha^{k+1})$ and $a_{m+k} = \log N(X_F^{m+k}, \alpha^{m+k+1})$ for all positive integers m and k. It follows from Observation 3.2 that $a_k \le a_{m+k}$ and from Lemma 3.3 that $a_{m+k} \le a_m + a_k$ for all positive integers m and k. Therefore, $a_k \le a_{m+k} \le a_m + a_k$, and $a_k/k \le a_{m+k}/k \le a_m/k + a_k/k$ follows for all positive integers m and k. By Theorem 3.5, the limit $\lim_{k\to\infty}(a_k/k)$ exists, therefore, for each positive integer m,

$$\lim_{k\to\infty} \frac{a_k}{k} \le \lim_{k\to\infty} \frac{a_{m+k}}{k} \le \lim_{k\to\infty} \frac{a_m}{k} + \lim_{k\to\infty} \frac{a_k}{k} = \lim_{k\to\infty} \frac{a_k}{k}.$$

Thus, for each positive integer m, $\lim_{k\to\infty} a_{m+k}/k = \lim_{k\to\infty} a_k/k$ and

$$\lim_{k\to\infty} \frac{\log N(X_F^{m+k}, \alpha^{m+k+1})}{k} = \lim_{k\to\infty} \frac{\log N(X_F^k, \alpha^{k+1})}{k}$$

follows for each positive integer m. Therefore, for any positive integer m, we get that

$$\begin{aligned}
h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) &= \lim_{k\to\infty} \frac{\log N(\vee_{i=0}^k \sigma^{-i}((\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+))}{k} \\
&= \lim_{k\to\infty} \frac{\log N(X_F^{m+k}, \alpha^{m+k+1})}{k} \\
&= \lim_{k\to\infty} \frac{\log N(X_F^k, \alpha^{k+1})}{k} \\
&= \mathrm{ent}(F, \alpha).
\end{aligned}$$

□

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LEMMA 3.14. *Let X be a non-empty compact metric space, let F be a closed relation on X and let σ be the shift map on X_F^+. Then*

$$h(\sigma) = \sup\{h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) \mid \alpha \textit{ is an open cover for } X, m \textit{ is a positive integer}\}$$

Proof. Let $A = \{h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) \mid \alpha$ is an open cover for X, m is a positive integer$\}$ and let $B = \{h(\sigma, \mathcal{U}) \mid \mathcal{U}$ is an open cover for $X_F^+\}$. Note that $h(\sigma) = \sup(B)$. We distinguish the following possible cases.

- A is not bounded in $\mathbb{R}$. Then $\sup(A) = \infty$ and since $A \subseteq B$, it follows that $\sup(B) = \infty$. Therefore, in this case, $\sup(A) = \sup(B)$.
- A is bounded in $\mathbb{R}$. Since $A \subseteq B$, it follows that $\sup(A) \leq \sup(B)$. To show that $\sup(A) \geq \sup(B)$, we show that for every open cover $\mathcal{U}$ for X_F^+, there are an open cover α for X and a positive integer m such that $h(\sigma, \mathcal{U}) \leq h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+)$. Let $\mathcal{U}$ be any open cover for X_F^+. To show that there are an open cover α for X and a positive integer m such that $h(\sigma, \mathcal{U}) \leq h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+)$, we show that there are an open cover α for X and a positive integer m such that $\mathcal{U} \leq (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+$. Since X_F^+ is a compact metric space, it follows that $\mathcal{U}$ has a finite subcover. Let $U_0, U_1, U_2, \ldots, U_k \in \mathcal{U}$ be such that $X_F^+ = U_0 \cup U_1 \cup U_2 \cup \cdots \cup U_k$. Let λ be a Lebesgue number (a positive number λ is a Lebesgue number of an open cover $\mathcal{U}$ of X if for each set $A \subseteq X$ with diameter less than λ, there is U in $\mathcal{U}$ such that $A \subseteq U$) of the open cover $\{U_0, U_1, U_2, \ldots, U_k\}$ for X_F^+, let α be an open cover for X and m a positive integer such that for each $A \in \alpha$, $\operatorname{diam}(A) < \lambda$ and $\operatorname{diam}(X_F^+)/2^m < \lambda$. It follows that for each element $U \in (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+$, $\operatorname{diam}(U) < \lambda$. Therefore, for each $U \in (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+$, there is an $i \in \{0, 1, 2, \ldots, k\}$ such that $U \subseteq B_i$. Thus, $\{U_0, U_1, U_2, \ldots, U_k\} \leq (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+$. Since $\{U_0, U_1, U_2, \ldots, U_k\} \subseteq \mathcal{U}$, it follows that also $\mathcal{U} \leq (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+$. □

LEMMA 3.15. *Let X be a non-empty compact metric space, let F be a closed relation on X such that $p_2(F) \subseteq p_1(F)$ and let σ be the shift map on X_F^+. Then*

$$\operatorname{ent}(F) = \sup\{h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) \mid \alpha \textit{ is an open cover for } X, m \textit{ is a positive integer}\}.$$

Proof. It follows from Lemma 3.13 that for each positive integer m,

$$\sup\{\operatorname{ent}(F, \alpha) \mid \alpha \text{ is an open cover for } X\} =$$
$$\sup\{h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) \mid \alpha \text{ is an open cover for } X\}.$$

For each positive integer m, let $A_m = \{h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) \mid \alpha$ is an open cover for $X\}$, let $A = \{h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) \mid \alpha$ is an open cover for X, m is a positive integer$\}$ and let $B = \{\operatorname{ent}(F, \alpha) \mid \alpha$ is an open cover for $X\}$. It follows from $A = \bigcup_{i=1}^{\infty} A_i$ and from $\sup(A_m) = \sup B$ for each positive integer m that $\sup(A) = \sup(B)$. □

THEOREM 3.16. *Let X be a non-empty compact metric space, let F be a closed relation on X such that $p_2(F) \subseteq p_1(F)$ and let σ be the shift map on X_F^+. Then*

$$\operatorname{ent}(F) = h(\sigma).$$

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Proof. The theorem follows from Lemmas 3.14 and 3.15:

$$\begin{aligned}
&\mathrm{ent}(F)\\
&\quad = \sup\{h(\sigma, (\alpha^{m+1} \times X^{\mathbb{Z}_+}) \cap X_F^+) \mid \alpha \text{ is an open cover for } X, m \text{ is a positive integer}\}\\
&\quad = h(\sigma).
\end{aligned}$$

□

The following well-known result is a corollary of Theorem 3.16.

COROLLARY 3.17. *Let X be a non-empty compact metric space, let $f : X \to X$ be a continuous function and let $\sigma : \varprojlim(X, f) \to \varprojlim(X, f)$ be the shift function. Then*

$$h(f) = h(\sigma).$$

We conclude this section by stating some open problems. Let X be a non-empty compact metric space and let F be a closed relation on X. If there is a positive integer m such that $X_F^m = \emptyset$, then, obviously, $\mathrm{ent}(F) = 0$. Note that there is a positive integer m such that $X_F^m = \emptyset$ if and only if $\bigcap_{n=0}^{\infty} F^n(X) = \emptyset$, where $F^n(x) = \{y \in X \mid \text{there is } \mathbf{x} \in X_F^n \text{ such that } \pi_0(\mathbf{x}) = x \text{ and } \pi_n(\mathbf{x}) = y\}$ and $F^n(X) = \bigcup_{x\in X} F^n(x)$. This gives rise to the following problems that came up during a discussion on the topic with our anonymous referee.

Problem 3.18. Let X be a non-empty compact metric space and let F be a closed relation on X. Is it true that

$$\mathrm{ent}(F) = \mathrm{ent}\bigg(F \cap \bigg(\bigg(\bigcap_{n=0}^{\infty} F^n(X)\bigg) \times \bigg(\bigcap_{n=0}^{\infty} F^n(X)\bigg)\bigg)\bigg)?$$

Note that for any non-empty closed relation F on a compact metric space, $p_2(F) \subseteq p_1(F)$ is a stronger condition than $\bigcap_{n=0}^{\infty} F^n(X) \neq \emptyset$. So, it is natural to state the following problem.

