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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A NEUMANN PROBLEM INVOLVING VARIABLE EXPONENT GROWTH CONDITIONS

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Abstract. In this paper we study a non-linear elliptic equation involving $p(x)$-growth conditions and satisfying a Neumann boundary condition on a bounded domain. For that equation we establish the existence of two solutions using as a main tool an abstract linking argument due to Brézis and Nirenberg.

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1. Introduction. The goal of this paper is to establish the existence of solutions for the Neumann problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = f(u), & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ with $1 < p(x) < N$ for all $x \in \overline{\Omega}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function given by the formula

$$f(t) = \begin{cases} |t|^{a-1}t, & \text{for } |t| \leq \left(\dfrac{1}{2}\right)^{\frac{1}{a-1}}, \\ t - |t|^{a-1}t, & \text{for } |t| > \left(\dfrac{1}{2}\right)^{\frac{1}{a-1}}, \end{cases}$$

where a is a positive real number.

The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics (see [**27**]), electrorheological fluids (see [**1**], [**5**], [**14**], [**26**]) or image restoration (see [**4**]). In what concern some recent studies on equations possessing variable exponent growth conditions we refer to [**10**, **11**, **16**–**23**] and the references therein.

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This paper is motivated by the studies in [17] and [18]. In [17] the following problem is studied

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = A|u|^{a-2}u + B|u|^{b-2}u, & \text{for } x \in \Omega, \\ u = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{2}$$

where $p \in C(\overline{\Omega})$ verifies $p(x) > 1$ for any $x \in \overline{\Omega}$ and $1 < a < \inf_\Omega p < \sup_\Omega p < b < \min\{N, \frac{N \cdot \inf_\Omega p}{N - \inf_\Omega p}\}$ and A, $B > 0$. Using Ekeland's variational principle and the mountain-pass lemma, the author shows that for A and B small enough problem (2) has two distinct solutions.

In [18] the following Neumann problem is analysed

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda(|u|^{q(x)-2}u - u), & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{for } x \in \partial\Omega, \end{cases} \tag{3}$$

where $p \in C(\overline{\Omega})$ verifies $p(x) > N$ for any $x \in \overline{\Omega}$, $\lambda > 0$ is a constant and $q \in C(\overline{\Omega})$ satisfies $2 < q(x) < \inf_{y \in \overline{\Omega}} p(y)$ for any $x \in \overline{\Omega}$. For problem (3) the author proves the existence of three solutions by using a result due to Ricceri [25].

In the present paper we continue the studies begun in [17] and [18]. Under suitable conditions we will prove the existence of two solutions for problem (1) by applying an abstract linking argument due to Brézis and Nirenberg [3]. More exactly, our key argument will be the following theorem.

THEOREM 1 (Brézis–Nirenberg [3]). *Assume X is a Banach space with the direct sum decomposition*

$$X = X_1 \oplus X_2,$$

with $\dim X_2 < \infty$. *Assume $J \in C^1(X, \mathbb{R})$ with $J(0) = 0$ satisfies (PS) condition (i.e., any sequence $\{u_n\} \subset X$ satisfying $\{J(u_n)\}$ is a bounded sequence in $\mathbb{R}$ and $\langle J'(u_n), v\rangle \le \epsilon_n \|v\|_X$ for any $v \in X$, with $\epsilon_n \to 0$, has a convergent subsequence). Moreover, for a positive constant $R > 0$, we have*

$$J(u) \ge 0, \quad \text{for all } u \in X_1 \text{ with } \|u\|_X \le R,$$

$$J(u) \le 0, \quad \text{for all } u \in X_2 \text{ with } \|u\|_X \le R.$$

Also assume that J is bounded below and $\inf_X J < 0$. *Then J has at least two non-trivial critical points.*

2. Preliminary results.

In this section we recall some background facts concerning the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. We refer the reader to the book by Musielak [24] and the papers by Edmunds [6–8], Kovacik and Rákosník [15] and Fan [9, 12].

Throughout this paper we assume that $p(x) > 1$, $p(x) \in C(\overline{\Omega})$.

Set

$$C_+(\overline{\Omega}) = \{h;\ h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

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For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We define a norm, th so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1(x) \le p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder-type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{4}$$

holds true.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ then the following relations hold true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{5}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{6}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \tag{7}$$

Next, we define

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega);\ \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega),\ \text{for any } x \in \{1, \ldots, N\}\right\}.$$

On $W^{1,p(x)}(\Omega)$ we consider the norm

$$\|u\| = |u|_{p(x)} + |\ |\nabla u|\ |_{p(x)}.$$

We remember that $(W^{1,p(x)}(\Omega), \|\cdot\|)$ is a reflexive and separable Banach space.

