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Higher Dimensional Spaces of Functions on the Spectrum of a Uniform Algebra

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Abstract. In this paper we introduce a nested family of spaces of continuous functions defined on the spectrum of a uniform algebra. The smallest space in the family is the uniform algebra itself. In the "finite dimensional" case, from some point on the spaces will be the space of all continuous complex-valued functions on the spectrum. These spaces are defined in terms of solutions to the nonlinear Cauchy–Riemann equations as introduced by the author in 1976, so they are not generally linear spaces of functions. However, these spaces do shed light on the higher dimensional properties of a uniform algebra. In particular, these spaces are directly related to the generalized Shilov boundary of the uniform algebra (as defined by the author and, independently, by Sibony in the early 1970s).

In all that follows, $\mathcal{A}$ will be a uniform algebra defined on a compact Hausdorff space X with spectrum M. We will regard the functions in $\mathcal{A}$ as being defined as continuous, complex-valued functions on all of M by their natural extension there. (For the definition of uniform algebras and their basic properties, please consult one of the standard introductions to the area such as [1, 6, 7, 9].)

We wish to define two families of spaces of functions on M which will help to elaborate the higher dimensional structure of $\mathcal{A}$ and M. These families will be denoted here by $\mathcal{A}_q$ and $\tilde{\mathcal{A}}_q$. It remains to be demonstrated whether or not $\mathcal{A}_q$ and $\tilde{\mathcal{A}}_q$ can be different from each other. These spaces are defined in terms of solutions of the generalized Cauchy–Riemann equations as defined by the author, so we first recall the definition and some basic properties of solutions of these equations (See [3, 4] for proofs and further details.)

Definition 1 Let n be a positive integer, and let Ω be an open subset of $\mathbb{C}^n$. Let q be a nonnegative integer. Then the generalized Cauchy–Riemann equation of order q is given by

$$\overline{\partial} f \wedge (\partial\overline{\partial} f)^q = 0$$

where f is, say, a C^∞ complex-valued function defined on Ω. In this case we will say that the function f is q-holomorphic on Ω.

The following result gives an alternative characterization of q-holomorphic functions.

Proposition 2 *Let $f \in C^\infty(\Omega)$ for some open subset Ω of $\mathbb{C}^n$. Let $\mathcal{M}(f)$ be the $n \times (n+1)$ matrix defined by adjoining the antiholomorphic gradient of f to the complex Hessian of f. So*

$$\mathcal{M}(f) = ((f_{\bar{z}_i z_j}) f_{\bar{z}})$$

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where

$$f_{\bar{z}} = (f_{\bar{z}_1}, \dots, f_{\bar{z}_n}).$$

Then f is q-holomorphic on Ω if and only if the rank of $\mathcal{M}(f)$ is less than or equal to q everywhere on Ω.

The following are some basic properties of q-holomorphic functions.

Proposition 3 *If ϕ is a holomorphic function into the domain of a q-holomorphic function f, then $f \circ \phi$ is q-holomorphic.*

Proposition 4 *If ϕ is a holomorphic function on $\mathbb{C}$ whose domain contains the image of a q-holomorphic function f, then $\phi \circ f$ is q-holomorphic.*

Proposition 5 *Let $f \in C^\infty(\Omega)$ for some open subset Ω of $\mathbb{C}^n$. If locally there exists coordinates $\zeta_1, \dots, \zeta_n$ on Ω such that f is holomorphic in $\zeta_1, \dots, \zeta_{n-q}$, then f is q-holomorphic.*

The converse of the above Proposition should be true with suitable nondegeneracy conditions on the derivatives. See [3] for an example. The next result is very useful, and shows that q-holomorphic functions on $\mathbb{C}^n$ are nontrivial when $q < n$.