Problem 3.19. Let X be a non-empty compact metric space, let F be a closed relation on X such that $\bigcap_{n=0}^{\infty} F^n(X) \neq \emptyset$ and let σ be the shift map on X_F^+. Is it true that also in this case,

$$\mathrm{ent}(F) = h(\sigma)?$$

An affirmative solution to the problem gives a nice generalization of Theorem 3.16.

4. *Returns*

In this section, we present returns for closed relations on compact metric spaces and use them to obtain non-trivial sufficient conditions for a relation on a compact metric space to have non-zero entropy. First, we introduce the notion of a (k, ε)-return on a set.

Definition 4.1. Let X be a non-empty compact metric space, let A be a non-empty subset of X, let F be a closed relation on X, let k be a positive integer and let $\varepsilon > 0$. We say that *F has a (k, ε)-return on A* if for each $a \in A$, there are positive integers j and j' such that $0 < j' \leq j \leq k$ and points $\mathbf{x}, \mathbf{y} \in X_F^j$ such that:

(1) $\mathbf{x}(0) = \mathbf{y}(0) = a$;

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(2) $\mathbf{x}(j), \mathbf{y}(j) \in A$; and
(3) $\rho(\mathbf{x}(j'), \mathbf{y}(j')) \geq \varepsilon$.

We say that F *has a* (k, ε)*-return* if there is a non-empty set $B \subseteq X$ such that F has a (k, ε)-return on B.

In Construction 4.1, we give detailed instructions on how to construct an (F, A, k, ε)-return function for any element $a \in A$ (which is defined later in Definition 4.2).

Construction 4.1. Let X be a non-empty compact metric space, let $A \subseteq X$, let F be a closed relation on X, let k be a positive integer and let $\varepsilon > 0$. Suppose that F has a (k, ε)-return on A.

- *Step 1.* For each $a \in A$, we choose and fix positive integers $j'(a)$ and $j(a)$ such that $0 < j'(a) \leq j(a) \leq k$ and points $\mathbf{x}(a, 0), \mathbf{x}(a, 1) \in X_F^{j(a)}$ such that:
  (1) $a = \mathbf{x}(a, 0)_0 = \mathbf{x}(a, 1)_0$;
  (2) $\rho(\mathbf{x}(a, 0)_{j'(a)}, \mathbf{x}(a, 1)_{j'(a)}) \geq \varepsilon$;
  (3) $\mathbf{x}(a, 0)_{j(a)}, \mathbf{x}(a, 1)_{j(a)} \in A$.

  Then we define
  $$\delta(a, 0) = \mathbf{x}(a, 0)_{j(a)} \quad \text{and} \quad \delta(a, 1) = \mathbf{x}(a, 1)_{j(a)}.$$
- *Step 2.* Choose and fix $a^* \in A$. We inductively define functions
  $$\Delta : \Sigma_2 \to A^{\mathbb{Z}_+} \quad \text{and} \quad J : \Sigma_2 \to \mathbb{Z}_+^{\mathbb{Z}_+}$$
  by $\Delta(\mathbf{s})_0 = a^*$ and for each non-negative integer n, we define
  $$\Delta(\mathbf{s})_{n+1} = \delta(\Delta(\mathbf{s})_n, \mathbf{s}_n)$$
  and $J(\mathbf{s})_0 = 0$, and for each non-negative integer n, we define
  $$J(\mathbf{s})_{n+1} = J(\mathbf{s})_n + j(\Delta(\mathbf{s})_n)$$
  for any sequence $\mathbf{s} \in \Sigma_2$.
- *Step 3.* We define the function $\Psi : \Sigma_2 \to X_F^+$ by
  $$\Psi(\mathbf{s})_{J(\mathbf{s})_n} = \Delta(\mathbf{s})_n$$
  for each $\mathbf{s} \in \Sigma_2$ and for each positive integer n, and
  $$\Psi(\mathbf{s})_{J(\mathbf{s})_n + i} = \mathbf{x}(\Delta(\mathbf{s})_n, \mathbf{s}_n)_i$$
  for each $\mathbf{s} \in \Sigma_2$, for each positive integer n and for each $i \in [0, j(\Delta(\mathbf{s})_n)]$.

Definition 4.2. Let X be a non-empty compact metric space, let $A \subseteq X$, let F be a closed relation on X, let k be a positive integer, let $\varepsilon > 0$ such that F has a (k, ε)-return on A and let $a^* \in A$. We call such a function $\Psi : \Sigma_2 \to X_F^+$ from Construction 4.1 a (F, A, k, ε)-return function for a^*.

The following observation follows directly from Construction 4.1.

Observation 4.2. Let X be a non-empty compact metric space, let F be a closed relation on X, let k be a positive integer, let $\varepsilon > 0$, let $A \subseteq X$ be such that F has a (k, ε)-return on A and let $a^* \in A$. Also, let Ψ be a (F, A, k, ε)-return function for a^*.

Note that for all sequences $\mathbf{s}, \mathbf{t} \in \Sigma_2$ and for each positive integer n:

(1) $J(\mathbf{s})_{n+1} \leq kn$;

(2) if $\mathbf{s}_i = \mathbf{t}_i$ for each $i \in [0, n]$, then $\Delta(\mathbf{s})_i = \Delta(\mathbf{t})_i$ and $J(\mathbf{s})_i = J(\mathbf{t})_i$ for each $i \in [0, n+1]$;

(3) $\Psi(\mathbf{s})_0 = a^*$;

(4) for $i = 0$,

$$\Psi(\mathbf{s})_{J(\mathbf{s})_n+i} = \mathbf{x}(\Delta(\mathbf{s})_n, s_n)_0 = \Delta(\mathbf{s})_n = \Psi(\mathbf{s})_{J(\mathbf{s})_n},$$

and for $i = j(\Delta(\mathbf{s})_n)$, it holds that $J(\mathbf{s})_n + i = J(\mathbf{s})_{n+1}$, and

$$\Psi(\mathbf{s})_{J(\mathbf{s})_n+i} = \mathbf{x}(\Delta(\mathbf{s})_n, \mathbf{s}_n)_{j(\Delta(\mathbf{s})_n)} = \Delta(\mathbf{s})_{n+1} = \Psi(\mathbf{s})_{J(\mathbf{s})_{n+1}}$$

follows.

LEMMA 4.3. *Let X be a non-empty compact metric space, let F be a closed relation on X, let k and m be positive integers, let $\varepsilon > 0$, let $A \subseteq X$ be such that F has a (k, ε)-return on A, and let $a^* \in A$. Also, let Ψ be a (F, A, k, ε)-return function for a^*, let $j' = j'(\Delta(\mathbf{s})_m)$ and let $\mathbf{s}, \mathbf{t} \in \Sigma_2$ be any sequences of* 0*s and* 1*s such that:*

- *for any $i \in [0, m-1]$, $\mathbf{s}_i = \mathbf{t}_i$ and*
- $\mathbf{s}_m \neq \mathbf{t}_m$.

