LINEAR TRANSFORMATIONS ON ALGEBRAS OF MATRICES: THE INVARIANCE OF THE ELEMENTARY SYMMETRIC FUNCTIONS

MARVIN MARCUS AND ROGER PURVES

1. Introduction. In this paper we examine the structure of certain linear transformations T on the algebra of n-square matrices M_n into itself. In particular if $A \in M_n$ let $E_r(A)$ be the rth elementary symmetric function of the eigenvalues of A. Our main result states that *if* $4 \leqslant r \leqslant n-1$ *and* $E_r(T(A)) = E_r(A)$ *for* $A \in M_n$ *then* T *is essentially* (*modulo taking the transpose and multiplying by a constant*) *a similarity transformation:*

$$T\colon A \to SAS^{-1}.$$

No such result as this is true for $r = 1, 2$ and we shall exhibit certain classes of counterexamples. These counterexamples fail to work for $r = 3$ and the structure of those T such that $E_3(T(A)) = E_3(A)$ for all $A \in M_n$ is unknown to us. In **(1)** it is established that those T which preserve the rank (determinant) of every matrix in M_n are essentially of the form $T\colon A \to PAQ$ where P and Q are non-singular, (PQ is unimodular). In the first part of what follows, we shall improve this result by requiring only that T preserves non-singularity. We remark that *in general we do not assume that T is multiplicative or anti-multiplicative anywhere in the paper.*

We shall collect here the notation to be used throughout. For $A \in M_n$ let A' = transpose of A, $\rho(A)$ = rank of A, $\mathrm{tr}(A)$ = trace of A, A_{ij} = the element in position (i, j) of A, O_n = the n-square zero matrix, and E_{ij} = the n-square matrix with 1 at position (i, j), 0 elsewhere. In addition if $A \in M_p$ and $B \in M_q$ we define $A \oplus B \in M_{p+q}$ to be the direct sum of A and B. If $1 \leqslant p \leqslant n$ then Q_{pn} will be the set of all sequences of p-tuples $\omega = (i_1, \ldots, i_p)$ where $1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n$. A transformation $T\colon M_n \to M_n$ will be called a direct product if there exists a scalar c and fixed U and V in M_n such that

$$T(A) = cUAV$$

or

$$T(A) = cUA'V$$

for all $A \in M_n$. This is motivated by the fact that the mapping $T\colon A \to UAV$ has a matrix representation $V' \times U$, the direct product of V' and U, with a

Received October 17, 1958. The work of the first author was supported by United States National Science Foundation Grant NSF-G5416. The work of the second author was supported by the National Research Council of Canada.

proper choice of co-ordinate system for M_n. We remark that the mapping $T: A \rightarrow A'$ cannot be accomplished by pre- and post-multiplication by fixed matrices U and V for all A. We shall also denote by e.v.(A) the set of all n eigenvalues of A counting multiplicities.

2. Linear maps of GL_n into itself.

As usual, GL_n is the group of n-square non-singular matrices in M_n. We shall determine all T such that $T(GL_n) \subseteq GL_n$.

LEMMA 2.1. *If $0 \neq A \in M_n$ then A is similar to a matrix B with $B_{ii} \neq 0$, $i = 1, \ldots, n$.*

Proof. We may assume A is in Jordan form. It is known in general that A is similar to a matrix with $\text{tr}(A)/n$ in position (i, i), $i = 1, \ldots, n$. Hence we may assume $\text{tr}(A) = 0$. If $A = E_{12}$ let u_1 be the vector with all entries 1 and let u_2 be the vector with first entry $1 - n$ and the remaining entries 1. Normalize u_1 and u_2 and let $u_3, \ldots, u_n$ be a completion to an orthonormal basis. Let U be the orthogonal matrix with u_i as column i. Then the (i, i) entry of $UE_{12}U'$ is $u_{i1}u_{i2} \neq 0$. The proof is now completed by induction on n. If $A \in M_{n+1}$ is in Jordan form with zero trace we consider first the case that A is diagonal. Since $A \neq 0$ we can assume $A_{11} \neq 0$ and moreover the matrix $C \in M_n$ obtained by deleting row and column 1 of A is not 0_n. By induction choose $V \in M_n$ such that $(VCV^{-1})_{ii} \neq 0$ for $i = 1, \ldots, n$. Then

$$(1 \oplus V)\,\mathrm{A}(1 \oplus V^{-1}) = \mathrm{A}_{11} \oplus VCV^{-1}$$

has all non-zero diagonal elements. If A is not diagonal we can clearly assume $A_{12} = 1$ and the submatrix C above is not 0_n. As before we select $V \in M_n$ such that

$$P = (1 \oplus V)\,A\,(1 \oplus V^{-1})$$

has all non-zero entries on the diagonal with the possible exception of P_{11}. If $P_{11} = 0$ and b_{11} is the $(1, 1)$ entry of VCV^{-1} then select $U \in M_2$ such that

$$U\begin{pmatrix} 0 & * \\ 0 & b_{11} \end{pmatrix} U^{-1}$$

has non-zero diagonal entries. Then

$$B = (U \oplus I_{n-1})\,P(U^{-1} \oplus I_{n-1})$$

is the required matrix.