Theorem 6 *Let $f \in C^\infty(\Omega)$ for some open subset Ω of $\mathbb{C}^n$, and suppose that f is q-holomorphic on Ω for some $q < n$. Then f satisfies the "maximum principle" on Ω. That is, if K is a compact subset of Ω, then $|f|$ achieves its maximum somewhere on the topological boundary of K.*

Example 1 Obviously holomorphic functions are q-holomorphic for all $q \geq 0$. The functions $|z|^2$ and $|w|^2$ are 1-holomorphic on $\mathbb{C}^2$. However, $|z|^2 + |w|^2$ is not 1-holomorphic on $\mathbb{C}^2$ (or on any subdomain of $\mathbb{C}^2$).

Having reviewed some of the basic properties of q-holomorphic functions, we now relate these functions to the theory of uniform algebras. We introduce here certain new spaces of functions defined on the spectrum of a uniform algebra.

Definition 7 Given a uniform algebra $\mathcal{A}$ on a compact set X with spectrum M, let $\tilde{\mathcal{A}}_q = \text{Closure}\{\phi(f_1, \dots, f_m) : f_1, \dots, f_m \in \mathcal{A}$, and ϕ is q-holomorphic in a neighborhood of the joint spectrum $\sigma(f_1, \dots, f_m)\}$.

Here closure denotes closure in the uniform norm on M, and the joint spectrum $\sigma(f_1, \dots, f_m)$ is the image of M in $\mathbb{C}^n$ under the mapping $x \to (f_1(x), \dots, f_m(x))$.

$\tilde{\mathcal{A}}_q$ gives us one way to measure the "codimension" of a uniform algebra $\mathcal{A}$ relative to $C(M)$, the space of all continuous complex-valued functions on M, by comparing $\tilde{\mathcal{A}}_q$ with $C(M)$ for various values of q.

Definition 8 $\operatorname{codim}(\mathcal{A}) = \min\{n : \tilde{\mathcal{A}}_n = C(M)\}$.

(If the set on the right-hand side is empty, we will say that $\operatorname{codim}(\mathcal{A})$ is infinite.)

Recall the following notion of an analytic polydisk in the spectrum:

Definition 9 A subset V of the spectrum M is called an "analytic polydisk" of dimension n if there exists a continuous one-to-one mapping ψ of the interior of the standard unit polydisk Δ^n onto V such that all of the functions in $\mathcal{A}$ are holomorphic on Δ^n when composed with ψ.

The existence of an analytic polydisk in the spectrum M forces $\operatorname{codim}(\mathcal{A})$ to be at least as large as the dimension of the polydisk.

Proposition 10 *Let n be a positive integer, and suppose that M contains an analytic polydisk of dimension n. Then* $\operatorname{codim}(\mathcal{A}) \geq n$.

Proof Let V be an analytic polydisk in M of dimension n, and let ψ be a corresponding map from Δ^n onto V. Let K be any compact subset of Δ^n, and let f be any function in $\tilde{\mathcal{A}}_{n-1}$. Suppose first that $f = \phi(f_1, \dots, f_m)$ for some $f_1, \dots, f_m \in \mathcal{A}$, and for some ϕ which is $(n-1)$-holomorphic near the joint spectrum of $f_1, \dots, f_m$.

Then $f_1 \circ \psi, \dots, f_m \circ \psi$ are holomorphic on Δ^n, hence by Proposition 3 $f \circ \psi$ is $(n-1)$-holomorphic on Δ^n. It follows by Theorem 6 that $f \circ \psi$ must achieve its maximum value somewhere on the boundary of K. Since functions such as f are dense in $\tilde{\mathcal{A}}_{n-1}$, it follows that all functions in $\tilde{\mathcal{A}}_{n-1}$ have this same property.

Let $K = \{z \in \Delta^n : |z_j| \leq 1/2 \text{ for all } j\}$. There is a function $g \in C(M)$ which is 1 on $\psi(0)$ and 0 on $\psi(\partial K)$. By the above, g is not in $\tilde{\mathcal{A}}_{n-1}$, so $\operatorname{codim}(\mathcal{A}) \geq n$ as claimed. ■

Conversely, if the spectrum M is embedded in $\mathbb{C}^n$, then $\operatorname{codim}(\mathcal{A})$ can be no larger than n.