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Set

$$\Lambda(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx.$$

Then

$$\|u\|^{p^-} \leq \Lambda(u) \leq \|u\|^{p^+}, \quad \forall\, u \in W^{1,p(x)}(\Omega) \text{ with } \|u\| > 1, \tag{8}$$

$$\|u\|^{p^+} \leq \Lambda(u) \leq \|u\|^{p^-}, \quad \forall\, u \in W^{1,p(x)}(\Omega) \text{ with } \|u\| < 1, \tag{9}$$

$$\|u_n - u\| \to 0 \;\Leftrightarrow\; \Lambda(u_n - u) \to 0. \tag{10}$$

Finally, we note that if $s(x) \in C(\overline{\Omega})$ and $1 < s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \geq N$.

3. The main result. In this paper we study the existence and multiplicity of weak solutions for problem (1). We say that $u \in W^{1,p(x)}(\Omega)$ is a *weak solution* of (1) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\, dx - \int_\Omega f(u) v\, dx = 0,$$

for any $v \in W^{1,p(x)}(\Omega)$.

The main result of this paper is given by the next theorem.

THEOREM 2. *Assume the following inequality holds true*

$$p^+ < a < \frac{Np^-}{N-p^-}, \tag{11}$$

where a is given in the definition of f. Then problem (1) has at least two non-trivial weak solutions.

We point out that in the context of Orlicz–Sobolev spaces a similar problem as (1) was studied recently by Halidias and Le [**13**]. Our result is more general than the result in [**13**] since the variable exponent Sobolev spaces are a special type of Musielak–Orlicz spaces which generalize the Orlicz–Sobolev spaces.

4. Proof of Theorem 2. Let $J : W^{1,p(x)}(\Omega) \to \mathbb{R}$ be the energy functional corresponding to problem (1)

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \int_\Omega F(u)\, dx,$$

where F is a primitive of f, i.e.,

$$F(t) = \int_0^t f(r)\, dr = \begin{cases} \dfrac{1}{a+1}|t|^{a+1}, & \text{for } |t| \leq \left(\dfrac{1}{2}\right)^{\frac{1}{a-1}} \\[2mm] \dfrac{t^2}{2} - \dfrac{1}{a+1}|t|^{a+1} - D, & \text{for } |t| > \left(\dfrac{1}{2}\right)^{\frac{1}{a-1}} \end{cases},$$

with D a positive constant such that F is continuous on $\mathbb{R}$.

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Standard arguments imply that $J \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ with

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx - \int_\Omega f(u)v\, dx,$$

for any $u,\ v \in W^{1,p(x)}(\Omega)$. Thus, we observe that the critical points of functional J correspond to the weak solutions of equation (1).

On the other hand, we point out that since $p^- \le p(x)$ for all $x \in \overline{\Omega}$, it follows that $W^{1,p(x)}(\Omega) \subset W^{1,p^-}(\Omega)$.

Set

$$V' = \left\{u \in W^{1,p^-}(\Omega);\ \int_\Omega u(x)\, dx = 0\right\}$$

and

$$V = V' \cap W^{1,p(x)}(\Omega).$$

Clearly, V' is the topological complement of $\mathbb{R}$ with respect to $W^{1,p^-}(\Omega)$ and V is the topological complement of $\mathbb{R}$ with respect to a subspace X of $W^{1,p(x)}(\Omega)$, i.e.,

$$W^{1,p^-}(\Omega) = V' \oplus \mathbb{R},$$
$$X = V \oplus \mathbb{R} \subset W^{1,p(x)}(\Omega).$$

The above considerations show that it is enough to find weak solutions for equation (1) in the subspace X of $W^{1,p(x)}(\Omega)$.

Remark 1. We remark that using the Poincaré–Wirtinger inequality (see **[2]**, p. 194) we have

$$|u|_{p^-} \le C \cdot \|\nabla u\|_{p^-}, \quad \forall\, u \in V', \tag{12}$$

where $C > 0$ is constant.

Our idea is to prove Theorem 2 by applying Theorem 1. With that end in view, we prove some auxiliary results which show that functional J satisfies the conditions from the hypotheses of Theorem 1.

Lemma 1. *Assume that condition (11) is fulfilled. Then J is bounded from below and* $\inf_X J < 0$.

