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WEIGHTED OCCUPATION TIME FOR BRANCHING PARTICLE SYSTEMS AND A REPRESENTATION FOR THE SUPERCRITICAL SUPERPROCESS

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ABSTRACT. We obtain a representation for the supercritical Dawson-Watanabe superprocess in terms of a subcritical superprocess with immigration, where the immigration at a given time is governed by the state of an underlying branching particle system. The proof requires a new result on the laws of weighted occupation times for branching particle systems.

1. **Introduction.** In this paper we obtain a representation theorem for the supercritical (Dawson-Watanabe) superprocess $(X, \mathbb{P}^{\mu})$ over a (Borel right) Markov process ξ with branching mechanism $\phi(z) = bz - cz^2/2$, where $b, c > 0$. We will show in §3 that X can be represented as the sum of two independent components. If $(\tilde{X}, \tilde{\mathbb{P}}^{\mu})$ is the superprocess over ξ with branching mechanism $\tilde{\phi}(z) = -bz - cz^2/2$, then the first is a copy of $\tilde{X}$ under $\tilde{\mathbb{P}}^{\mu}$. The second is produced by choosing at random a finite number of particles via a Poisson random measure with intensity $(2b/c)\mu$, letting these move like independent copies of ξ and perform binary branching at rate b, each particle constantly throwing off mass at rate c that continues to evolve according to the dynamics under which mass evolves for $\tilde{X}$. In terms of the "particle picture", the particles throwing off mass can be thought of as individuals with infinite lines of descent (*cf.* [13, 12]). The bulk of the mass represents individuals without infinite lines of descent and, as we would thus expect and indeed show in Proposition 3.1, $\tilde{X}$ evolves like X conditioned on extinction.

To prove our representation theorem we will apply a new result on the law of the "weighted occupation time" for branching particle systems. This result describes the joint law of

$$\int_0^t ds\langle Z_s, g_{t-s}\rangle$$

and Z_t, for any branching particle system Z and collection of measurable functions $\{g_s\}$. To be more precise, denote by $\mathbb{H}^{\nu}$ the law of Z started with initial state ν (an integer-valued

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measure). We will show in §2 that

$$\mathbb{H}^{\nu} \exp - \int_0^t ds\langle Z_s, g_{t-s}\rangle - \langle Z_t, f\rangle = \exp -\langle \nu, V_t^g f\rangle,$$

where $V_t^g f$ is the unique solution to the integral equation

$$\exp -V_t^g f = P_t e^{-f} + \int_0^t ds\, P_{t-s}[\eta(\exp -V_s^g f) - g_s \exp -V_s^g f];$$

here (P_t) is the transition semigroup of the underlying spatial motion and η is an operator characterising the branching mechanism of the process. As far as we are aware, no such result has appeared before in the literature. The case where Z is critical binary branching Brownian motion in $\mathbb{R}^d$ and $g_t := 1_A$, for some bounded Borel A, was considered by Cox and Griffeath [2], where various asymptotic results are obtained and a statement similar to ours concerning the moments of the occupation time are justified heuristically. The analogous result for (a special class of) superprocesses was first obtained by Iscoe [10], and later generalised by Fitzsimmons [9] and Dynkin [4, 5].

The representation theorem was motivated by, and is in some sense a generalization of, the so-called *immortal particle representation* for the critical (*i.e.* $b = 0$) superprocess conditioned on non-extinction (in the sense of [8]). Evans [7] proves that this superprocess can be represented as the sum of two independent components. The first is a copy of the unconditioned superprocess: this is how the initial mass evolves. The second is produced by choosing at random an "immortal particle" according to the normalized initial measure, letting this move like an independent copy of the underlying spatial motion and throw off pieces of mass that continue to evolve according to the dynamics under which mass evolves for the unconditioned superprocess. The immortal particle representation was predicted by heuristic arguments of Aldous [1] as part of his work on continuum random trees, and by a Feynman-Kac type formula of Roelly-Coppoletta and Rouault [15].