LEMMA 2.2. *If $0 \neq A \in M_n$ then there is a $Z \in M_n$ such that*

$$\text{e.v.}(A + Z) \cap \text{e.v.}(Z) = 0.$$

Proof. By Lemma 2.1. choose $P \in M_n$ such that $(P^{-1}AP)_{ii \neq 0}$ for $i = 1, \ldots, n$. Let X be defined as follows:

$$X_{ii} = 1, \qquad i = 1, \ldots, n$$
$$X_{ij} = -(P^{-1}AP)_{ij}, \qquad i > j$$
$$X_{ij} = 0, \qquad i < j.$$

Then X has all n eigenvalues 1 and $P^{-1}AP + X$ has eigenvalues $1 + (P^{-1}AP)_{ii}$ $i = 1, \ldots, n$ none of which are 1. Then $Z = PXP^{-1}$ has the required property.

LEMMA 2.3. *If $T(GL_n) \subseteq GL_n$ then T is non-singular.*

Proof. We have that if

$$\det (xI_n - [T(I_n)]^{-1} T(A)) = 0$$

for some x then

$$\det (xI_n - A) = 0$$

for that x. In other words the distinct elements of e.v. $([T(I)]^{-1}T(A))$ form a subset of the distinct eigenvalues of A. Now suppose $0 \neq A \in M_n$ and $T(A) = 0$. Choose $Z \in M_n$ by Lemma 2.2 such that

$$\text{e.v. } (Z) \cap \text{e.v.}(A + Z) = 0$$

Then

$$[T(I_n)]^{-1} T(A + Z) = [T(I_n)]^{-1} T(Z)$$

and the distinct eigenvalues of $[T(I_n)]^{-1}T(Z)$ form a subset of the distinct eigenvalues of both $A + Z$ and Z. This shows that $A = 0$ if $T(A) = 0$ and T is non-singular.

LEMMA 2.4. *If $T(GL_n) \subseteq GL_n$ and $T(I_n) = I_n$ then* e.v. $(T(A)) =$ e.v.(A) *for all $A \in M_n$.*

Proof. As in the proof of Lemma 2.3, we know that if $T(A)$ has a set of n distinct eigenvalues then

$$\text{e.v.}(A) = \text{e.v.}(T(A)).$$

Since T^{-1} exists we can say that if B has n distinct eigenvalues then

$$\text{e.v.}(B) = \text{e.v.}(T^{-1}(B)).$$

If $T(A)$ has multiple eigenvalues choose a sequence B_j converging to $T(A)$ such that B_j has distinct eigenvalues. The proof is completed using the fact that the eigenvalues depend continuously on the elements.

THEOREM 2.1. *If $T(GL_n) \subseteq GL_n$ then there exist U and V in GL_n such that either*

$$T : A \to UAV \text{ for all } A \in M_n$$

or

$$T : A \to UA'V \text{ for all } A \in M_n.$$

https://doi.org/10.4153/CJM-1959-039-4 Published online by Cambridge University Press

Proof. By Lemma 2.4 the map

$$\phi : A \to [T(I_n)]^{-1}T(A)$$

satisfies

$$\text{e.v. } (\phi(A)) = \text{e.v.}(A)$$

for all $A \in M_n$. But by (**1**: Theorem 2),

$$\phi(A) = UAU^{-1}$$

or

$$\phi(A) = UA'U^{-1}.$$

Multiplication on the left by $T(I_n)$ completes the proof.

3. Linear maps preserving the symmetric functions. We now determine the structure of those linear T on M_n to M_n such that for each $A \in M_n$

$$E_r(A) = E_r(T(A)).$$

For each r let the class of all such T be denoted by $\mathfrak{A}_r$. It is clear that if T, $S \in \mathfrak{A}_r$ then $TS \in \mathfrak{A}_r$. Also if $T \in \mathfrak{A}_r$ and T^{-1} exists then $T^{-1} \in \mathfrak{A}_r$; for since any B is in the range of T we have

$$E_r(B) = E_r(TT^{-1}(B)) = E_r(T^{-1}(B)).$$

Our first result shows that $\mathfrak{A}_r$ is actually a multiplicative group for $r \geqslant 2$.