Proof. Clearly, by the definition of function F we observe that $F(t) \le 0$ for t large enough. Since F is continuous on $\mathbb{R}$ we deduce that there exists a constant $k > 0$ such that

$$\int_\Omega F(u)\, dx \le k, \quad \forall\, u \in X.$$

Thus, we find

$$J(u) \ge \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - k \ge -k > -\infty, \quad \forall\, u \in X,$$

or J is bounded from below.

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On the other hand, there exists a constant $t_1 > 0$ small enough such that $\int_\Omega F(t_1)\ dt = \int_\Omega \frac{1}{a+1} {t_1}^{a+1}\ dt = \frac{1}{a+1} {t_1}^{a+1}|\Omega| > 0$. Using that fact we get

$$J(t_1) < 0.$$

Since any constant function is an element of X we infer that $\inf_X J < 0$. The proof of Lemma 1 is complete. □

LEMMA 2. *Assume that condition (11) is fulfilled. Then J satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subseteq X$ be such that

$$|J(u_n)| \leq M \tag{13}$$

and

$$|\langle J'(u_n), \varphi\rangle| \leq \epsilon_n \|\varphi\|, \quad \forall \varphi \in X, \tag{14}$$

where $\epsilon_n \to 0$.

We claim that $\{u_n\}$ is bounded in X. Arguing by contradiction and passing to a subsequence, we assume that $\|u_n\| \to \infty$ and $\|u_n\| > 1$.

Set

$$v_n(x) := \frac{u_n(x)}{\|u_n\|}.$$

Since $\{v_n\}$ is bounded in X and X is a reflexive Banach space we can assume that, passing eventually to a subsequence, v_n converges weakly to v in X. Next, since X is compactly embedded in $L^{p(x)}(\Omega)$ we infer that v_n converges strongly to v in $L^{p(x)}(\Omega)$.

By (13) we have

$$\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)}\, dx - \int_\Omega F(u_n)\, dx \leq M. \tag{15}$$

On the other hand, it is obvious that

$$t^{p(x)} \geq \rho^{p^-} \cdot \left(\frac{t}{\rho}\right)^{p(x)}, \quad \forall\ t > 0,\ \rho > 1,\ x \in \Omega.$$

Choosing $t = |\nabla u_n(x)|$ and $\rho = \|u_n\| > 1$ we get,

$$\left|\frac{|\nabla u_n(x)|}{\|u_n\|}\right|^{p(x)} \cdot \|u_n\|^{p^-} \leq |\nabla u_n(x)|^{p(x)}, \quad \forall\ x \in \Omega. \tag{16}$$

Using (16) we deduce that

$$\int_\Omega \frac{1}{p(x)} |\nabla v_n(x)|^{p(x)}\, dx \leq \frac{1}{\|u_n\|^{p^-}} \int_\Omega \frac{1}{p(x)} |\nabla u_n(x)|^{p(x)}\, dx. \tag{17}$$

Dividing (15) by $\|u_n\|^{p^-} > 1$ and using (17) we obtain

$$\int_\Omega \frac{1}{p(x)} |\nabla v_n(x)|^{p(x)}\, dx \leq \int_\Omega \frac{F(u_n)}{\|u_n\|^{p^-}}\, dx + \frac{M}{\|u_n\|^{p^-}}, \quad \forall\ n. \tag{18}$$

Next, we prove that

$$\int_\Omega \frac{F(u_n)}{\|u_n\|^{p^-}}\,dx \to 0. \tag{19}$$

The definition of F implies that there exists a constant $M_1 > 0$ such that

$$\frac{F(t)}{|t|^{p^-}} \le 0, \quad \forall\, |t| > M_1, \quad \text{a.e. } x \in \Omega.$$

Hence

$$\begin{aligned}\int_\Omega \frac{F(u_n)}{\|u_n\|^{p^-}}\,dx &\le \int_{\{x\in\Omega; |u_n(x)|\le M_1\}} \frac{F(u_n)}{\|u_n\|^{p^-}}\,dx + \int_{\{x\in\Omega; |u_n(x)|\ge M_1\}} \frac{F(u_n)}{|u_n(x)|^{p^-}}\,\frac{|u_n(x)|^{p^-}}{\|u_n\|^{p^-}}\,dx\\ &\le \int_{\{x\in\Omega; |u_n(x)|\le M_1\}} \frac{F(u_n)}{\|u_n\|^{p^-}}\,dx.\end{aligned}$$

The above results assure that (19) holds true.