2. **Weighted occupation times for branching particle systems.** Let $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, \xi_t, P^x)$ be a Borel right Markov process with Lusin state space $(E, d, \mathcal{E})$ and semigroup (P_t). We assume that (P_t) is conservative (*i.e.* $P_t 1 = 1$). Denote by $N(E)$ the class of finite integer-valued Borel measures on E, and by $\mathcal{N}(E)$ the Borel σ-algebra generated by the weak* topology on $N(E)$. We write $b\mathcal{E}$ (resp. $p\mathcal{E}$, $bp\mathcal{E}$) for the class of bounded (resp. non-negative, bounded and non-negative) $\mathcal{E}$-measurable real valued functions on E. Let φ be the probability generating function of a non-negative integer-valued random variable: $\varphi(z) = \sum p_i z^i$ $(0 \leq z \leq 1)$ for some non-negative sequence p_i, $i = 0, 1, 2, \ldots$ with $\sum p_i = 1$. We will assume that

$$\varphi'(1) \equiv \sum i p_i < \infty. \tag{1}$$

This assumption allows us to extend φ to the entire real line in such a way that the extended function, which we denote by $\tilde{\varphi}$, has bounded and continuous first derivatives on $\mathbb{R}$ and is therefore uniformly Lipschitz continuous on $\mathbb{R}$. We can (and will) also regard

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$\tilde{\varphi}$ as an operator on $b\mathcal{E}$ (considered as a Banach space with sup norm) by defining $[\tilde{\varphi}(f)](x) := \tilde{\varphi}\big(f(x)\big)$, for $f \in b\mathcal{E}$, $x \in E$. Then $\tilde{\varphi}$, considered as an operator on $b\mathcal{E}$ in this sense, is uniformly Lipschitz continuous on $b\mathcal{E}$.

Let $b \in \mathbb{R}_+$ and define the operator η on $b\mathcal{E}$ by $\eta(f) := b[\tilde{\varphi}(f) - f]$. Note that η uniquely determines φ and b, and is also uniformly Lipschitz continuous on $b\mathcal{E}$.

Let $Z = (W, \mathcal{G}, \mathcal{G}_t, \Theta_t, Z_t, \mathbb{H}^\nu)$ be a branching particle system with ξ as its underlying spatial motion, φ as the generating function of its offspring distribution, and with constant branching rate b. Then Z is a Borel right Markov branching process with (Lusin) state space $\big(N(E), \mathcal{N}(E)\big)$ and Laplace functionals given (see, for example, [5]) by

$$\mathbb{H}^\nu \exp -\langle Z_t, f\rangle = \exp -\langle \nu, V_t f\rangle, \tag{2}$$

for $f \in p\mathcal{E}$, where $\hat{V}_t f := \exp -V_t f$ satisfies the integral equation

$$\hat{V}_t f = P_t e^{-f} + \int_0^t ds\, P_s \eta(\hat{V}_{t-s} f). \tag{3}$$

We refer to η as the *branching mechanism* of Z, and to Z as a *branching particle system over ξ with branching mechanism η*. That $\hat{V}_t f$ is the unique solution to (3) follows from the following uniqueness lemma, which we record also for later reference. It is a modification of (part of) a well known theorem, originally due to Segal [17], a nice proof of which appears in [14, Theorem 6.1.2].

LEMMA 2.1. *Let X be an arbitrary Banach space, and let $f: [0,T] \times X \longrightarrow X$ be continuous in t on $[0,T]$ and uniformly Lipschitz continuous on X. Let (T_t) be a semigroup of bounded linear operators on X, uniformly bounded on $[0,T]$. Suppose that, for $u_0 \in X$, the integral equation*

$$u(t) = T_t u_0 + \int_0^t T_{t-s} f\big(s, u(s)\big)\, ds$$

has a solution $u: [0,T] \longrightarrow X$. Then it is unique.

The proof is a simple application of Gronwall's inequality (*cf.* [14]). (We state Lemma 2.1 in sufficient generality to allow the reader to extend the results of this section to branching particle systems with a more general time-dependent branching mechanism $\eta_t(z) = b_t[\varphi_t(z) - z]$, where b_t and φ_t depend continuously on t and η_t is uniformly Lipschitz on every bounded time interval.)

To apply the lemma to our case, note that (P_t) is a contraction semigroup on $b\mathcal{E}$, and is therefore bounded on intervals.

The main result of this section describes the joint law of the weighted occupation time

$$\int_0^t ds \langle Z_s, g_{t-s}\rangle$$

and Z_t under $\mathbb{H}^\nu$.