Then:

(1) *for each $i \in [0, J(\mathbf{s})_m]$,*

$$\Psi(\mathbf{s})_i = \Psi(\mathbf{t})_i;$$

(2) *for each $i \in [0, j(\Delta(\mathbf{s})_m)]$,*

$$\Psi(\mathbf{s})_{J(\mathbf{s})_m+i} = \mathbf{x}(\Delta(\mathbf{s})_m, \mathbf{s}_m)_i \quad \text{and} \quad \Psi(\mathbf{t})_{J(\mathbf{s})_m+i} = \mathbf{x}(\Delta(\mathbf{s})_m, \mathbf{t}_m)_i;$$

(3) $\rho(\Psi(\mathbf{s})_{J(\mathbf{s})_m+j'}, \Psi(\mathbf{t})_{J(\mathbf{s})_m+j'}) \geq \varepsilon$ *and*

$$J(s)_m + j' \leq J(s)_m + j(\Delta(s)_m) = J(s)_{m+1} \leq mk.$$

Proof. The lemma follows directly from the construction of the function Ψ and from Observation 4.2. □

COROLLARY 4.4. *Let X be a non-empty compact metric space, let F be a closed relation on X, let k and m be positive integers, let $\varepsilon > 0$, let $A \subseteq X$ be such that F has a (k, ε)-return on A, and let $a^* \in A$. Also, let Ψ be a (F, A, k, ε)-return function for a^*, and let $\mathbf{s}, \mathbf{t} \in \Sigma_2$ be any sequences of* 0*s and* 1*s. Then the following hold.*

(1) *If there is a non-negative integer $i \leq m$ such that $\mathbf{t}_i \neq \mathbf{s}_i$, then there is a positive integer $j < mk$ such that*

$$\rho(\Psi(\mathbf{s})_j, \Psi(\mathbf{t})_j) \geq \varepsilon.$$

(2) *Let α be an open cover for X consisting of sets with diameter less than ε and let $A^* \in \alpha$ be such that $a^* \in A^*$. Then for any $U \in A^* \times \alpha^{mk-1}$,*

$$\{\pi_{[0,mk]}(\Psi(\mathbf{s})), \pi_{[0,mk]}(\Psi(\mathbf{t}))\} \not\subseteq U.$$

(3) $N(\pi_{[0,mk]}(\Psi(\Sigma_2)), A^* \times \alpha^{mk-1}) \geq 2^{m-1}$.

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Proof. Item (1) follows directly from Lemma 4.3, item (2) follows from item (1) and item (3) follows from item (2). □

Theorem 4.5 is our first main result of the paper. It says that for a closed relation F on a compact metric space X, the existence of a (k, ε)-return on a subset of X implies that the entropy of G is non-zero.

THEOREM 4.5. *Let X be a non-empty compact metric space, let $A \subseteq X$, let F be a closed relation on X, let k be a positive integer and let $\varepsilon > 0$ such that F has a (k, ε)-return on A. Then*

$$\mathrm{ent}(F) \geq \frac{\log(2)}{k}.$$

Proof. Let $a^* \in A$ and let Ψ be a (F, A, k, ε)-return function for a^*. Let α be an open cover for X such that for any $A \in \alpha$, the diameter of A is less than ε. It follows from Corollary 4.4 that for each positive integer m, $N(X_F^{mk-1}, \alpha^{mk}) \geq 2^{m-1}$. Thus,

$$\mathrm{ent}(F) \geq \lim_{m\to\infty} \frac{\log(N(X_F^{mk-1}, \alpha^{mk}))}{mk-1} \geq \lim_{m\to\infty} \frac{\log(2^{m-1})}{mk-1} = \lim_{m\to\infty} \frac{(m-1)\log(2)}{mk-1} = \frac{\log(2)}{k}.$$

□

Note that it follows from the following observation that there are closed relations F on compact metric spaces X such that $\mathrm{ent}(F) \neq 0$ and for each non-empty $A \subseteq X$, F has no (k, ε)-return on A. So, sufficient conditions from Theorem 4.5 are not necessary conditions for non-zero entropy of F. Thus, we do not have a characterization of non-zero entropy.

Observation 4.6. If F is a graph of a continuous function from X to X, then for any positive integer k and for any $\varepsilon > 0$, F does not have a (k, ε)-return since for each positive integer k and for all $\mathbf{x}, \mathbf{y} \in X_F^+$, $\mathbf{x}(0) = \mathbf{y}(0)$ implies that $\mathbf{x}(j) = \mathbf{y}(j)$ for all $j \leq k$.

In particular, if φ is a homeomorphism on X with non-zero entropy, then $\Gamma(\varphi)$ is a closed relation on X with non-zero entropy that admits no (k, ε)-returns. An example of such a space X and a homeomorphism φ on X is the Cantor set $\{0, 1\}^{\mathbb{Z}}$ and the shift homeomorphism σ on it. Also, note that such a space X may be connected. For example, let X be the pseudo-arc. It was proved by Kennedy in [13] that there is a homeomorphism φ on X such that $h(\varphi) \neq 0$.

THEOREM 4.7. *Let X be a non-empty compact metric space, let $A \subseteq X$, let F be a closed relation on X, let k be a positive integer and let $\varepsilon > 0$ such that has a (k, ε)-return on A. Then there is a countable subset $A' \subseteq A$ such that F has a (k, ε)-return on A'.*

Proof. Let $a^* \in A$ and let Ψ be a (F, A, k, ε)-return function for a^*. While Σ_2 is uncountable, for each positive integer m, the set $\{\Delta(\mathbf{s})_i : \mathbf{s} \in \Sigma_2, i \leq m\}$ is finite and so, the set $\{\Delta(\mathbf{s})_i : \mathbf{s} \in \Sigma_2, i \in \mathbb{Z}_+\}$ is a countable set A' such that F has a (k, ε)-return on A'. □

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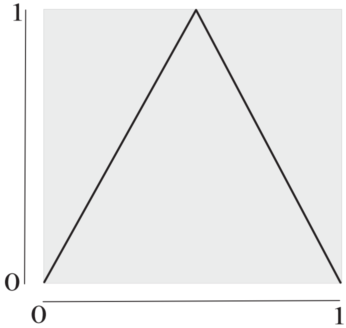

FIGURE 1. The graph of the function f from Example 4.9.

Observation 4.8. For any integer $N \geq 2$, one can also define that F has a (k, ε, N)-return on A if for every $a \in A$, there exist $j \in \{2, 3, 4, \ldots, k\}$ and the set

$$\{\mathbf{x}^\ell \mid \ell \in \{0, \ldots, N-1\}\} \subseteq X_F^j$$

such that for each $\ell \in \{0, 1, 2, \ldots, N-1\}$, $\mathbf{x}^\ell(0) = a$ and $\mathbf{x}^\ell(j) \in A$, and for all $\ell, \ell' \in \{0, 1, 2, \ldots, N-1\}$ such that $\ell \neq \ell'$, there exists a positive integer $j' \leq j$ such that $\rho(\mathbf{x}^\ell(j'), \mathbf{x}^{\ell'}(j')) \geq \varepsilon$. One can then embed Σ_N into X_F^+ and obtain that the entropy of the relation F is at least $\log(N)/k$.

Next, we give some examples showing how easy it is to prove, using the notion of returns, that the entropy of a function is positive. First, we show how the notion of returns can be used to give a simple proof that the tent map has non-zero entropy.

Example 4.9. Let $I = [0, 1]$ and let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) = \begin{cases} 2x & \text{for } x \leq \frac{1}{2}, \\ 2 - 2x & \text{for } x > \frac{1}{2}, \end{cases}$$

see Figure 1.

Let $F = \Gamma(f)^{-1}$. It is easy to check that for each $x \in [0, 1]$, both $(x, \frac{1}{2}x, \frac{1}{4}x)$ and $(x, 1 - \frac{1}{2}x, \frac{1}{2} + \frac{1}{4}x)$ are elements of I_F^2. Since $|\frac{1}{4}x - (\frac{1}{2} + \frac{1}{4}x)| = \frac{1}{2} > \frac{1}{3}$, $\Gamma(f)^{-1}$ has a $(3, \frac{1}{3})$-return on $A = [0, 1]$. It follows from Theorem 4.5 that $h(f) = \mathrm{ent}(F) > 0$.

The following example presents how returns may be used to produce a homeomorphism on a Cantor fan with positive entropy (a continuum X is *a Cantor fan* if X is homeomorphic to the continuum $\bigcup_{c \in C} S_c$, where $C \subseteq [0, 1]$ is the standard Cantor set and for each $c \in C$, S_c is the straight line segment in the plane from $(0, 0)$ to $(c, 1)$).