By (18) and (19) we have

$$\int_\Omega \frac{1}{p(x)}|\nabla v_n|^{p(x)}dx \to 0, \tag{20}$$

which implies $\|\nabla v_n\|_{p(x)} \to 0$. Since $\|_{p(x)}$ is (weakly) inferior semi-continuous, we find

$$0 \le \|\nabla v\|_{p(x)} \le \liminf_{n\to\infty} \|\nabla v_n\|_{p(x)} = 0.$$

Therefore $\nabla v(x) = 0$ a.e. $x \in \Omega$ which yields $v \in \mathbb{R}$. It follows that

$$\lim_{n\to\infty} \|\nabla(v_n - v)\|_{p(x)} = \lim_{n\to\infty} \|\nabla v_n\|_{p(x)} = 0. \tag{21}$$

Relation (21) and the fact that v_n converges strongly to v in $L^{p(x)}(\Omega)$ imply that actually v_n converges strongly to v in X. That fact combined with $\|v_n\| = 1$ shows that $v \neq 0$ and consequently $|u_n(x)| \to \infty$ as $n \to \infty$ a.e. $x \in \Omega$.

Next, choosing $\varphi = u_n$ in (14) and taking into account that relation (13) holds true, we find

$$\begin{aligned}&\int_\Omega [p^- F(u_n(x)) - f(u_n(x))\cdot u_n(x)]\,dx + \int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega \frac{p^-}{p(x)}|\nabla u_n|^{p(x)}\,dx\\ &\quad\le M\cdot p^- + \epsilon_n \cdot \|u_n\|.\end{aligned}$$

Dividing the above inequality by $\|u_n\|$ we obtain

$$\int_\Omega \frac{p^- F(u_n(x)) - f(u_n(x))\cdot u_n(x)}{|u_n(x)|}\cdot |v_n(x)|\,dx \le \frac{Mp^- + \varepsilon_n\|u_n\|}{\|u_n\|}.$$

Passing to the limit in the above relation we have

$$\liminf_{n\to\infty} \int_\Omega \frac{p^- F(u_n(x)) - f(u_n(x))\cdot u_n(x)}{|u_n(x)|}\cdot |v_n(x)|\,dx \le 0.$$

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The above inequality and the Fatou Lemma imply

$$\int_{\Omega} \liminf_{n\to\infty} \frac{p^- F(u_n(x)) - f(u_n(x))\cdot u_n(x)}{|u_n(x)|} \cdot |v_n(x)|\, dx \le 0. \tag{22}$$

On the other hand, analysing the definition of functions f and F we deduce

$$\lim_{|t|\to\infty} \frac{p^- F(t) - f(t)t}{|t|} = \lim_{|t|\to\infty} \frac{\frac{p^-}{2}t^2 - \frac{p^-}{a+1}|t|^{a+1} - p^- D - t^2 + |t|^{a+1}}{|t|} = \infty,$$

since by relation (11) we have $a+1 > p^-$. It follows that there exists a constant $\alpha > 0$ such that

$$\lim_{|t|\to\infty} \frac{p^- F(t) - f(t)t}{|t|} \ge \alpha > 0.$$

The above inequality, relation (22) and the fact that $|u_n(x)| \to \infty$ as $n \to \infty$ a.e. $x \in \Omega$ imply

$$\int_{\Omega} |v(x)|\, dx \le 0.$$

But $v \neq 0$ is a constant function as we have already noticed and that is a contradiction with the above relation. In this way we have proved that $\{u_n\}$ is bounded in X. Then there exists $u \in X$ such that u_n converges weakly to u in X. Since X is compactly embedded in any $L^{s(x)}(\Omega)$ for any $s \in (\overline{\Omega})$ with $1 < s(x) < (Np^-)/(N-p^-)$ for all $x \in \overline{\Omega}$ we deduce that u_n converges strongly to u in $L^{s(x)}(\Omega)$. That information and the form of f and F imply that

$$\lim_{n\to\infty} \int_{\Omega} f(u_n)(u_n - u)\, dx = 0.$$

In order to prove that u_n converges strongly to u in X we choose $\varphi = u_n - u$ in (14). This yields

$$\left| \int_{\Omega} (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)\, dx \right| \le \int_{\Omega} |f(u_n)||u_n - u|\, dx + \varepsilon \|u_n - u\| + \left| \int_{\Omega} |\nabla u|^{p(x)-2}\nabla u(\nabla u_n - \nabla u)\, dx \right|.$$