THEOREM 2.2. *Let $f \in p\mathcal{E}$ and, for each s, $g_s \in bp\mathcal{E}$. Assume that the mapping $(x,s) \longmapsto g_s(x)$ is jointly measurable in (x,s). Then, in the above notation,*

$$\mathbb{H}^\nu \exp - \int_0^t ds \langle Z_s, g_{t-s}\rangle - \langle Z_t, f\rangle = \exp -\langle \nu, V_t^g f\rangle, \tag{4}$$

where $\hat{V}_t^g f := \exp -V_t^g f$ *is the unique solution to the integral equation*

$$\hat{V}_t^g f = P_t e^{-f} + \int_0^t ds\, P_{t-s}[\eta(\hat{V}_s^g f) - g_s \hat{V}_s^g f]. \tag{5}$$

PROOF. Without loss of generality we can assume that g_s is independent of time: $g_s = g$, say, $\forall s$. To extend the argument to time dependent g_s, just set $g(x,s) = g_s(x)1_{s\le t}$ and consider the branching particle system with the same branching mechanism η, but over the space-time process associated with ξ, and with initial measure $\nu \times \delta_0$. The hypothesis ensures that $g \in bp\mathcal{E}^*$, where $\mathcal{E}^*$ is the Borel σ-algebra on $E \times \mathbb{R}_+$.

It follows from the branching property of Z that, for each $t \ge 0$, there exists $V_t^g f \in p\mathcal{E}$ such that

$$\mathbb{H}^\nu \exp - \int_0^t ds\langle Z_s, g\rangle - \langle Z_t, f\rangle = \exp -\langle \nu, V_t^g f\rangle. \tag{6}$$

To show that $\hat{V}_t^g f$ is the unique solution to (5) it is sufficient to prove that it satisfies (5): the uniqueness follows from Lemma 2.1. (Note that since η is uniformly Lipschitz continuous on $b\mathcal{E}$, the mapping $f \longmapsto \eta(f) - fg$ is also uniformly Lipschitz continuous on $b\mathcal{E}$.)

We will first show, by conditioning on the first branch point, that $\hat{V}_t^g f$ satisfies the integral equation

$$\hat{V}_t^g f = P_t^g e^{-f} + \int_0^t ds\, P_{t-s}^g \eta(\hat{V}_s^g f), \tag{7}$$

where (P_t^g) is the semigroup on $b\mathcal{E}$ defined by

$$P_t^g h(x) := P^x\Big(\exp - \int_0^t ds\, g(\xi_s)\Big) h(\xi_t). \tag{8}$$

Then we will establish the equivalence of (7) and (5) to complete the proof.

By conditioning on the first branch point

$$\inf\{t > 0 : Z(t) > Z(0)\} \tag{9}$$

we get

$$\hat{V}_t^g f = e^{-bt} P_t^g e^{-f} + \int_0^t b e^{-br} P_r^g \phi(\hat{V}_{t-r}^g f)\, dr. \tag{10}$$

Now by repeated application of (10),

$$\begin{aligned}
\int_0^t bP_r^g(\hat{V}_{t-r}^g f)\, dr &= \int_0^t bP_r^g\Big[e^{-b(t-r)} P_{t-r}^g e^{-f} + \int_0^{t-r} be^{-bs} P_s^g \phi(\hat{V}_{t-r-s}^g f)\, ds\Big]\, dr \\
&= (1 - e^{-bt}) P_t^g e^{-f} + \int_0^t bP_r^g \int_0^{t-r} be^{-bs} P_s^g \phi(\hat{V}_{t-r-s}^g f)\, ds \\
&= P_t^g e^{-f} - \hat{V}_t^g f + \int_0^t be^{-br} P_r^g \phi(\hat{V}_{t-r}^g f)\, dr \\
&\quad + \int_0^t bP_r^g \int_0^{t-r} be^{-bs} P_s^g \phi(\hat{V}_{t-r-s}^g f)\, ds
\end{aligned} \tag{11}$$

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To simplify the last expression on the RHS of (11) we change the order of integration and integrate by parts:

$$\begin{aligned}\int_0^t bP_r^g \int_0^{t-r} be^{-bs}P_s^g\phi(\hat{V}_{t-r-s}^g f)\,ds\,dr &= \int_0^t be^{-bs}P_s^g \int_0^{t-s} bP_r^g\phi(\hat{V}_{t-r-s}^g f)\,dr\,ds\\ &= \int_0^t be^{-bs}\int_s^t bP_r^g\phi(\hat{V}_{t-r}^g f)\,dr\,ds\\ &= \int_0^t bP_r^g\phi(\hat{V}_{t-r}^g f)\,dr - \int_0^t be^{-bs}P_s^g\phi(\hat{V}_{t-s}^g f)\,ds.\end{aligned}$$

Equation (7) follows. To show that (7) and (5) are equivalent we use the following version of the Feynman-Kac formula (*cf.* [18, III.39]).