Proposition 11 *Let $\mathcal{A}$ be a uniform algebra on a compact set X. Suppose that for some n the spectrum M of $\mathcal{A}$ is contained in $\mathbb{C}^n$ in the sense that there exist n functions $f_1, \dots, f_n \in \mathcal{A}$ such that the mapping $F = (f_1, \dots, f_n)$ from M to the joint spectrum is one-to-one. Then* $\operatorname{codim}(\mathcal{A}) \leq n$.

Proof Let P be any complex polynomial in $2n$ variables. Consider $g(z_1, \dots, z_n) = P(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$. By Proposition 5, any such g is n-holomorphic on $\mathbb{C}^n$. Hence $g \circ F \in \tilde{\mathcal{A}}_n$. But functions such as g are dense in the space of continuous functions on the joint spectrum. It follows that $\tilde{\mathcal{A}}_n = C(M)$. So $\operatorname{codim}(\mathcal{A}) \leq n$ as claimed. ■

What functions are in $\tilde{\mathcal{A}}_q$? Consider the following.

Definition 12 Given a uniform algebra $\mathcal{A}$ on a compact set X with spectrum M, let $\mathcal{A}_q = \text{Closure}\{P(f_1, \dots, f_r, \bar{f}_{r+1}, \dots, \bar{f}_{r+q}) : r \geq 0, P \text{ is a polynomial}, f_j \in \mathcal{A}\}$.

Example 2 Let $\mathcal{A} = P(\Delta^2)$, the uniform closure of the complex polynomials on $\mathbb{C}^2$ restricted to Δ^2. The spectrum M of $\mathcal{A}$ is, of course, Δ^2. Note that $|z|^2 \in \mathcal{A}_1$ and $|w|^2 \in \mathcal{A}_1$, but $|z|^2 + |w|^2 \in C(M) \setminus \mathcal{A}_1$.

Proposition 13 *$\mathcal{A}_q$ is a subspace of $\tilde{\mathcal{A}}_q$.*

Proof Let P be a complex polynomial in $r+q$ variables. Then the function $z \mapsto P(z_1,\dots,z_r,\bar{z}_{r+1},\dots,\bar{z}_{r+q})$ is q-holomorphic on $\mathbb{C}^{r+q}$. It follows from the definition of $\tilde{\mathcal{A}}_q$ that $P(f_1,\dots,f_r,\bar{f}_{r+1},\dots,\bar{f}_{r+q}) \in \tilde{\mathcal{A}}_q$. Hence $\mathcal{A}_q \subseteq \tilde{\mathcal{A}}_q$. ■

Note that $\mathcal{A}_0 = \mathcal{A}$, hence the standard holomorphic functional calculus implies that $\mathcal{A}_0 = \tilde{\mathcal{A}}_0$. It is not immediately clear, at least to this author, whether the same always holds true when $q > 0$. So for now, we could also define an alternative measure of the "codimension" of a uniform algebra as follows.

Definition 14 $\operatorname{CODIM}(\mathcal{A}) = \min\{n : \mathcal{A}_n = C(M)\}$.

Proposition 15 $\operatorname{CODIM}(\mathcal{A}) \geq \operatorname{codim}(\mathcal{A})$. *Hence if the spectrum M contains a polydisk of dimension n, then* $\operatorname{CODIM}(\mathcal{A}) \geq n$. *Furthermore, if, as defined above, M is "contained" in $\mathbb{C}^n$, then* $\operatorname{CODIM}(\mathcal{A}) \leq n$.

Proof That CODIM $\geq$ codim follows immediately from Proposition 13. The last assertion follows from the fact that the functions from $\tilde{\mathcal{A}}_n$ used in the proof of Proposition 11 are actually in the subspace $\mathcal{A}_n$. ■

To further understand the structure of the spectrum M in terms of properties of $\mathcal{A}_q$ (or $\tilde{\mathcal{A}}_q$) it is useful to recall the notion of a "boundary" for a space of functions.