All the above pieces of information show that

$$\left| \int_{\Omega} \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)(\nabla u_n - \nabla u)\, dx \right| \to 0.$$

The last relation, the fact that u_n converges strongly to u in $L^{p(x)}(\Omega)$ and Theorem 3.1 in [**10**] imply that u_n converges strongly to u in X, i.e., J satisfies the (PS) condition. The proof of the lemma is complete. □

LEMMA 3. *Assume that condition (11) is fulfilled. Then there exists* $\rho > 0$ *such that for all* $u \in V$ *with* $\|u\| \le \rho$ *we have* $J(u) \ge 0$ *and* $J(e) \le 0$ *for all* $e \in \mathbb{R}$ *with* $|e| \le \rho$.

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Proof. We choose $u \in V$ with $\|u\| = \rho$, where ρ is small enough and will be specified later. The definition of F and relation (11) imply that for all $\epsilon > 0$ there exist $\delta > 0$ and $\gamma > 0$ such that

$$F(t) \leq \varepsilon |t|^{p^+}, \quad \forall\ |t| \leq \delta, \quad \text{a.e. } x \in \Omega$$

and

$$F(t) \leq \varepsilon |t|^{p^+} + \gamma |t|^{a+1}, \quad \forall\ |t| \in \mathbb{R}, \quad \text{a.e. } x \in \Omega. \tag{23}$$

Since $p^- \leq p(x)$ for all $x \in \overline{\Omega}$ we have that $L^{p(x)}(\Omega)$ is continuously embedded in $L^{p^-}(\Omega)$. Thus, there exists $k_0 > 0$ such that

$$|u|_{p^-} \leq k_0 |u|_{p(x)}, \quad u \in L^{p(x)}(\Omega).$$

Assuming $\|u\| \leq 1$ it follows $\|\nabla u\|_{p(x)} \leq 1$. Hence by (6) we deduce that

$$\int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx \geq \frac{1}{p^+} \|\nabla u\|_{p(x)}^{p^+} \geq C \|\nabla u\|_{p^-}^{p^+}. \tag{24}$$

Inequalities (12) and (24) imply

$$\int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx \geq C \|u\|_{W^{1,p^-}(\Omega)}^{p^+}. \tag{25}$$

Relations (23) and (11) yield

$$\int_\Omega F(u)\, dx \leq \epsilon |u|_{p^+}^{p^+} + \gamma_1 |u|_{a+1}^{a+1} \leq \epsilon c_1 \|u\|_{W^{1,p^-}(\Omega)}^{p^+} + \gamma_2 \|u\|_{W^{1,p^-}(\Omega)}^{a+1}, \tag{26}$$

where γ_1 and γ_2 are positive constants.

Choosing ϵ small enough and using relations (25) and (26) we obtain

$$J(u) \geq C \|u\|_{W^{1,p^-}(\Omega)}^{p^+} - \gamma_1 \|u\|_{W^{1,p^-}(\Omega)}^{a+1}. \tag{27}$$

Relations (27) and (11) show that there exists $\theta > 0$ such that

$$J(u) \geq 0, \quad \forall\ u \in V \ \text{ with } \|u\|_{W^{1,p^-}(\Omega)} \leq \theta.$$

Since $V \subset X \subset W^{1,p(x)}(\Omega) \subset W^{1,p^-}(\Omega)$, there exists $C_0 > 0$ such that

$$\|u\|_{W^{1,p^-}(\Omega)} \leq C_0 \|u\|, \quad \forall\ u \in V.$$

Taking $\rho > 0$ small enough, $\|u\| \leq \rho$ implies $\|u\|_{W^{1,p^-}(\Omega)} \leq \theta$, for all $u \in V$ and therefore

$$J(u) \geq 0, \quad \forall\ u \in V \ \text{ with } \|u\| \leq \rho.$$

Finally, for $t \in \mathbb{R}$ considering the constant function which belongs to X we have $J(t) = -\int_\Omega F(t)\, dx$. But $F(t) \geq 0$ for $|t|$ small enough. It follows that for $t \in \mathbb{R}$ small enough we have $J(t) \leq 0$. The proof of the lemma is complete. □

Proof of Theorem 2 completed. By Lemmas 1, 2 and 3 we remark that the hypotheses of Theorem 1 are fulfilled. Thus, we conclude that problem (1) has two non-trivial weak solutions. Theorem 2 is verified. □

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