LEMMA 3.1. *If $r \geqslant 2$ and $T \in \mathfrak{A}_r$ then T^{-1} exists. Thus $\mathfrak{A}_r$ is a multiplicative group for $r \geqslant 2$.*

Proof. Suppose $T(A) = O_n$ and $A \neq O_n$. Then

$$E_r(A + X) = E_r(T(A + X)) = E_r(T(X)) = E_r(X)$$

for any $X \in M_n$. By Lemma 2.1 there exists $P \in GL_n$ such that $(P^{-1} AP)_{ii} \neq 0$ for $i = 1, \ldots, n$.
Define $X \in M_n$ as follows:

$$\begin{array}{ll} X_{ii} = x & i = 1, \ldots, r-1 \\ X_{ii} = 0 & i = r, \ldots, n \\ X_{ij} = 0 & i < j \\ X_{ij} = -(P^{-1}AP)_{ij} & i > j. \end{array}$$

Then

$$f_r(x) = E_r(P^{-1}AP + X) = E_r(A + PXP^{-1}) = E_r(PXP^{-1}) = E_r(X) = 0.$$

Thus the coefficient of x^{r-1} in the polynomial $f_r(x)$ must be 0. This means that the sum of the last $n - r + 1$ entries on the main diagonal of $P^{-1}AP$ is 0. Similarly we can show that the sum of any $n - r + 1$ is 0. But since $r \geqslant 2$, $n - r + 1 < n$ and it is clear that $(P^{-1}AP)_{ii} = 0$ $(i = 1, \ldots, n)$. This completes the proof.

LEMMA 3.2. *If $A \in M_n$ and $A \neq 0$ then*

$$\deg \det (xA + B) \leqslant 1 \text{ for all } B \in M_n$$

if, and only if, $\rho(A) = 1$.

Proof. We can clearly assume that A is in Jordan canonical form and the "if" part of the result is obvious.

In the other direction we show first that A has at most one non-zero eigenvalue. Suppose

$$\lambda_{i_1}, \ldots, \lambda_{i_k}$$

are the non-zero eigenvalues of A in positions (i_t, i_t), $t = 1, \ldots, k$. Let B be a diagonal matrix with 0 at positions (i_t, i_t) $t = 1, \ldots, k$ and 1 elsewhere on the main diagonal. Then

$$\deg \det (x A + B) = k = 1.$$

Suppose now that A has the single non-zero eigenvalue λ which we may assume is in position (1, 1). To show that $\rho(A) = 1$ it will suffice to show that the elements along the superdiagonal of A are all 0. This is clear for $n = 2$. If $n > 2$ let α be the largest integer such that there is a 1 at position $(\alpha, \alpha + 1)$ of A. Define B as follows:

$$\begin{array}{ll} B_{ii} = 0 & i = \alpha, \alpha + 1 \\ B_{ii} = 1 & i \neq \alpha, \alpha + 1 \\ B_{\alpha+1,\alpha} = 1 & \\ B_{ij} = 0 & \text{elsewhere.} \end{array}$$

Then

$$\det (xA + B) = -\lambda x^2 - x.$$

Thus there must be a 0 at $(\alpha, \alpha + 1)$ and a repetition of this procedure shows that there are no 1's along the superdiagonal when $\lambda \neq 0$.

Now assume that $\lambda = 0$ and that the (1, 2) entry of A is 1. Define α as above and if $\alpha > 2$ define B as follows:

$$\begin{array}{ll} B_{ii} = 0 & i = 1, 2, \alpha, \alpha + 1 \\ B_{ii} = 1 & \text{elsewhere on the main diagonal} \\ B_{21} = 1 & \\ B_{ij} = 0 & \text{elsewhere off the main diagonal.} \end{array}$$

Then

$$\det (xA + B) = x^2.$$

In this way all elements $(i, i + 1)$ for $2 < i < n - 1$ are shown to be 0. To settle position (2, 3) use the test matrix

$$B = E_{31} \oplus I_{n-3}.$$

for $E_{31} \in M_3$. This completes the proof.

https://doi.org/10.4153/CJM-1959-039-4 Published online by Cambridge University Press

LEMMA 3.3. *If* $3 \leqslant r < n$ *and* $A \in M_n$, $A \neq O_n$ *then the condition*

$$\deg E_r\,(xA + B) \leqslant 1$$

for all $B \in M_n$ *implies that* A *has at most one non-zero eigenvalue.*

Proof. We can again assume A is in Jordan canonical form with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $z_1, \ldots, z_n$ be indeterminates and let B be the diagonal matrix with $B_{ii} = z_i$ $i = 1, \ldots, n$. Then

$$E_r(xA + B) = \sum_{\omega=(i_1,\ldots,i_r)\in Q_{rn}} \prod_{k=1}^{r} (x\lambda_{i_k} + z_{i_k})$$