Definition 16 If X is a compact Hausdorff space and $\mathcal{F}$ is any collection of continuous complex-valued functions on X, then we will say that a subset E of X is a "boundary" for $\mathcal{F}$ if for every $f \in \mathcal{F}$ there exists $x \in E$ such that $|f(x)| = \|f\| = \max\{|f(x)| : x \in X\}$.

A fundamental fact in the theory of uniform algebras is that for any uniform algebra $\mathcal{A}$ on a compact Hausdorff space X with spectrum M there is a smallest closed boundary (the "Shilov boundary") $K \subseteq X \subseteq M$ for $\mathcal{A}$ considered as a space of functions on X or M. That is, K is a closed subset of X which is boundary for $\mathcal{A}$, and if K_0 is any other closed boundary in M for $\mathcal{A}$, then $K_0 \supseteq K$. We will denote the Shilov boundary of $\mathcal{A}$ by $\partial_0\mathcal{A}$.

The notion of Shilov boundary was independently generalized to a higher dimensional concept by Sibony [8] and the author [2]. A full discussion of these ideas may be found in Tonev's book [10].

Notation If $\mathcal{A}$ is a uniform algebra with spectrum M, and if $\mathcal{G} \subseteq \mathcal{A}$, we will let $\#\mathcal{G}$ denote the cardinality of the set $\mathcal{G}$ and we will let $V(\mathcal{G}) = \{x \in M : \text{ for all } g \in \mathcal{G}, g(x) = 0\}$.

Definition 17 Let $\mathcal{A}$ be a uniform algebra with spectrum M. If q is a nonnegative integer, we define the q-th order Shilov boundary of $\mathcal{A}$, denoted $\partial_q\mathcal{A}$, to be the smallest closed subset of M with the property that if $\mathcal{G} \subseteq \mathcal{A}$ and $\#\mathcal{G} \leq q$ and $V(\mathcal{G}) \neq \varnothing$, then for every $f \in \mathcal{A}$ there exists $x \in \partial_q\mathcal{A} \cap V(\mathcal{G})$ such that

$$|f(x)| = \max\{|f(y)| : y \in V(\mathcal{G})\}.$$

Note that when $q = 0$, this definition of $\partial_q\mathcal{A}$ is consistent with the previous definition of the original Shilov boundary. It is an easy exercise to show that $\partial_q\mathcal{A}$ always exists. Part of the motivation for the definition of the generalized Shilov boundary may be seen in the fact that when $n \geq 2$, $\partial_0 P(\Delta^n)$ is a proper subset of the topological boundary of Δ^n in $\mathbb{C}^n$. However, for all $n > 0$, $\partial_{n-1}P(\Delta^n)$ is exactly the topological boundary of Δ^n.

It turns out that the generalized Shilov boundary is related to the space of functions $\mathcal{A}_q$.

Theorem 18 *If $\mathcal{A}$ is a uniform algebra and q is a nonnegative integer, then $\partial_q\mathcal{A}$ is the smallest closed boundary for the space $\mathcal{A}_q$.*

Proof Let K be a subset of the spectrum M which is a closed boundary for $\mathcal{A}_q$. We will first prove that K contains $\partial_q\mathcal{A}$. Let $\mathcal{G} \subset \mathcal{A}$ with $\#\mathcal{G} \leq q$ and $V(\mathcal{G}) \neq \varnothing$. Let $f \in \mathcal{A}$.

We may suppose that $\mathcal{G} = \{g_1, \dots, g_q\}$ for q functions from $\mathcal{A}$ with the property that $\sum_{j=1}^q |g_j|^2 \leq 1/2$ on M. Assume that f is not identically zero on $V(\mathcal{G})$. (If not, work with $f + 1$ instead.) Choose $x^* \in V(\mathcal{G})$ such that $|f(x^*)| = \max\{|f(y)| : y \in V(\mathcal{G})\} > 0$.