Example 4.10. Let $X = [-1, 1]$ and let $F = \Gamma(f_1) \cup \Gamma(f_2)$, where $f_1 : [-1, 1] \to [-1, 1]$ and $f_2 : [-1, 1] \to [-1, 1]$ are defined by

$$f_1(x) = -x \quad \text{and} \quad f_2(x) = x$$

for each $x \in X$. By [11, Example 2.7], the Mahavier product X_F^+ is a Cantor fan. Note that $A = \{1\}$ is a $(3, 2)$-return with sequences $(1, -1, 1)$ and $(1, 1, 1)$. So, the entropy of the

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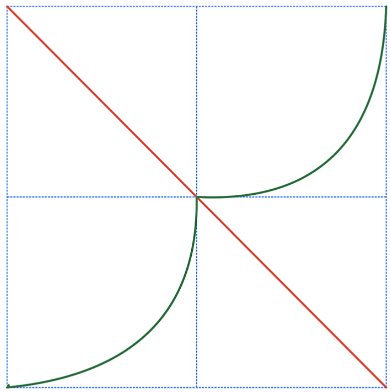

FIGURE 2. The relation F from Example 4.11.

shift map σ_F on X_F^+ is non-zero. The same argument as in [4, Example 4.15] shows that the inverse limit $\varprojlim(X_F^+, \sigma_F)$ is also a Cantor fan. Therefore, the shift map σ on $\varprojlim(X_F^+, \sigma_F)$ is a homeomorphism on a Cantor fan with positive entropy.

In the following example, the same simple argument is used to produce a transitive homeomorphism on a Cantor fan with positive entropy. (The function $f : X \to X$ is *transitive* if for all non-empty open sets U and V in X, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$.) Note that the homeomorphism from Example 4.10 is not transitive.

Example 4.11. Let $X = [-1, 1]$ and let $F = \Gamma(f_1) \cup \Gamma(f_2)$, where $f_1 : [-1, 1] \to [-1, 1]$ and $f_2 : [-1, 1] \to [-1, 1]$ are defined by

$$f_1(x) = -x \quad \text{and} \quad f_2(x) = \begin{cases} x^{1/3}; & x \in [-1, 0], \\ x^2; & x \in [0, 1] \end{cases}$$

for each $x \in X$; see Figure 2. By [4, Example 4.15], the Mahavier product X_F^+ is a Cantor fan and the shift map σ_F on X_F^+ is transitive. Note that $A = \{1\}$ is a $(3, 2)$-return with sequences $(1, -1, 1)$ and $(1, 1, 1)$. So, the entropy of the shift map σ_F on X_F^+ is positive. It follows from [4, Example 4.15] that the inverse limit $\varprojlim(X_F^+, \sigma_F)$ is also a Cantor fan. Therefore, the shift map σ on $\varprojlim(X_F^+, \sigma_F)$ is a transitive homeomorphism on a Cantor fan with positive entropy.

The closed relation in the following example was shown in [2] to produce a dynamical system with positive entropy that does not have periodic points or finitely generated Cantor sets. (Let X be a compact metric space and let F be a closed relation on X. We say that F *has finitely generated Cantor sets*, if there is a Cantor set C in X_F^+ such that for some finite collection $G \subseteq F$, $X_G^+ = C$.) The notion of well-aligned sets was introduced in [2] as a way to generalize the proof that the relation in Example 4.12 produces a dynamical system with positive entropy. While the existence of well-aligned sets may be hard to verify in many different settings, the use of returns introduced in this paper serves a different

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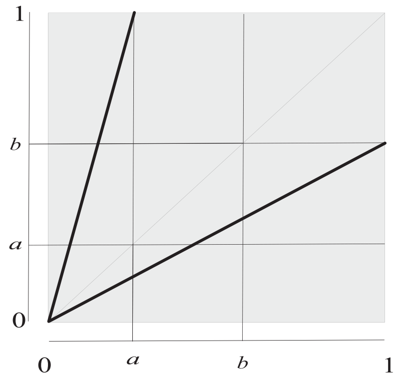

FIGURE 3. The relation F from Example 4.12.

purpose in that it can greatly simplify proofs of positive entropy in this setting of shift maps on Mahavier products as well as in more traditional settings as in Example 4.9. However, it is especially useful when there are no finitely generated Cantor sets. Our examples also tend to illustrate that it is much easier to verify the existence of a return than the existence of well-aligned sets.

Example 4.12. Let $I = [0, 1]$. In [2], it was shown that if $0 < a < 1$ and $0 < b < 1$, and

$$F = \left\{(x, y) \in [0, 1] \times [0, 1] \mid y = \frac{1}{a}x \text{ or } y = bx\right\},$$

see Figure 3, then $\mathrm{ent}(F) \neq 0$.

We present a much streamlined proof of this fact by showing that if $0 < a \leq b < 1$, then F has a (k, ε)-return on the set $A = [ab, a]$. For the case $0 < b \leq a < 1$, a similar argument shows that F^{-1} has a (k, ε)-return on $A = [ab, b]$.

So assume $0 < a \leq b < 1$. Note if $b > a$, then there is an $m \in \mathbb{N}$ such that $b^{m-1} > a$ and $b^m \leq a$, and thus $ab<b^m \leq a$. So if $y \in (a, b]$, then there is an $m_y \in \mathbb{N}$ such that $m_y \leq m$ and $b^{m_y-1}y > a$ and $b^{m_y}y \leq a$, and thus $ab \leq b^{m_y}y \leq a$.

We show that if $k = m + 1$ and $\varepsilon = (1/a - b)ab/2$, then F has a (k, ε)-return on $[ab, a]$.

Let $x \in [ab, a]$. If $(b/a)x \leq a$, then $(b/a)x \geq (b/a)ab = b \cdot b \geq ab$ and $(x, bx) \in F$ with $bx < x \leq a$, so $(bx, (b/a)x) \in F$. It follows that $(x, bx, (b/a)x) \in I_F^2$. It is easy to see that also $(x, (1/a)x, (b/a)x) \in I_F^2$. Observe that $x \in [ab, a]$, $(b/a)x \in [ab, a]$ and $|(1/a)x - bx| = ((1/a) - b)x \geq ((1/a) - b)ab > \varepsilon$. So, for $(b/a)x \leq a$, F has an $(m + 1, \varepsilon)$-return on $[ab, a]$.

If $(b/a)x > a$, then $(b/a)x \leq (b/a)a = b$. So, there is a positive integer m_x such that $m_x \leq m$ and $ab \leq b^{m_x}(b/a)x \leq a$. Now

$$\left(x, bx, \frac{b}{a}x, b\frac{b}{a}x, b^2\frac{b}{a}x, \ldots, b^{m_x}\frac{b}{a}x\right) \in I_F^{m_x+2}$$

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and

$$\left(x, \frac{1}{a}x, \frac{b}{a}x, b\frac{b}{a}x, b^2\frac{b}{a}x, \ldots, b^{m_x}\frac{b}{a}x\right) \in I_F^{m_x+2}.$$

Again observe that $x \in [ab, a]$, $b^{m_x}(b/a)x \in [ab, a]$ and $|(1/a)x - bx| = ((1/a) - b)x \geq ((1/a) - b)ab > \varepsilon$. So, for $(b/a)x > a$, F has an $(m+1, \varepsilon)$-return on $[ab, a]$. So, by Theorem 4.5, it follows that $\mathrm{ent}(F) \neq 0$.

Observation 4.13. Let F be the closed relation on $I = [0, 1]$ from Example 4.12 for $a = \frac{1}{2}$ and $b = \frac{1}{3}$. Note that it follows from Theorem 3.16 that the shift mapping σ_F on I_F^+ has non-zero entropy and that it follows from [3, Theorem 4.3] that σ_F is transitive. It follows from [3, Theorem 5.17] that the inverse limit $\varprojlim(I_F^+, \sigma_F)$ is a Lelek fan (let X be a Cantor fan and let Y be a subcontinuum of X. We say that Y is *a Lelek fan* if the end-points of Y are dense in Y) and from [3, Observation 5.3] that the shift homeomorphism σ on $\varprojlim(I_F^+, \sigma_F)$ is transitive. Therefore, the shift homeomorphism σ on $\varprojlim(I_F^+, \sigma_F)$ is a transitive homeomorphism on a Lelek fan with non-zero entropy. This answers the question [21, Question 3.11] by Oprocha in the affirmative.

We conclude the section by stating and proving various corollaries to Theorem 4.5.