$$= \sum_{t=0}^{r} \left(\sum_{\omega\in Q_{rn}} \sum_{s_t\subseteq\omega} \prod_{\alpha\in s_t} \lambda_\alpha \prod_{\beta\in\omega-s_t} z_\beta \right) x^t$$

where

$$\sum_{s_t\subseteq\omega}$$

means the sum over all subsets s_t of ω with t members and

$$\prod_{\beta\in\omega-s_t}$$

means the product over those elements of ω not in s_t. Hence for $t \geqslant 2$ we have that the coefficient of x^t in the above sum must be 0 for any choice of $z_1, \ldots, z_n$. From this it is not difficult to show that the tth elementary symmetric function of any $n - r + t$ of the λ_j is 0. Choosing $t = 2$ we have that if all the λ_j are equal they must all be 0. Assume then that for some μ, σ, $\lambda_\sigma \neq \lambda_\mu$. Since $r \geqslant 3$ we have that $k = n - r + 2 < n$. Let

$$\lambda_{i_1}, \ldots, \lambda_{i_{k-1}}$$

be a choice of $k - 1$ of the eigenvalues with $i_j \neq \sigma, \mu$ for $j = 1, \ldots, k - 1$. Then

$$0 = E_2\,(\lambda_\sigma, \lambda_{i_1}, \ldots, \lambda_{i_{k-1}}) = \lambda_\sigma E_1(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}}) + E_2\,(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})$$

and a similar relation holds for λ_μ.
We then have

$$(\lambda_\sigma - \lambda_\mu)\, E_1\,(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}}) = 0.$$

If $r > 3$ then $k - 1 < n - 2$ and this last relation implies that $\lambda_i = 0$ for $i \neq \sigma, \mu$. In this case

$$\lambda_\sigma\lambda_\mu = 0$$

and A has at most one non-zero eigenvalue. To settle the case $r = 3$ let $E_t(\hat{\lambda}_j)$ denote the tth elementary symmetric function of all the λ_i for $i \neq j$. We first note that

$$E_2(\lambda_1, \ldots, \lambda_n) = \lambda_j E_1(\hat{\lambda}_j) + E_2(\hat{\lambda}_j) = \lambda_j E_1(\hat{\lambda}_j).$$

Summing on j we have

$$n\,E_2(\lambda_1, \ldots, \lambda_n) = 2\,E_2(\lambda_1, \ldots, \lambda_n) = 0.$$

Thus

$$\lambda_j E_1(\hat{\lambda}_j) = 0.$$

Setting

$$s = \sum_{j=1}^{n} \lambda_j$$

we have

$$\lambda_j^2 = \lambda_j s, \ \lambda_j(\lambda_j - s) = 0$$

and thus the non-zero eigenvalues of A are all equal to s. This completes $r = 3$.

LEMMA 3.4. *Assume* $4 \leqslant r \leqslant n + 3$ *and let* $A \in M_{n+3}$, $A \neq O_n$. *Then*

$$\deg E_r(xA + B) \leqslant 1 \quad \textit{for all} \quad B \in M_{n+3}$$

if, and only if, $\rho(A) = 1$.

Proof. The "if" part of the theorem is clear. To prove the "only if" part we can assume A is in Jordan canonical form and proceed by induction on n. For $n = 1$ or $r = n + 3$ Lemma 3.2 gives the result. Thus assume $r < n + 4$ and by Lemma 3.3 we know that A has at most one non-zero eigenvalue λ which we can assume is in position (1, 1). Call the (2, 3) entry ϵ (either 1 or 0). Define B to be the matrix with 1 in position (3, 2) and $r - 3$ 1's in any of the diagonal positions (i, i) for $i > 3$, 0's elsewhere. Then

$$E_r(xA + B) = \lambda\, \epsilon\, x^2.$$

Consider first the situation in which $\lambda \neq 0$. Then $\epsilon = 0$ and row 2 and column 2 of A are both zero. If we restrict B to those matrices with row 2 and column 2 zero we can apply the induction hypothesis to conclude that the submatrix of A obtained by deleting row 2 and column 2 has rank 1. Thus $\rho(A) = 1$ as well. In case $\lambda = 0$ let ϵ_1 and ϵ_2 be the (1, 2) and $(n + 3, n + 4)$ entries of A respectively. Define B as follows:

$$\begin{aligned} &B_{21} = B_{n+4,n+3} = 1. && \\ &B_{ii} = 1, && 3 \leqslant i < r - 2 \\ &B_{ij} = 0 && \text{elsewhere.} \end{aligned}$$

Then

$$E_r(xA + B) = \epsilon_1 \epsilon_2 x^2$$

and we may assume without loss of generality that $\epsilon_2 = 0$. But then we can apply the induction argument as before to obtain $\rho(A) = 1$.