For each nonnegative integer n define a polynomial P_n in $2q+1$ variables by $P_n = z_1(1 - \sum_{j=1}^q z_{j+1}z_{j+q+1})^n$, and let $p_n = P_n(f, g_1, \dots, g_q, \bar{g}_1, \dots, \bar{g}_q) \in \mathcal{A}_q$. Since $p_n \in \mathcal{A}_q$ and since K is a boundary for $\mathcal{A}_q$, $\forall n\ \exists x_n \in K$ such that

$$|p_n(x_n)| = \max\{|p_n(y)| : y \in M\} \geq |p_n(x^*)| = |f(x^*)|.$$

Let x be a limit point in K of $\{x_n\}$. Then $x \in V(\mathcal{G})$. For, if not, there exists k such that $g_k(x) \neq 0$. Since x is a limit point of $\{x_n\}$, there is a sequence of large n such that $|g_k(x_n)| > (1/2)|g_k(x)|$. Since $\sum |g_j|^2 \leq 1/2$, we have for these large n that $|1 - \sum |g_j(x_n)|^2| = 1 - \sum |g_j(x_n)|^2 \leq 1 - |g_k(x_n)|^2 < 1 - (1/4)|g_k(x)|^2$. Let $r = 1 - (1/4)|g_k(x)|^2$. Then $0 < r < 1$, so we can choose a large N such that $r^N < |f(x^*)|/\|f\|$. Then $|p_N(x_N)| = |f(x_N)|(1 - \sum |g_j(x_N)|^2)^N < \|f\|(|f(x^*)|/\|f\|) = |f(x^*)|$. This contradicts the property of p_N established above and shows that $x \in V(\mathcal{G})$, as claimed.

Since $x \in K$, to prove that K contains $\partial_q\mathcal{A}$ it remains to show that $|f(x)| = |f(x^*)|$. We already know that $|f(x)| \leq |f(x^*)|$. But since x is a limit point of $\{x_n\}$, and since for all n we know that $|f(x_n)| \geq |f(x_n)||1 - \sum |g_j|^2| = |p_n(x_n)| \geq |f(x^*)|$, it follows that $|f(x)| \geq |f(x^*)|$ as desired.

Now we wish to show that $\partial_q\mathcal{A}$ is a boundary for $\mathcal{A}_q$. So let P be a polynomial in $r + q$ variables, let $f_1, \dots, f_r$ and $h_1, \dots, h_q$ be functions in $\mathcal{A}$, and let $p =$

$P(f_1,\dots,f_r,\bar{h}_1,\dots,\bar{h}_q) \in \mathcal{A}_q$. We must show that there exists $x \in \partial_q\mathcal{A}$ such that $|p(x)| = \|p\| = \max\{|p(y)| : y \in M\}$.

Choose $x^* \in M$ such that $|p(x^*)| = \|p\|$. Let $g_j = h_j - h_j(x^*)$ for $j = 1,\dots,q$, and let $\mathcal{G} = \{g_1,\dots,g_q\}$. Note that $x^* \in V(\mathcal{G})$. Since for each j we have $h_j = g_j + h_j(x^*)$, we can, by rearranging terms, find a polynomial Q such that $p = Q(f_1,\dots,f_r,\bar{g}_1,\dots,\bar{g}_q)$.

Let $f = Q(f_1,\dots,f_r,g_1,\dots,g_q) \in \mathcal{A}$. Then, by definition of the generalized Shilov boundary, there exists $x \in \partial_q\mathcal{A} \cap V(\mathcal{G})$ such that: $|f(x)| = \max\{|f(y)| : y \in V(\mathcal{G})\}$. In particular, $|f(x)| \geq |f(x^*)|$.