LEMMA 2.3 (FEYNMAN-KAC). *In the above notation*

$$P_t^g f = P_t f - \int_0^t ds\,P_s(gP_{t-s}^g f).$$

PROOF. By the Markov property,

$$\begin{aligned}\int_0^t ds\,P_s(gP_{t-s}^g f) &= \int_0^t ds\,P_s\Big[gP^{\cdot}\Big(\exp - \int_0^{t-s} g(\xi_r)\,dr\Big)f(\xi_{t-s})\Big]\\ &= \int_0^t ds\,P^x g(\xi_s)P^{\xi_s}\Big[\Big(\exp - \int_0^{t-s} g(\xi_r)\,dr\Big)f(\xi_{t-s})\Big]\\ &= P^x\int_0^t ds\,g(\xi_s)\Big(\exp - \int_s^t g(\xi_r)\,dr\Big)f(\xi_t)\\ &= P^x f(\xi_t)\int_0^t d\Big(\exp - \int_s^t g(\xi_r)\,dr\Big)\\ &= P_t f - P_t^g f.\end{aligned}$$

∎

Now (7) becomes

$$\begin{aligned}\hat{V}_t^g f &= P_t^g e^{-f} + \int_0^t ds\,P_s^g\eta(\hat{V}_{t-s}^g f)\\ &= P_t^g e^{-f} + \int_0^t ds\Big\{P_s\eta(\hat{V}_{t-s}^g f) - \int_0^s dr\,P_r[gP_{s-r}^g\eta(\hat{V}_{t-s}^g f)]\Big\}\\ &= P_t^g e^{-f} + \int_0^t ds\,P_s\eta(\hat{V}_{t-s}^g f) - \int_0^t dr\,P_r\Big[g\int_r^t ds\,P_{s-r}^g\eta(\hat{V}_{t-s}^g f)\Big]\\ &= P_t^g e^{-f} + \int_0^t ds\,P_s\eta(\hat{V}_{t-s}^g f) - \int_0^t dr\,P_r[(\hat{V}_{t-r}^g f - P_{t-r}^g e^{-f})g]\\ &= P_t e^{-f} + \int_0^t ds\,P_{t-s}[\eta(\hat{V}_s^g f) - g\hat{V}_s^g f],\end{aligned}$$

and the theorem is proved. ∎

3. **The representation theorem.** Let ξ be a Borel right Markov process with Lusin state space $(E, \mathcal{E})$ and conservative semigroup (P_t). Denote by $M(E)$ the class of finite Borel measures on E. Let $X = (W, \mathcal{G}, \mathcal{G}_t, \Theta_t, X_t, \mathbb{P}^\mu)$ and $\tilde{X} = (\tilde{W}, \tilde{\mathcal{G}}, \tilde{\mathcal{G}}_t, \tilde{\Theta}_t, \tilde{X}_t, \tilde{\mathbb{P}}^\mu)$ be superprocesses over ξ with respective branching mechanisms $\phi(z) = bz - cz^2/2$ and

$\tilde{\phi}(z) = -bz - cz^2/2$ $(b, c > 0)$, and denote their respective transition semigroups by (Q_t) and $(\tilde{Q}_t)$. (For details concerning the existence and regularity of superprocesses in this context, see [9].) Denote by (U_t) and $(\tilde{U}_t)$ the cumulant semigroups associated with X and $\tilde{X}$ respectively. Thus, for each $f \in bp\mathcal{E}$, U_tf and $\tilde{U}_tf$ are the unique solutions to the integral equations

$$U_tf = P_tf + \int_0^t ds\, P_s\phi(U_{t-s}f) \tag{12}$$

and

$$\tilde{U}_tf = P_tf + \int_0^t ds\, P_s\tilde{\phi}(\tilde{U}_{t-s}f), \tag{13}$$

respectively. The Laplace functionals of X and $\tilde{X}$ are given by

$$\mathbb{P}^\mu \exp -\langle X_t, f\rangle = \exp -\langle \mu, U_tf\rangle, \tag{14}$$

and

$$\tilde{\mathbb{P}}^\mu \exp -\langle \tilde{X}_t, f\rangle = \exp -\langle \mu, \tilde{U}_tf\rangle. \tag{15}$$

The relationship between X and $\tilde{X}$ is given by the following proposition.