COROLLARY 4.14. *Let X be a non-empty compact metric space, let F be a closed relation on X and let k be a positive integer. If there are two sets J and K in X_F^k with $\rho(J, K) > 0$ (where $\rho(J, K) = \inf\{\rho(x, y) \mid x \in J, y \in K\}$) and such that $\pi_k(J) \cup \pi_k(K) \subseteq \pi_0(J) \cap \pi_0(K)$, then F has a $(k, \rho(J, K))$-return. In particular, F has non-zero entropy.*

Proof. Let $\varepsilon = \rho(J, K)$. We show that F has a (k, ε)-return on X. Since $\rho(J, K) > 0$, if $(x_0, x_1, x_2, \ldots, x_k) \in J$ and $(y_0, y_1, y_2, \ldots, y_k) \in K$ with $x_0 = y_0$, then there is an integer $i \in \{1, 2, 3, \ldots k\}$ such that $\rho(x_i, y_i) > \varepsilon$. Let $x \in \pi_0(J) \cap \pi_0(K)$. Then there are elements $(x_0, x_1, x_2, \ldots, x_k) \in J$ and $(y_0, y_1, y_2, \ldots, y_k) \in K$ with $x = x_0 = y_0$ and since $\pi_k(J) \cup \pi_k(K) \subseteq \pi_0(J) \cap \pi_0(K)$, we have $\{x_k, y_k\} \subseteq \pi_0(J) \cap \pi_0(K)$. So F has a (k, ε)-return on $\pi_0(J) \cap \pi_0(K)$. By Theorem 4.5, $\mathrm{ent}(F) \neq 0$. □

COROLLARY 4.15. *Let X be a non-empty compact metric space and let F be a closed relation on X. Also, let k_x and k_y be two positive integers, let $\mathbf{x} \in X_F^{k_x}$ and $\mathbf{y} \in X_F^{k_y}$ be such that $\mathbf{x}(k_x) = \mathbf{x}(0) = \mathbf{y}(0) = \mathbf{y}(k_y)$, and let j be a positive integer such that $0 < j \leq \min\{k_x, k_y\}$ and $\mathbf{x}(j) \neq \mathbf{y}(j)$. Then there are a positive integer k and an $\varepsilon > 0$ such that F has a (k, ε)-return. In particular, F has non-zero entropy.*

Proof. We show that there is a set $A \subseteq X$, a positive integer k and an $\varepsilon > 0$ such that F has a (k, ε)-return on A. Let $\mathbf{s} = \mathbf{x} \star \mathbf{y}$ and $\mathbf{t} = \mathbf{y} \star \mathbf{x}$. Then $\mathbf{s}, \mathbf{t} \in X_F^{k_x+k_y}$ such that $\mathbf{s}(0) = \mathbf{t}(0) = \mathbf{s}(k_x + k_y) = \mathbf{t}(k_x + k_y)$ and $\mathbf{s}(j) \neq \mathbf{t}(j)$. Therefore, for $A = \{\mathbf{s}(0)\}$, F has a $(k_x + k_y, \varepsilon)$-return on A, where $\varepsilon = \frac{1}{2}\rho(\mathbf{s}(j), \mathbf{t}(j))$. By Theorem 4.5, $\mathrm{ent}(F) \neq 0$. □

In the last corollary (Corollary 4.17) to Theorem 4.5, so-called well-aligned sets are used to detect non-zero entropy. They form a more visual or geometric apparatus for spotting non-zero entropy. Before stating and proving the corollary, we give the following definitions to describe this apparatus.

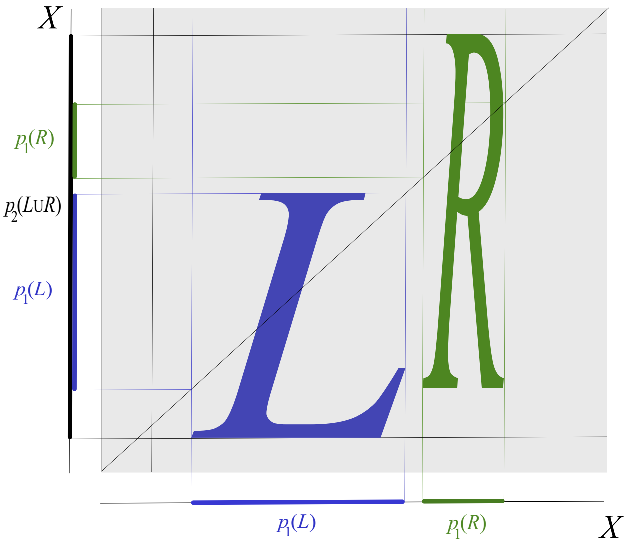

FIGURE 4. The sets L and R from Definition 4.3.

Definition 4.3. Let X be a non-empty compact metric space, let F be a closed relation on X, and let L and R be non-empty subsets of F. We say that *the sets L and R are well aligned in F*, if:

(1) $p_2(L) \cap p_2(R) \neq \emptyset$;

(2) there is $\varepsilon > 0$ such that for all $t \in p_2(L) \cap p_2(R)$, there are $\ell \in p_1(p_2^{-1}(t) \cap L)$ and $r \in p_1(p_2^{-1}(t) \cap R)$ such that $\rho(r, \ell) \geq \varepsilon$;

(3) $p_1(L) \cup p_1(R) \subseteq p_2(L \cup R)$;

(4) there is a positive integer N such that for each $t \in p_2(L \cup R)$, there are a positive integer $i_0 \leq N$ and a point $(a_0, a_1, a_2, \ldots, a_{i_0-1}, a_{i_0}) \in X^{i_0}_{F^{-1}}$ such that:
   (a) $a_0 = t$ and
   (b) $a_{i_0} \in p_2(L) \cap p_2(R)$;

see Figure 4.

Example 4.16. Let $F = \{(0, 1), (0, \frac{3}{4}), (\frac{3}{4}, 0), (1, 0)\}$ be a closed relation on $[0, 1]$. It is easy to see that $\mathrm{ent}(F) \neq 0$. Let $L = \{(\frac{3}{4}, 0)\}$, $R = \{(1, 0), (0, 1), (0, \frac{3}{4})\}$. Then the sets L and R are well aligned in F.

Definition 4.4. Let X be a non-empty compact metric space, let F be a closed relation on X. We say that *the relation F is well aligned* if there are $L, R \subseteq F$ such that the sets L and R are well aligned in F.

COROLLARY 4.17. *Let X be a non-empty compact metric space and let F be a closed relation on X. If F or F^{-1} is a well-aligned relation, then* $\mathrm{ent}(F) \neq 0$.

Proof. Suppose that F is a well-aligned relation. Let L and R be non-empty closed subsets of F such that L and R are well aligned in F; that is:

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(1) $p_2(L) \cap p_2(R) \neq \emptyset$;
(2) there is $\varepsilon > 0$ such that for all $t \in p_2(L) \cap p_2(R)$, there are $\ell \in p_1(p_2^{-1}(t) \cap L)$ and $r \in p_1(p_2^{-1}(t) \cap R)$ such that $\rho(r, \ell) \geq \varepsilon$;
(3) $p_1(L) \cup p_1(R) \subseteq p_2(L \cup R)$;
(4) there is a positive integer N such that for each $t \in p_2(L \cup R)$, there are a positive integer $i_0 \leq N$ and a point $(a_0, a_1, a_2, \ldots, a_{i_0-1}, a_{i_0}) \in X_F^{i_0}$ such that:
    (a) $a_0 = t$ and
    (b) $a_{i_0} \in p_2(L) \cap p_2(R)$.

Choose and fix such a positive integer N and $\varepsilon > 0$. Let $A = p_2(L \cup R)$. Then F^{-1} has an $(N, \varepsilon/2)$-return on A. Thus, by Theorem 4.5, $\mathrm{ent}(F^{-1}) \neq 0$. It follows that $\mathrm{ent}(F) \neq 0$. If F^{-1} is a well-aligned relation, the proof is similar. □

5. *Finite relations*

Finite relations on compact metric spaces X can have positive entropy. This may happen even in the case where X is not finite. We show in this final section that, unlike topological entropy for closed relations on compact metric spaces in general, in the case of finite relations, positive entropy is equivalent to the shift map on the Mahavier product being Li–Yorke chaotic as well as equivalent to DC-2 distributional chaos for the shift map, as well as equivalent to G having a (k, ε)-return. Before stating and proving our theorems, we present the following definitions.