But for $y \in V(\mathcal{G})$ we have $g_j(y) = 0 = \bar{g}_j(y)$ for $j = 1,\dots,q$. Hence for such y we get

$$\begin{aligned} f(y) &= Q(f_1(y),\dots,f_r(y),g_1(y),\dots,g_q(y)) \\ &= Q(f_1(y),\dots,f_r(y),\bar{g}_1(y),\dots,\bar{g}_q(y)) \\ &= p(y). \end{aligned}$$

Since $x, x^* \in V(\mathcal{G})$, we get $|p(x)| = |f(x)| \geq |f(x^*)| = |p(x^*)| = \|p\| \geq |p(x)|$. Therefore $|p(x)| = \|p\|$ as desired. ■

Another concept from the standard theory of uniform algebras which also applies in the present context is the notion of a "peak set" or "peak point" for which we give a generalized definition here.

Definition 19 If X is a compact Hausdorff space and $\mathcal{F}$ is any collection of continuous complex-valued functions on X, then we will say that a closed subset K of X is a "peak set" for $\mathcal{F}$ if there exists $f \in \mathcal{F}$ such that $f \equiv 1$ on K and $|f| < 1$ on $X \setminus K$. If $K = \{x\}$ for some point $x \in X$, x is called a "peak point."

The following result follows immediately from the above definitions.

Proposition 20 *If $\mathcal{A}$ is a uniform algebra with spectrum M and q is a nonnegative integer, then any boundary $E \subseteq M$ for $\mathcal{A}_q$ must contain the set $\{x \in M : x \text{ is a peak point for } \mathcal{A}_q\}$.*

It is a result of Bishop ([5], or see, for example, [9]) that for a uniform algebra $\mathcal{A}$ on a compact *metric* space, the set of peak points for $\mathcal{A}$ is in fact a boundary for $\mathcal{A}$. It is therefore the minimal boundary for $\mathcal{A}$, and its closure must be $\partial_0\mathcal{A}$.

Example 3 Let $\mathcal{A} = P(\Delta^2)$. Then $\partial_1\mathcal{A} = \partial\Delta^2 = \{x \in \Delta^2 : x \text{ is a peak point for } \mathcal{A}_1\}$. In fact, if (z_0, w_0) is a point in the topological boundary of Δ^2, then, say, $|z_0| = 1$ and $|w_0| \leq 1$. In that case the following function in $\mathcal{A}_1$ "peaks" at (z_0, w_0):

$$f(z,w) = P(z,w,\bar{w}) = (1/8)(\bar{z}_0 z + 1)(4 - |w - w_0|^2).$$

We can easily derive a weaker version of Bishop's result for the peak points of $\mathcal{A}_q$.

Theorem 21 *Let $\mathcal{A}$ be a uniform algebra defined on a compact metric space X with spectrum M. Then*

$$\partial_q \mathcal{A} = \mathrm{Closure}\{x \in M : \textit{ x is a peak point for } \mathcal{A}_q\}.$$

Proof Since by Theorem 18 we have that $\partial_q \mathcal{A}$ is a boundary for $\mathcal{A}_q$, by Proposition 20 we have that

$$\partial_q \mathcal{A} \supseteq \mathrm{Closure}\{x \in M : \text{ x is a peak point for } \mathcal{A}_q\}.$$

On the other hand, let $f_1, \ldots, f_r, g_1, \ldots, g_q$ be functions in $\mathcal{A}$, and let P be any polynomial in $r + q$ variables. Let $p = P(f_1, \ldots, f_r, \bar{g}_1, \ldots, \bar{g}_q) \in \mathcal{A}_q$. Let $\mathcal{B}$ be the uniform algebra on M generated by $\mathcal{A}$ together with $\mathcal{G} = \{\bar{g}_1, \ldots, \bar{g}_q\}$. Note that $\mathcal{B} \subseteq \mathcal{A}_q$.

Since $p \in \mathcal{B}$, by Bishop's result there is a peak point $x \in M$ for $\mathcal{B}$ such that $|p(x)| = \|p\|$. But any peak point for $\mathcal{B}$ is also a peak point for $\mathcal{A}_q$. So, in particular, $|p|$ achieves its maximum on $\mathrm{Closure}\{x \in M : \text{ x is a peak point for } \mathcal{A}_q\}$.