LEMMA 3.5. *If* $4 \leqslant r \leqslant n$ *and* $T \in \mathfrak{A}_r$ *and* $\rho(A) = 1$ *for* $A \in M_n$ *then* $\rho(T(A)) = 1$.

Proof. Consider the polynomial $f_r(x) = E_r(xT(A) + B)$. Since $T^{-1} \in \mathfrak{A}_r$

we have $f_r(x) = E_r(xA + T^{-1}(B))$. Since $\rho(A) = 1$, $\deg f_r(x) \leqslant 1$ for all B, and by Lemma 3.4 $\rho(T(A)) = 1$.

LEMMA 3.6. *If* $4 \leqslant r \leqslant n$ *and* $T \in \mathfrak{A}_r$ *then for every* $A \in M_n$

$$\rho(T(A)) = \rho(A).$$

Proof. Let $\rho(A) = k$ and select $A_j\ j = 1, \ldots, k$ such that $\rho(A_j) = 1$ and

$$A = \sum_{j=1}^{k} A_j.$$

Then by Lemmas 3.5 and 3.1

$$\rho(T(A)) \leqslant k = \rho(A) = \rho(T^{-1}(T(A)) \leqslant \rho(T(A)).$$

We are now in a position to prove our main result concerning the structure of $\mathfrak{A}_r$.

THEOREM 3.1. *If* $4 \leqslant r \leqslant n - 1$ *and* $T \in \mathfrak{A}_r$ *then there exist* U *and* V *in* M_n *such that either*

(i) $$T : A \to UAV \quad \text{for all} \quad A \in M_n$$

or

(ii) $$T : A \to UA'V \quad \text{for all } A \in M_n$$

where

(iii) $$UV = e^{i\phi}I_n,\ r\phi \equiv 0\ (2\pi).$$

Proof. The existence of U and V satisfying (i) and (ii) is an immediate consequence of Lemma 3.6 and Theorem 2.1. It is clear that it suffices to show that $E_r(PB) = E_r(B)$ for all $B \in M_n$ implies that $P = e^{i\phi}I_n$ with $r\phi \equiv 0(2\pi)$. Letting $C_r(B)$ denote the rth compound of B we have

$$\operatorname{tr} C_r(PB) = \operatorname{tr} C_r(B) \text{ for all } B \in M_n.$$

Hence

$$\operatorname{tr}\{[C_r(P) - I_{\binom{n}{r}}]\, C_r(B)\} = 0.$$

This implies immediately that

$$C_r(P) = I_{\binom{n}{r}}.$$

By the polar factorization theorem let $P = UH$ where U is unitary and H is positive definite Hermitian (p. d. h.). Then

$$C_r(U)C_r(H) = I_{\binom{n}{r}}$$

implies that $C_r(U)$ is both unitary and p. d. h. Hence every eigenvalue of $C_r(U)$ is 1 and this in turn implies that every eigenvalue of U is $e^{i\phi}$ for $r\phi \equiv 0\ (2\pi)$. Similarly we show $H = I_n$ and the result is at hand.

https://doi.org/10.4153/CJM-1959-039-4 Published online by Cambridge University Press

4. The structure of $\mathfrak{A}_j$ for $j = 1, 2, 3$. At this point Theorem 3.1 together with the results in **(1)** completely settle the question of the structure of $\mathfrak{A}_r$ when $r \geqslant 4$. It is easy to construct singular $T \in \mathfrak{A}_1$ (map A into the diagonal matrix B with $B_{ii} = A_{ii}$). Thus not much can be said about $\mathfrak{A}_1$. In examining $\mathfrak{A}_2$ we are led to two kinds of counterexamples: (i) those transformations $S \in \mathfrak{A}_2$ which permute the entries of every $A \in M_n$ in some fixed way; (ii) those transformations $C \in \mathfrak{A}_2$ which map A into $K \circ A$ where $K \in M_n$ and $K \circ A$ is the Hadamard product of K and A $((K \circ A)_{ij} = K_{ij}A_{ij}\, i, j = 1, \ldots, n)$. We shall show that there exist non-trivial examples of both types (i) and (ii) in $\mathfrak{A}_2$ but that no such examples exist in $\mathfrak{A}_3$. We remark here that Lemma 3.4 fails for $r = 3$; for take $A = E_{12} + E_{34} \in M_4$ and note that although $E_3\,(x\,A + B)$ is at most linear in x for $B \in M_4$, $\rho(A) = 2$. Thus there is no hope for proving Theorem 3.1 via Lemma 3.4 for $r = 3$.

Denote by S_r that subset of $\mathfrak{A}_r$ consisting of transformations that rearrange the elements of every $A \in M_n$ in some fixed way. Similarly, let H_r denote that subset of $\mathfrak{A}_r$ consisting of transformations of the type $A \to K \circ A$, $K \in M_n$.