Definition 5.1. Let X be a non-empty compact metric space, let $f : X \to X$ be a continuous function and let $x, y \in X$ such that $x \neq y$. The set $\{x, y\}$ is called *a Li–Yorke pair for f* if

$$\liminf(\rho(f^n(x), f^n(y))) = 0 \quad \text{and} \quad \limsup(\rho(f^n(x), f^n(y))) > 0.$$

Definition 5.2. Let X be a non-empty compact metric space, let $f : X \to X$ be a continuous function and let $S \subseteq X$. We say that the set S is *a scrambled set or a Li–Yorke set in* (X, f) if for all $x, y \in S$,

$$x \neq y \Longrightarrow \{x, y\} \quad \text{is a Li–Yorke pair for } f.$$

Definition 5.3. Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. The dynamical system (X, f) is called *Li–Yorke chaotic* if X contains an uncountable scrambled set.

The following is a well-known result.

THEOREM 5.1. *Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. If $h(f) > 0$, then the dynamical system (X, f) is Li–Yorke chaotic.*

Proof. See the proof of [6, 2 from Corollary 2.4, p. 10]. □

See [6] for more references and information about Li–Yorke chaotic topological dynamical systems.

The approach used in the following theorem was suggested by our referee. Let $X = \{0, 1, 2, \ldots, m-1\}$ be a finite set and F be a relation on X. We define the zero-one $m \times m$-matrix M_F on (X, F) as follows: for all $x, y \in X$, $M_{xy} = 1$ if $(x, y) \in F$ and $M_{xy} = 0$ if $(x, y) \notin F$. Observe that for all $x, y \in X$ and for each positive integer N, M^N_{xy} counts the number of distinct paths of length N from x to y, that is, M^N_{xy} is the cardinality of the set $\{\mathbf{x} \in X^N_F \mid \mathbf{x}(0) = x \text{ and } \mathbf{x}(N) = y\}$.

THEOREM 5.2. *Let $X = \{0, 1, 2, \ldots, m-1\}$ be a finite set and F be a relation on X. If there are $x \in X$ and a positive integer k such that $M^k_{xx} > 1$, then* $\mathrm{ent}(F) \geq (\log N)/k$.

Proof. Let $x \in X$ and let k be a positive integer such that $M^k_{xx} > 1$. Let $N = M^k_{xx}$. Then F has $(k, 1, N)$-return on $\{x\}$ (see Observation 4.8 for the definition of (k, ε, N)-returns) and so, the entropy of the relation F is at least $(\log N)/k$. □

For any point $x \in X$, we say that x is recurrent in (X, F) if $M^k_{xx} \geq 1$ for some positive integer k. Using this, one can easily prove the following theorem.

THEOREM 5.3. *Let $X = \{0, 1, 2, \ldots, m-1\}$ be a finite set and F be a relation on X. The following statements are equivalent.*

(1) *There are $x \in X$ and a positive integer k such that $M^k_{xx} > 1$.*
(2) $\mathrm{ent}(F) \neq 0$.

Proof. Let $x \in X$ and let k be a positive integer such that $M^k_{xx} > 1$. It follows from Theorem 5.2 that $\mathrm{ent}(F) \neq 0$. This proves the implication from statement (1) to (2). To prove the implication from statement (2) to (1), suppose that for every $x \in X$ and for every positive integer k, $M^k_{xx} \leq 1$. First, we show that for each recurrent point x in (X, F), there is exactly one element $y \in X$ such that $(x, y) \in F$. To show this, let $x \in X$ be a recurrent point in (X, F). Also, let k be a positive integer such that $M^k_{xx} \geq 1$. It follows from our assumption that $M^k_{xx} = 1$. Therefore, there is exactly one element $y \in X$ such that $(x, y) \in F$.

For each recurrent point x in (X, F), we denote by $f(x)$ the point in X such that $(x, f(x)) \in F$. Note that for each recurrent point x in (X, F), the point $f(x)$ is also a recurrent point in (X, F). It follows that for any recurrent point x in (X, F) and for any positive integer m, there is exactly one point $\mathbf{x} \in X^m_F$ such that $\mathbf{x}(0) = x$.

Next, we examine non-recurrent points in (X, F). For any non-recurrent point x in (X, F), either:

(1) there is a positive integer m such that for each $\mathbf{x} \in X^m_F$, $\mathbf{x}(0) \neq x$; or
(2) there are a positive integer m and a point $\mathbf{x} \in X^m_F$ such that $\mathbf{x}(0) = x$ and $\mathbf{x}(m)$ is a recurrent point.

Let R be the set of recurrent points in (X, F) and let

$$A = \{x \in X \setminus R \mid \text{there is a positive integer } m \text{ such that for each } \mathbf{x} \in X^m_F, \mathbf{x}(0) \neq x\}$$

and

$$B = \{x \in X \setminus R \mid \text{there are a positive integer } m \text{ and } \mathbf{x} \in X^m_F \text{ such that } \mathbf{x}(0) = x \text{ and } \mathbf{x}(m) \in R\}.$$

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For each $x \in A$, let m_x be a positive integer such that for each $\mathbf{x} \in X_F^{m_x}$, $\mathbf{x}(0) \neq x$, and let $m_A = \max\{m_x \mid x \in A\}$. Let $n = |X \setminus R|$. Note that for each $x \in B$, for each positive integer $m > n$ and for each $\mathbf{x} \in X_F^m$,

$$\mathbf{x}(0) = x \Longrightarrow \mathbf{x}(m) \in R.$$

For each $x \in B$, let $\ell_x = |\{\mathbf{x} \in X_F^{n+1} \mid \mathbf{x}(0) = x\}|$. Then $|\{\mathbf{x} \in X_F^m \mid \mathbf{x}(0) = x\}| = \ell_x$ for each positive integer $m > n$. Let $m_0 = \max\{n+1, m_A\}$. It follows that for each positive integer $m > m_0$, $|X_F^m| \leq |R| + \sum_{x\in B} \ell_x$. Therefore, for any open cover α for X and for any positive integer m,

$$m \geq m_0 \Longrightarrow N(X_F^m, \alpha^{m+1}) \leq |R| + \sum_{x\in B} \ell_x.$$

Let α be any open cover for X. Then

$$\mathrm{ent}(F, \alpha) = \lim_{m\to\infty} \frac{\log N(X_F^m, \alpha^{m+1})}{m} \leq \lim_{m\to\infty} \frac{\log(|R| + \sum_{x\in B} \ell_x)}{m} = 0$$

and it follows that $\mathrm{ent}(F) = 0$. □

LEMMA 5.4. *Let X be a non-empty compact metric space and let F be a non-empty closed relation on X. If for each positive integer k, for each $\varepsilon > 0$ and for each $a \in X$, F has no (k, ε) returns on $\{a\}$, then the following hold.*

(1) *For each $a \in X$ and for each positive integer m, there is at most one point $\mathbf{x} \in X_F^m$ such that $\mathbf{x}(0) = \mathbf{x}(m) = a$.*

(2) *For each positive integer m, for each $a \in X$ and for each $\mathbf{x} \in X_F^m$ such that $\mathbf{x}(0) = \mathbf{x}(m) = a$, we use $\overline{\mathbf{x}} \in X_F^+$ to denote the periodic point that is defined by:*
   (a) $\pi_{[0,m]}(\overline{\mathbf{x}}) = \mathbf{x}$*; and*
   (b) $\overline{\mathbf{x}}(i+m) = \overline{\mathbf{x}}(i)$ *for all* $i \in \mathbf{Z}_+$.