Since functions of the form of p are dense in $\mathcal{A}_q$, it follows that $\mathrm{Closure}\{x \in M : x \text{ is a peak point for } \mathcal{A}_q\}$ is a closed boundary for $\mathcal{A}_q$ and must (by Theorem 18) contain $\partial_q \mathcal{A}$. ■

Open Questions

The above results suggest a number of questions for further study. Here are a few whose answers are not immediately obvious to the author.

If $\mathcal{A}$ and $\mathcal{B}$ are two uniform algebras, it is possible to define the tensor product algebra $\mathcal{A} \otimes \mathcal{B}$ (see, for example, [9]). For algebras such as $P(\Delta^n)$, we clearly have $\mathrm{CODIM}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{CODIM}(\mathcal{A}) + \mathrm{CODIM}(\mathcal{B})$. Is this always true? Is it true for "codim" instead of "CODIM"?

In the simple examples such as $\mathcal{A} = P(\Delta^n)$, we clearly have $\mathrm{CODIM}(\mathcal{A}) = \mathrm{codim}(\mathcal{A})$. Is this true for all uniform algebras?

Note that $\tilde{A}_q = \mathcal{A}_q$ for all uniform algebras when $q = 0$. This is simply a direct consequence of the usual holomorphic functional calculus. Is this result also true when $q > 0$? If so, this would, of course, imply that CODIM and codim are always the same.

In simple examples where $\mathrm{CODIM}(\mathcal{A}) = n$, we can find n functions $g_1, \ldots, g_n \in \mathcal{A}$ such that the uniform algebra generated by $\mathcal{A} \cup \{\bar{g}_1, \ldots, \bar{g}_n\}$ on the spectrum M is $C(M)$. Is this always true?

Theorem 21 shows that

$$\mathrm{Closure}\{x \in M : \text{ x is a peak point for } \mathcal{A}_q\}$$

is a boundary for $\mathcal{A}_q$ when $\mathcal{A}$ is a uniform algebra defined on a metric space. Is the possibly smaller set

$$\{x \in M : \text{ x is a peak point for } \mathcal{A}_q\}$$

always a boundary for $\mathcal{A}_q$? Bishop's result says that it is when $q = 0$.

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References

[1] H. Alexander and J. Wermer, *Several Complex Variables and Banach Algebras,* Third Edition. Graduate Texts in Mathematics 35, Springer-Verlag, New York, 1998.
[2] R. Basener, *A generalized Shilov boundary and analytic structure.* Proc. Amer. Math. Soc. **47**(1975), 98–104.
[3] R. Basener, *Nonlinear Cauchy-Riemann equations and q-pseudoconvexity.* Duke Math. J. **43**(1976), no. 1, 203–213.
[4] R. Basener, *Several dimensional properties of the spectrum of a uniform algebra.* Pacific J. Math. **74**(1978), no. 2, 297–306.
[5] E. Bishop, *A minimal boundary for function algebras.* Pacific J. Math. **9**(1959), 629–642.
[6] A. Browder, *Introduction to Function Algebras.* Benjamin, New York, 1969.
[7] T. Gamelin, *Uniform Algebras.* Prentice-Hall, Englewood Cliffs, NJ, 1969.
[8] N. Sibony, *Multi-dimensional analytic structure in the spectrum of a uniform algebra.* In: Spaces of Analytic Functions, Springer Lecture Notes in Math. 512, Springer Verlag, 1976, pp. 139–175.
[9] E. L. Stout, *The Theory of Uniform Algebras.* Bogden and Quigley, Tarrytown-on-Hudson, NY, 1971.
[10] T. Tonev, *Big-Planes, Boundaries and Function Algebras.* North-Holland Mathematics Studies 172, North-Holland Publishing, Amsterdam, 1992.

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