THEOREM 4.1. *If $S \in S_2$ then $S = \sigma_1\, \sigma_2\, \sigma_3$ where*

(i) *σ_3 is a permutation of the main diagonal entries only.*

(ii) *σ_2 is a permutation of the set of pairs of entries symmetrically located across the main diagonal.*

(iii) *σ_1 interchanges symmetrically located entries.*

The proof of Theorem 4.1 is a straightforward enumeration of the possibilities for images under S of matrices of the types $E_{ii} + E_{jj}$, $i < j$ and $E_{ij} + E_{ji}$, $i < j$. We omit the details.

THEOREM 4.2. *No element of S_2 of the types* (i), (ii), (iii) *in* Theorem 4.1 *is a direct product except the identity map and the transpose map.*

Proof. This is done by showing that any map of the types σ_1, σ_2, σ_3 described in Theorem 4.1 maps some non-singular N into a singular matrix. First, suppose σ_3 maps the (j, j) entry into the (i_j, i_j) entry. Choose a permutation π of $1, \ldots, n$ such that $\pi(j) = j$ and $\pi(i) \neq i$ for $i \neq j$. Let N be the permutation matrix corresponding to π and observe that $\sigma_3(N)$ is singular. Next, suppose σ_2 maps (i, j) and (j, i) into (k, l) and (l, k) respectively. Let

$$N = E_{ij} + E_{ji} + \sum_{t \neq i, j} E_{tt}$$

and note that N is non-singular and $\sigma_2(N)$ is singular. Next, suppose σ_1 interchanges (i, j) and (j, i) and leaves fixed (k, l) and (l, k). It is not difficult to exhibit non-singular $N \in M_3$ or M_4 for which $\sigma_1(N)$ is singular and we proceed to show that the examples in M_n for $n > 4$ can be reduced to one of the cases $n = 3$ or $n = 4$. Suppose first that none of the equalities: $i = k$, $i = l$, $j = k$, $j = l$ holds. Then set $N_1 = E_{ij} + E_{ji} + E_{kl} + E_{lk}$ and let the permutation π of $1, \ldots, n$ be $(i\ j)\ (2\ i)$ with corresponding permutation matrix

https://doi.org/10.4153/CJM-1959-039-4 Published online by Cambridge University Press

P. Then $P\sigma(N_1)P' = E_{12} + E_{21} + E_{kl} + E_{lk}$. Similarly obtain a permutation matrix Q such that $QP\sigma_1(N_1)P'Q' = E_{12} + E_{21} + E_{34} + E_{43}$. We are then confronted essentially with the case $n = 4$. If any of the equalities $i = k$, $i = l$, $j = k$, $j = l$ holds we can reduce the situation to the case $n = 3$ by a similar device.

We may describe the structure of H_2 as follows:

THEOREM 4.3. *If $C \in H_2$, $C : A \to K \circ A$ then $K_{ij} = (K_{ji})^{-1}$ for $i \neq j$ and either $K_{ii} = 1$ $(i = 1, \ldots, n)$ or $K_{ii} = -1$ for $i = 1, \ldots, n$.*

We omit the proof which consists of a straightforward consideration of the possibilities for the 2-square sub-determinants of K.

We remark at this point that it seems plausible that $\mathfrak{A}_2$ is generated by taking only products of elements of S_2, H_2 and maps of the form $A \to PAP^{-1}$, $P \in GL_n$. We have been unable to prove this, however.

The situations for S_3 and H_3 are somewhat more involved but we shall use a sequence of lemmas to show that:

S_3 consists only of the identity map, the transpose map, and maps of the form $A \to PAP'$ for P a permutation matrix; H_3 consists only of the identity map and the map $A \to K \circ A = \theta DAD^{-1}$ where D is a diagonal matrix and θ is a cube root of 1. It is not known to us whether there exist other elements of $\mathfrak{A}_3$ which are not direct products.

LEMMA 4.1. *If $A \in M_n$ and A has n elements 1, the rest 0, then for $n > r \geqslant 1$,*

$$E_r(A) = \binom{n}{r}$$

if, and only if, $A = I_n$.