   Then for each positive integer m, for each $a \in X$, for each $\mathbf{x} \in X_F^m$ such that $\mathbf{x}(0) = \mathbf{x}(m) = a$ and for each $\mathbf{y} \in X_F^+$, the following holds:

$$\mathbf{y}(i) = a \quad \textit{for infinitely many non-negative integers } i \Longrightarrow \mathbf{y} = \overline{\mathbf{x}}.$$

Proof. Suppose that for each positive integer k, for each $\varepsilon > 0$ and for each $a \in X$, F has no (k, ε) returns on $\{a\}$. The first part of the lemma is obvious. To prove the second part of Lemma 5.4, let m be a positive integer, let $a \in X$ and let $\mathbf{x} \in X_F^m$ such that $\mathbf{x}(0) = \mathbf{x}(m) = a$. Also, let $\mathbf{y} \in X_F^+$ be such that $\mathbf{y}(i) = a$ for infinitely many non-negative integers i. To show that $\mathbf{y} = \overline{\mathbf{x}}$, suppose that $\mathbf{y} \neq \overline{\mathbf{x}}$. Then there is a positive integer i such that $\mathbf{y}(i) \neq \overline{\mathbf{x}}(i)$. Choose and fix such a positive integer i and let $m_1 > i$ and $m_2 > i$ be positive integers such that $\mathbf{y}(m_1) = \overline{\mathbf{x}}(m_2) = a$. Then

$$\pi_{[0,m_1]}(\mathbf{y}) \star \pi_{[0,m_2]}(\overline{\mathbf{x}}), \pi_{[0,m_2]}(\overline{\mathbf{x}}) \star \pi_{[0,m_1]}(\mathbf{y}) \in X_F^{m_1+m_2}$$

are such points that:

(1) $(\pi_{[0,m_1]}(\mathbf{y}) \star \pi_{[0,m_2]}(\overline{\mathbf{x}}))(0) = (\pi_{[0,m_1]}(\mathbf{y}) \star \pi_{[0,m_2]}(\overline{\mathbf{x}}))(m_1+m_2) = a$ and $(\pi_{[0,m_2]}(\overline{\mathbf{x}}) \star \pi_{[0,m_1]}(\mathbf{y}))(0) = (\pi_{[0,m_2]}(\overline{\mathbf{x}}) \star \pi_{[0,m_1]}(\mathbf{y}))(m_1+m_2) = a$; and

(2) $(\pi_{[0,m_1]}(\mathbf{y}) \star \pi_{[0,m_2]}(\overline{\mathbf{x}}))(i) \neq (\pi_{[0,m_2]}(\overline{\mathbf{x}}) \star \pi_{[0,m_1]}(\mathbf{y}))(i)$.

It follows that F has an $(m_1 + m_2, (d(\mathbf{y}(i), \overline{\mathbf{x}}(i)))/2)$-return on $\{a\}$, which is a contradiction. $\square$

LEMMA 5.5. *Let X be a non-empty compact metric space and let F be a finite subset of $X \times X$. If $\mathrm{ent}(F) \neq 0$, then X_F^+ is uncountable.*

Proof. Assume that $\mathrm{ent}(F) \neq 0$. First, let $Z = p_1(F) \cup p_2(F)$. Then Z is finite and F is a relation on Z, so we can use Theorem 5.3 for (Z, F). Let $x \in X$ and let k be a positive integer such that $M_{xx}^k > 1$. It follows from Theorem 5.3 that such a point x and such a positive integer k do exist. Let $n = M_{xx}^k$ and let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1} \in Z_F^k$ be n distinct points such that for each $i \in \{0, 1, 2, \ldots, n-1\}$, $\mathbf{x}_i(0) = x$ and $\mathbf{x}_i(k) = x$. Next, let $Y = \{\mathbf{x}_i(j) \mid i \in \{0, 1, 2, \ldots, n-1\}, j \in \{0, 1, 2, \ldots, k\}\}$ and let

$$G = \{(\mathbf{x}_i(j), \mathbf{x}_i(j+1)) \mid i \in \{0, 1, 2, \ldots, n-1\}, j \in \{0, 1, 2, \ldots, k-1\}\}.$$

Note that $Y \subseteq X$ and that $G \subseteq F$ is a relation on Y such that $p_2(G) \subseteq p_1(G)$. Also note that $\mathrm{ent}(G) \neq 0$ since G has a (k, ε)-return on Y. By Theorem 3.16, the topological entropy of the shift map σ on Y_G^+ is non-zero. Therefore, by Theorem 5.1, the dynamical system (Y_G^+, σ) is Li–Yorke chaotic, which implies that there is an uncountable scrambled set in Y_G^+. Therefore, Y_G^+ is uncountable. Since $Y_G^+ \subseteq X_F^+$, it follows that X_F^+ is uncountable. $\square$

THEOREM 5.6. *Let X be a non-empty compact metric space and let F be a finite subset of $X \times X$. The following statements are equivalent.*

(1) $\mathrm{ent}(F) \neq 0$.

(2) *There are a set $A \subseteq X$, a positive integer k and an $\varepsilon > 0$ such that F has a (k, ε)-return on A.*

(3) *There are:*
   - (a) *positive integers k_x and k_y;*
   - (b) *points $\mathbf{x} \in X_F^{k_x}$ and $\mathbf{y} \in X_F^{k_y}$ such that $\mathbf{x}(k_x) = \mathbf{x}(0) = \mathbf{y}(0) = \mathbf{y}(k_y)$; and*
   - (c) *a positive integer j such that $0 < j \leq \min\{k_x, k_y\}$ and $\mathbf{x}(j) \neq \mathbf{y}(j)$.*

(4) *X_F^+ is uncountable.*

Proof. The implication from statement (1) to (4) is Lemma 5.5, the implication from statement (2) to (1) is Theorem 4.5 and the implication from statement (3) to (1) is Corollary 4.15.

Now we prove the implication from statement (4) to (1). Assume $\mathrm{ent}(F) = 0$. By Theorem 4.5, there do not exist a non-empty set $A \subseteq X$, a positive integer k and an $\varepsilon > 0$ such that F has a (k, ε) return on A. According to Lemma 5.4, for each $a \in p_1(G)$, there is at most one point in X_F^+ with first coordinate a and in which a occurs as a coordinate infinitely many times. Let B be the set of all points in X_F^+ whose first coordinate occurs infinitely many times as a coordinate. Since F is finite, B is finite. Also, for each element $\mathbf{x} \in X_F^+$, some coordinate must be repeated infinitely many times, so there is a non-negative integer k such that $\sigma^k(\mathbf{x}) \in B$. Since $\sigma^{-1}(\mathbf{x})$ is finite for each $\mathbf{x} \in X_F^+$ and since $X_F^+ = \bigcup_{k=0}^{\infty} \sigma^{-k}(B)$, it follows that X_F^+ is countably infinite or finite.

We prove the implication from statement (1) to (2) similarly as the implication from statement (4) to (1): if we assume that there does not exist a set $A \subseteq X$, an integer k and

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an $\varepsilon > 0$ such that F has a (k, ε)-return on A, then again Lemma 5.4 can be used as above in the proof of the implication from statement (4) to (1) to show that X_F^+ is countable, and thus, by Lemma 5.5, it follows that $\text{ent}(F) = 0$.

Finally, we prove the implication from statement (2) to (3). Let $A \subseteq X$, let k be a positive integer and let $\varepsilon > 0$ such that F has a (k, ε)-return on A. Also, let $a^* \in A$ and let $\Psi : \Sigma_2 \to X_F^+$ be an (F, A, k, ε)-return function for a^*. Since F is finite, there are at least two different sequences $\mathbf{s}, \mathbf{t} \in \Sigma_2$ of 0s and 1s for which there are strictly increasing sequences (i_n) and (j_n) of positive integers such that:

- $i_0 = 1$ and $j_0 = 1$;
- for each positive integer ℓ, $\Psi(\mathbf{s})_{i_\ell}, \Psi(\mathbf{t})_{j_\ell} \in A$; and
- for all positive integers ℓ_1 and ℓ_2, $\Psi(\mathbf{s})_{i_{\ell_1}} = \Psi(\mathbf{s})_{i_{\ell_2}} = \Psi(\mathbf{t})_{j_{\ell_1}} = \Psi(\mathbf{t})_{j_{\ell_2}}$.