PROPOSITION 3.1. *The superprocess X conditioned on extinction has the same law as $\tilde{X}$.*

PROOF. Set

$$T = \inf\{t \geq 0 : \langle X_t, 1\rangle = 0\}.$$

By the Markov property,

$$\begin{aligned} \mathbb{P}^\mu\{\exp -\langle X_t, f\rangle \mid T < \infty\} &= \mathbb{P}^\mu\{T < \infty\}^{-1}\mathbb{P}^\mu\{\exp -\langle X_t, f\rangle(T < \infty)\} \\ &= \mathbb{P}^\mu\{T < \infty\}^{-1}\mathbb{P}^\mu\{\exp -\langle X_t, f\rangle \mathbb{P}^{X_t}(T < \infty)\}. \end{aligned} \tag{16}$$

To calculate $\mathbb{P}^\mu\{T < \infty\}$, let f be a constant, λ say, and solve (12) for $U_t\lambda$. Now plug this into (14), let $\lambda \longrightarrow \infty$ and then $t \longrightarrow \infty$ to get

$$\mathbb{P}^\mu\{T < \infty\} = \exp -\langle \mu, 2b/c\rangle. \tag{17}$$

Therefore, by (17), (16) and (14),

$$\begin{aligned} \mathbb{P}^\mu\{\exp -\langle X_t, f\rangle | T < \infty\} &= (\exp\langle \mu, 2b/c\rangle)\mathbb{P}^\mu\{\exp -\langle X_t, f + 2b/c\rangle\} \\ &= \exp -\langle \mu, U_t(f + 2b/c) - 2b/c\rangle. \end{aligned}$$

It is easy to check that $U_t(f + 2b/c) - 2b/c$ satisfies (13), so by uniqueness it must equal $\tilde{U}_tf$, as required. ∎

We now construct an $M(E) \times N(E)$-valued branching process $(W, Y, \mathbb{Q}^{\mu,\nu})$ as follows. First, let $(Y, \mathbb{Q}^{\mu,\nu})$ be a branching particle system over ξ with branching mechanism $\chi(z) = bz(z-1)$ and initial measure ν. We remind the reader that χ corresponds to binary branching at rate b. Note that for this branching particle system the condition

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(1) is satisfied. Then, conditional on $\{Y_t,\ t \geq 0\}$, let $(W, \mathbb{Q}^{\mu,\nu})$ be a superprocess over ξ with branching mechanism $\tilde{\phi}$, initial measure μ, and with immigration; where the immigration at time t is according to the measure cY_t. (Superprocesses with immigration were introduced by Dawson [3]; see also [15].) To write down the Laplace functionals of this process, first note that

(18)

$$\mathbb{Q}^{\mu,\nu}\{\exp -\langle W_t, f\rangle - \langle Y_t, h\rangle \mid Y_t, t \geq 0\} = \exp -\langle \mu, \tilde{U}_t f\rangle - \int_0^t ds \langle cY_s, \tilde{U}_{t-s} f\rangle - \langle Y_t, h\rangle.$$

Now take expectations under $\mathbb{Q}^{\mu,\nu}$ to get

(19)

$$\mathbb{Q}^{\mu,\nu} \exp -\langle W_t, f\rangle - \langle Y_t, h\rangle = [\exp -\langle \mu, \tilde{U}_t f\rangle] \mathbb{Q}^{\mu,\nu} \exp - \int_0^t ds \langle Y_s, c\tilde{U}_{t-s} f\rangle - \langle Y_t, h\rangle.$$

We denote the transition semigroup of (W, Y) by (R_t). Denote by N_μ the law of the Poisson random measure on E with intensity $(2b/c)\mu$. The Laplace functionals of N_μ (see, for example, [11]) are given by

$$\int_{N(E)} N_\mu(d\nu) \exp -\langle \nu, h\rangle = \exp -\left\langle \frac{2b}{c}\mu, 1 - e^{-h} \right\rangle. \tag{20}$$

We are now ready to state the theorem.