Proof. It is clear that since the rth order subdeterminants of A are integers that

$$E_r(A) \leqslant \operatorname{tr}\{[C_r(A)]\,[C_r(A)]'\}.$$

Hence

$$E_r(A) \leqslant \operatorname{tr} C_r(AA') = E_r(\alpha_1^2, \ldots, \alpha_n^2)$$

where α_j^2, $j = 1, \ldots, n$ are the eigenvalues of AA'. If $\rho(A) = k$ and $k < r$ it is clear that

$$0 = E_r(A) < \binom{n}{r}.$$

Otherwise if $k \geqslant r$

$$\begin{aligned} E_r(A) &\leqslant E_r(\alpha_1^2, \ldots, \alpha_n^2) = E_r(\alpha_1^2, \ldots, \alpha_k^2) \\ &\leqslant \binom{k}{r} k^{-r}\{E_1(\alpha_1^2, \ldots, \alpha_k^2)\}^r \\ &= \binom{k}{r} k^{-r}\{\operatorname{tr}(AA')\}^r = \binom{k}{r} k^{-r} n^r. \end{aligned}$$

We consider two cases:

(i) $k = n$. Then A is a permutation matrix and all eigenvalues lie on the unit circle. Then it is easily seen that

$$E_r(A) = \binom{n}{r}$$

implies all the eigenvalues are equal and the only permutation matrix with this property is I_n.

(ii) $k < n$. We shall show this is impossible. If $k = 1$, then $r = 1$ and $E_1(A) = \text{tr}\ (A) = n$. But I_n is the only matrix satisfying this and this is a contradiction. On the other hand, if $k \geqslant 2$ then

$$E_r(A) \leqslant \binom{k}{r} k^{-r} n^r < \binom{n}{r} = E_r(A)$$

and the proof is complete.

LEMMA 4.2. *If $S \in S_3$ and $n \geqslant 4$ then S either interchanges (i, j) and (j, i) for $i \neq j$ or leaves them fixed.*

Proof. Since

$$E_3(S(I_n)) = \binom{n}{3}$$

we have $S(I_n) = I_n$ by Lemma 4.1. Thus we may modify S to obtain

$$\sigma : A \rightarrow PS(A)P'$$

where $P \in M_n$ is such a permutation matrix that σ holds the main diagonal elements fixed. Now let

$$N_0 = 0_{n-2} \oplus J_2$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We show first that $\sigma(N_0) = N_0$. If this were not the case we have two possible alternatives:

(i) $\sigma(N_0)$ has a 1 at some position (k, l) such that $k < 1$ and $(k, l) \neq (n - 1, n)$.

(ii) $\sigma(N_0)$ has a 1 at some position (k, l) such that $k > 1$ and $(k, l) \neq (n, n - 1)$.

In (i) let D be a diagonal matrix in M_{n-2} with 1 at (k, k) and $(n - 3)$ zero's elsewhere on the diagonal. Then

$$E_3(D \oplus J_2) = -1.$$

However $\sigma(D \oplus J_2)$ has at most two non-zero rows and hence

$$E_3(\sigma(D \oplus J_2)) = 0.$$

In a similar way we eliminate the alternative (ii). Hence σ either interchanges or leaves fixed the entries at $(n-1, n)$ and $(n, n-1)$. A similar argument for the other pairs of symmetrically located entries completes the proof.

LEMMA 4.3. *If* $S \in S_r$, $r \geqslant 2$ *and*

$$S : A \rightarrow UAV$$

or

$$S : A \rightarrow UA'V$$

then U *and* V *are permutation matrices.*

We omit the proof.

THEOREM 4.4. *If* $S \in S_3$ *and* $n = p + 2$, $p \geqslant 1$ *then either*

$$S : A \rightarrow PAP' \quad \textit{for all} \quad A \in M_n$$

or

$$S : A \rightarrow PA'P' \quad \textit{for all} \quad A \in M_n$$

where $P \in M_n$ *is a permutation matrix.*

Proof. The proof is by induction on the integer p. For $p = 1$ the result in (**1**, Theorem 2) shows that S is a direct product (modulo taking the transpose), and Lemma 4.3 combined with argument used in the latter part of the proof of Theorem 3.1 establishes that S has the above form. Now we modify S as in Lemma 4.2 to obtain $\sigma \in S_3$ where σ holds diagonal elements fixed. Assume the result for all integers up to $p > 1$. Then if $C \in M_{n-1} = M_{(p-1)+2}$ we have by Lemma 4.2 that

$$\sigma(0 \oplus C) = 0 \oplus \sigma(C)$$

and

$$\begin{aligned} E_3(\sigma(C)) &= E_3(0 \oplus \sigma(C)) = E_3(\sigma(0 \oplus C)) \\ &= E_3(0 \oplus C) = E_3(C). \end{aligned}$$

By the induction hypothesis and the fact that σ holds the diagonal elements fixed we see that if we consider σ as a mapping of $M_{n-1} \rightarrow M_{n-1}$ in the obvious way then

$$\sigma : C \rightarrow C \text{ for all } C \in M_{n-1}$$

or

$$\sigma : C \rightarrow C' \text{ for all } C \in M_{n-1}.$$

Now it is clear that if $A \in M_n = M_{p+2}$ and $C_i \in M_{n-1}$ is the principal submatrix obtained by deleting row and column i of A then the above argument shows that

$$\sigma(C_i) = C_i$$

or

$$\sigma(C_i) = C_i'.$$

Thus for each $A \in M_n$ it follows that

$$\sigma(A) = A$$

or

$$\sigma(A) = A',$$

and the proof is complete.