Since $\mathbf{s} \neq \mathbf{t}$, it follows that there is a positive integer m such that $\Psi(\mathbf{s})_m \neq \Psi(\mathbf{t})_m$. Fix such a positive integer m. Then, let:

- ℓ_1 and ℓ_2 be such positive integers that $i_{\ell_1} > m$ and $j_{\ell_2} > m$ and let
$$k_x = i_{\ell_1} \quad \text{and} \quad k_y = j_{\ell_2};$$
- $\mathbf{x} = \pi_{[1,k_x]}(\Psi(\mathbf{s}))$ and $\mathbf{y} = \pi_{[1,k_y]}(\Psi(\mathbf{t}))$; and
- $j = m$.

Note that for constructed k_x, k_y, $\mathbf{x}$, $\mathbf{y}$ and j, statement (3) follows. This completes the proof. □

The following is a corollary of Theorem 5.6.

Corollary 5.7. *Let X be a non-empty compact metric space and let F be a finite subset of $X \times X$. If $\text{ent}(F) \neq 0$, then there are elements $\mathbf{s}_1$ and $\mathbf{s}_2$ of F such that*

$$p_1(\mathbf{s}_1) = p_1(\mathbf{s}_2) \quad \text{and} \quad p_2(\mathbf{s}_1) \neq p_2(\mathbf{s}_2),$$

and there are elements $\mathbf{t}_1$ and $\mathbf{t}_2$ of F such that

$$p_2(\mathbf{t}_1) = p_2(\mathbf{t}_2) \quad \text{and} \quad p_1(\mathbf{t}_1) \neq p_1(\mathbf{t}_2).$$

Proof. Let $\text{ent}(F) \neq 0$. By Theorem 5.6, there are:

(1) positive integers k_x and k_y;
(2) points $\mathbf{x} \in X_F^{k_x}$ and $\mathbf{y} \in X_F^{k_y}$ such that $\mathbf{x}(k_x) = \mathbf{x}(0) = \mathbf{y}(0) = \mathbf{y}(k_y)$; and
(3) a positive integer j such that $0 < j \leq \min\{k_x, k_y\}$ and $\mathbf{x}(j) \neq \mathbf{y}(j)$.

Fix such k_x, k_y, $\mathbf{x}$ and $\mathbf{y}$, and let j_1 be the smallest among all positive integers $j \in \{1, 2, 3, \ldots, k_x\}$ such that $\mathbf{x}(j) \neq \mathbf{y}(j)$. Let

$$\mathbf{s}_1 = (\mathbf{x}(j_1 - 1), \mathbf{x}(j_1)) \quad \text{and} \quad \mathbf{s}_2 = (\mathbf{y}(j_1 - 1), \mathbf{y}(j_1)).$$

Then $p_1(\mathbf{s}_1) = p_1(\mathbf{s}_2)$ and $p_2(\mathbf{s}_1) \neq p_2(\mathbf{s}_2)$. This proves the first part of the claim. Next, we prove the second part of the claim. Suppose that there are no elements $\mathbf{t}_1$ and $\mathbf{t}_2$ in F such that $p_2(\mathbf{t}_1) = p_2(\mathbf{t}_2)$ and $p_1(\mathbf{t}_1) \neq p_1(\mathbf{t}_2)$. Then F is a graph of a single-valued function $f : p_2(F) \to p_1(F)$. Note that $\varphi : p_2(F) \to X_{F^{-1}}$, defined by $\varphi(x) = (x, f(x), f^2(x), f^3(x), \ldots)$ for any $x \in p_2(F)$, is a homeomorphism. Since F is finite, it follows that $p_2(F)$ is finite. Therefore, $X_{F^{-1}}^+$ is finite, which is a contradiction since

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by Theorem 5.6, $X_{F^{-1}}^{+}$ is uncountable (since $\mathrm{ent}(F^{-1})$ is also non-zero). This completes the proof. □

Definition 5.4. Let X be a non-empty compact metric space, let $f : X \to X$ be a continuous function and let $x, y \in X$ such that $x \neq y$. The set $\{x, y\}$ is called *a DC2-pair for f* if

$$\liminf \left(\frac{1}{n} \cdot \sum_{i=1}^{n} \rho(f^i(x), f^i(y))\right) = 0 \quad \text{and} \quad \limsup \left(\frac{1}{n} \cdot \sum_{i=1}^{n} \rho(f^i(x), f^i(y))\right) > 0.$$

Definition 5.5. Let X be a non-empty compact metric space, let $f : X \to X$ be a continuous function and let $S \subseteq X$. We say that the set S is *a DC2-scrambled set in* (X, f) if for all $x, y \in S$,

$$x \neq y \Longrightarrow \{x, y\} \quad \text{is a DC2-pair for } f.$$

Definition 5.6. Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. The dynamical system (X, f) is called *DC2-chaotic* if X contains an uncountable DC2-scrambled set.

Observation 5.8. Let X be a non-empty compact metric space and let $f : X \to X$ be a continuous function. If (X, f) is DC2-chaotic, then (X, f) is Li–Yorke chaotic. See [8] for more information.

Putting these facts together, we see that the case where F is a finite subset of $X \times X$ produces a dynamical system (X_F^+, σ) where several forms of chaos are equivalent. See the following corollary. Note that it is a known fact that, in general, Li–Yorke chaos does not imply positive entropy, see [23] for more information.

COROLLARY 5.9. *Suppose X is a non-empty compact metric space and F is a finite subset of $X \times X$. The following statements are equivalent.*

(1) $\mathrm{ent}(F) \neq 0$.
(2) (X_F^+, σ) *is Li–Yorke chaotic.*
(3) (X_F^+, σ) *has a DC2-scrambled Cantor set.*

Proof. First, we show the implication from statement (1) to (3). Suppose that $\mathrm{ent}(F) \neq 0$. A similar argument as in the first part of the proof of Lemma 5.5 gives that $h(\sigma) \neq 0$, where σ is the shift map on X_F^+. By [9, Theorem 1.1, p. 138] (X_F^+, σ) is DC2-chaotic and it follows from [9, Remark 3, p. 148] that (X_F^+, σ) has a DC2-scrambled Cantor set.

The implication from statement (3) to (2) follows from Observation 5.8.

Finally, we prove the implication from statement (2) to (1). Suppose that (X_F^+, σ) is Li–Yorke chaotic. Then X_F^+ contains an uncountable scrambled set. It follows that X_F^+ is uncountable. By Theorem 5.6, $\mathrm{ent}(F) \neq 0$. □

The following examples, where a countable closed subset F of $[0, 1] \times [0, 1]$, such that:

(1) $\mathrm{ent}(F) = 0$; and
(2) X_F^+ is uncountable,

is presented, together with two problems, is a good place to finish the paper.

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Example 5.10. Let $X = [0, 1]$ and let $F = \{(0, 0)\} \cup \{(1/2^i, 1/2^{i+1}) \mid i \in \{0, 1, 2, 3, \ldots\}\} \cup \{(1/2^i, 1/2^{i+1}) \mid i \in \{0, 1, 2, 3, \ldots\}\} \cup \{(1/2^i, 1/2^{i+2}) \mid i \in \{0, 1, 2, 3, \ldots\}\}$. Then $\mathrm{ent}(F) = 0$ since $F \subseteq \{(x, y) \in [0, 1] \times [0, 1] \mid y \leq x\}$ (see [10]), and X_F^+ is uncountable since each coordinate a of an element of X_F^+ can be followed by either $\frac{1}{2}a$ or $\frac{1}{4}a$.

Problem 5.11. Let X be a non-empty compact metric space and let F be a countable closed relation on X such that $\mathrm{ent}(F) \neq 0$. Is it true that either F or F^{-1} has a (k, ε)-return for some positive integer k and some $\varepsilon > 0$?

Problem 5.12. Let $X = [0, 1]$ and let $f : X \to X$ be a continuous function such that $h(f) \neq 0$. Is it true that $\Gamma(f)$ has a (k, ε)-return for some positive integer k and some $\varepsilon > 0$?

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