THEOREM 3.2. *The law of W under $\mathbb{Q}^{\delta_\mu \times N_\mu}$ is the same as the law of X under $\mathbb{P}^\mu$.*

Our strategy for proving Theorem 3.2 will be first to show that the one-dimensional distributions coincide; then we show that W under $\mathbb{Q}^{\delta_\mu \times N_\mu}$ is a Markov process, and the result follows. To do this we will need the following criterion for a function of a Markov process to be also Markov, due to Rogers and Pitman [16, Theorem 2]. We state the result as it appears in [7].

LEMMA 3.3. *Consider two measurable spaces F and G and a Markov process Z with state space F and transition semigroup (S_t). Let Γ be the Markov kernel from F to G which is induced by a measurable function $\gamma\colon F \longrightarrow G$ according to the formula*

$$\int f(x)\Gamma(\cdot, dx) = f \circ \gamma,$$

f measurable, and let Λ be a Markov kernel from G to F. Suppose that:

(i) the kernel $\Lambda\Gamma$ is the identity kernel on G;

(ii) for each $t \geq 0$, the Markov kernel $T_t := \Lambda S_t \Gamma$ from G to G satisfies the identity $\Lambda S_t = T_t \Lambda$;

(iii) the process Z has initial distribution $\Lambda(y, \cdot)$ for some $y \in G$.

Then $\gamma \circ Z$ is a Markov process with initial state y and transition semigroup (T_t).

REMARK. Dynkin [6] has found an alternative approach for proving Theorem 3.2, namely by using an analytic characterisation of the Laplace functionals of the finite dimensional distributions of a superprocess to show directly that the finite dimensional distributions of the two processes coincide.

PROOF OF THEOREM 3.2. First we show that for $f \in bp\mathcal{E}$,

$$\mathbb{Q}^{\delta_\mu \times N_\mu} \exp -\langle W_t, f\rangle = \mathbb{P}^\mu \exp -\langle X_t, f\rangle. \tag{21}$$

By (19), this can be rewritten as

$$\mathbb{Q}^{\delta_\mu \times N_\mu} \exp - \int_0^t ds \langle Y_s, c\tilde{U}_{t-s}f\rangle = \exp -\langle \mu, U_t f - \tilde{U}_t f\rangle. \tag{22}$$

Now to apply Theorem 2.2, set $g_t = c\tilde{U}_t f$ and write V_t^1 for V_t^g. The measurability of g_t follows from [9, Proposition 2.3(a)]. Therefore, by Theorem 2.2 and (20),

$$\begin{aligned}\mathbb{Q}^{\delta_\mu \times N_\mu} \exp - \int_0^t ds \langle Y_s, c\tilde{U}_{t-s}f\rangle &= \int_{N(E)} N_\mu(d\nu) \exp -\langle \nu, V_t^1(0)\rangle \\ &= \exp -\langle (2b/c)\mu, 1 - \exp -V_t^1(0)\rangle,\end{aligned}$$

and so it is sufficient to show that

$$\exp -V_t^1(0) = 1 - \frac{c}{2b}(U_t f - \tilde{U}_t f). \tag{23}$$

It follows from (5) that $\hat{V}_t^1(0) := \exp -V_t^1(0)$ is the unique solution to the integral equation

$$\hat{V}_t^1(0) = 1 + \int_0^t ds\, P_s\Big[\chi\big(\hat{V}_{t-s}^1(0)\big) - c\big(\hat{V}_{t-s}^1(0)\big)(\tilde{U}_{t-s}f)\Big], \tag{24}$$

and from (12) and (13) that the right hand side of (23) also satisfies (24), as required.

We have thus proved that the one-dimensional distributions coincide, and all that remains to be shown is that W under $\mathbb{Q}^{\delta_\mu \times N_\mu}$ is Markov. To do this we apply Lemma 3.3. Denote by Γ the Markov kernel induced by the projection from $M(E) \times N(E)$ onto $M(E)$ and by Λ the Markov kernel from $M(E)$ to $M(E) \times N(E)$ given by $\Lambda(\mu, \cdot) = \delta_\mu \times N_\mu$. Clearly, $\Lambda\Gamma$ is the identity kernel on $M(E)$. It follows from (21) that $Q_t = \Lambda R_t \Gamma$, so by Lemma 3.3 all we need to show is that $\Lambda R_t = Q_t \Lambda$. This would follow if for all $h \in bp\mathcal{E}$,

$$\mathbb{Q}^{\delta_\mu \times N_\mu}\{\exp -\langle Y_t, h\rangle \mid W_t\} = \exp -\Big\langle \frac{2b}{c} W_t, 1 - e^{-h}\Big\rangle, \tag{25}$$