THEOREM 4.5. *If $C \in H_3$ then there exists $D \in M_n$ such that*

$$C : A \to \theta DAD^{-1} \quad \textit{for all} \quad A \in M_n$$

where D is a diagonal matrix and $\theta^3 = 1$.

Proof. It suffices to show that there exist diagonal U and V in M_n such that $C(A) = UAV$ or $C(A) = UA'V$ for then it is clear that $U_{ii} = \theta^{-1}V_{ii}^{-1}$ for $i = 1, \ldots, n$ and $\theta^3 = 1$. Now for each $\omega \in Q_{3n}$ it is clear that we may consider C as a mapping of $M_3 \to M_3$ by restricting C to the principal submatrix of each $A \in M_n$ corresponding to the indices of ω. Call the restricted mapping $C_\omega : M_3 \to M_3$; and since C_ω preserves determinant it is a direct product:

$$C_\omega : A \to U_\omega A V_\omega \quad \text{for} \quad A \in M_3.$$

It is easy to check that U_ω and V_ω are diagonal by examining the images of $E_{ii} \in M_3$, $i = 1, 2, 3$ and using the fact that $C_\omega(A)$ is a Hadamard product. Thus on each 3-square principal submatrix C has the desired form. It will clearly suffice to show that $C : A \to K \circ A$ has the property $\rho(K) = 1$. For then K has the form $K_{ij} = a_i b_j$ $i, j = 1, \ldots, n$. We show that every 2-square submatrix of K is singular. Let $(\alpha_i\ \beta_j)$ denote the submatrix of K involving rows $\alpha_1\ \alpha_2$ and columns β_1, β_2. Suppose $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ involves fewer than 4 distinct integers. Then it is clear that $(\alpha_i\ \beta_j)$ is a part of some principal 3-square submatrix whose row and column indices we will designate by

$$\theta = \{\gamma_1\gamma_2\gamma_3\}.$$

By the above argument C_θ has the form

$$C_\theta : A \to U_\theta A V_\theta;\ A \in M_3$$

where U_θ and V_θ are diagonal with diagonal elements u_1, u_2, u_3 and v_1, v_2, v_3 respectively. It follows that for some i_1, i_2, j_1, j_2 that

$$K_{\alpha_s\beta_t} = u_{i_s}v_{j_t}\ s, t = 1, 2$$

and hence that $(\alpha_i\ \beta_j)$ is singular. In case $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ consists of 4 distinct integers we consider the two 3-square principal submatrices corresponding to

$$\mu = \{\alpha_1, \alpha_2, \beta_1\} \quad \text{and} \quad \sigma = \{\alpha_1, \alpha_2, \beta_2\}.$$

Again we see that

$$C_\mu : A \to U_\mu A V_\mu,\ A \in M_3$$
$$C_\sigma : A \to U_\sigma A V_\sigma,\ A \in M_3$$

where U_μ, U_σ, V_μ and V_σ are diagonal with main diagonals

$$(u_1, u_2, u_3),\ (u'_1, u'_2, u'_3),\ (v_1, v_2, v_3),\ (v'_1, v'_2, v'_3)$$

respectively. We then obtain for some i_1, j_1, i_2.

$$K_{\alpha_1\beta_1} = u_{i_1}v_{j_1} \qquad K_{\alpha_1\alpha_1} = u_{i_1}v_{i_1}$$
$$K_{\alpha_2\beta_1} = u_{i_2}v_{j_1} \qquad K_{\alpha_2\alpha_1} = u_{i_2}v_{i_1}$$

and for some n_1, n_2, m_2,

$$K_{\alpha_1\beta_1} = u'_{n_1}v'_{m_2} \qquad K_{\alpha_1\alpha_1} = u'_{n_1}v'_{n_1}$$
$$K_{\alpha_2\beta_2} = u'_{n_2}v'_{m_2} \qquad K_{\alpha_2\alpha_1} = u'_{n_2}v'_{n_1}.$$

From these equalities we see that

$$K_{\alpha_1\beta_1}/K_{\alpha_2\beta_1} = K_{\alpha_1\beta_2}/K_{\alpha_2\beta_2}$$

and again $(\alpha_i\, \beta_j)$ is singular.

Reference

1. M. Marcus and B. N. Moyls, *Linear transformations on algebras of matrices*, Can. J. Math., *11* (1959), 61–6.

University of British Columbia