$\mathbb{Q}^{\delta_\mu \times N_\mu}$–almost surely; or equivalently, if for all $h, f \in bp\mathcal{E}$,

$$\mathbb{Q}^{\delta_\mu \times N_\mu} \exp -\Big\langle \frac{2b}{c} W_t, 1 - e^{-h}\Big\rangle - \langle W_t, f\rangle = \mathbb{Q}^{\delta_\mu \times N_\mu} \exp -\langle Y_t, h\rangle - \langle W_t, f\rangle. \tag{26}$$

By (19), (20) and Theorem 2.2 the right hand side of (26) is equal to

$$\exp -\langle \mu, \tilde{U}_t f\rangle - \Big\langle \frac{2b}{c}\mu, 1 - \exp -V_t^1 h\Big\rangle, \tag{27}$$

where $\hat{V}_t^1 h := \exp -V_t^1 h$ is the unique solution to the integral equation

$$\hat{V}_t^1 h = P_t e^{-h} + \int_0^t ds\, P_s[\chi(\hat{V}_{t-s}^1 h) - c(\hat{V}_{t-s}^1 h)(\tilde{U}_{t-s}f)]. \tag{28}$$

Similarly, the left hand side of (26) is equal to

$$\exp -\Big\langle \mu, \tilde{U}_t\Big(\frac{2b}{c}(1-e^{-h})+f\Big)\Big\rangle - \Big\langle \frac{2b}{c}\mu, 1-\exp -V_t^2 h\Big\rangle, \tag{29}$$

where $\hat{V}_t^2 h := \exp -V_t^2 h$ is the unique solution to the integral equation

$$\hat{V}_t^2 h = P_t e^{-h} + \int_0^t ds\, P_s\Big[\chi(\hat{V}_{t-s}^2 h) - c(\hat{V}_{t-s}^2 h)\Big(\tilde{U}_{t-s}\Big(\frac{2b}{c}(1-e^{-h})+f\Big)\Big)\Big]. \tag{30}$$

Finally, it is easy to check using (28), (30), (13) and Lemma 2.1 that

$$\tilde{U}_t f - \frac{2b}{c}\hat{V}_t^1 h = \tilde{U}_t\Big(f - \frac{2b}{c}e^{-h}\Big) \tag{31}$$

and

$$\tilde{U}_t\Big(f + \frac{2b}{c}(1-e^{-h})\Big) - \frac{2b}{c}\hat{V}_t^2 h = \tilde{U}_t\Big(f - \frac{2b}{c}e^{-h}\Big). \tag{32}$$

It follows that (26) holds, and the theorem is proved. ∎

In particular, Theorem 3.2 gives us a representation for the total mass process $M_t := \langle X_t, 1\rangle$, a diffusion with infinitesimal generator

$$Af = \frac{c}{2}x\frac{d^2 f}{dx^2} + bx\frac{df}{dx}, \quad f \in C_c^\infty(\mathbb{R}_+).$$

Let $(\omega, \zeta, Q^{w,z})$ be the process with generator

$$Bg(w,z) = \frac{c}{2}w\frac{\partial^2 g}{\partial w^2}(w,z) + (cz - bw)\frac{\partial g}{\partial w}(w,z) + bz[g(w,z+1) - g(w,z)],$$

$g \in C_c^\infty(\mathbb{R}_+^2)$. It is easy to check that the process $(\langle W_\cdot, 1\rangle, \langle Y_\cdot, 1\rangle)$ under $\mathbb{Q}^{\mu,\nu}$ has the same law as (ω, ζ) under $Q^{\langle\mu,1\rangle,\langle\nu,1\rangle}$. If μ_x is Poisson with rate $2bx/c$, then by Theorem 3.2 the process $(\omega_t, t \geq 0)$ under Q^{x,μ_x} has the same law as the process $(M_t, t \geq 0)$ started at x. For an independent proof of this fact, relying only on the theory of diffusion processes, see [12].

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